Boundary terms in the action (Gibbons-Hawking-York)

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If we start from the Einstein-Hilbert action (setting $16\pi G=1$)

$$I_G = \int_{\mathcal{M}} \sqrt{-g}\, R$$

and vary the metric $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$, we obtain

$$\delta I_G = \int_{\mathcal{M}} \sqrt{-g} \left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} + \int_{\mathcal{M}} \sqrt{-g}\, \nabla_\mu v^\mu$$

$$= \int_{\mathcal{M}} \sqrt{-g} \overbrace{\left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \right)} \delta g^{\mu\nu} + \int_{\partial\mathcal{M}} \overbrace{\sqrt{h}\, n_\mu v^\mu}^{?}$$

$h_{\mu\nu} = g_{\mu\nu} \pm n_\mu n_\nu$: metric induced at $\partial\mathcal{M}$

n^μ: outward normal to $\partial\mathcal{M}$

where

$$v_\mu = g^{\nu\rho} \left(\nabla_\rho \delta g_{\mu\nu} - \nabla_\mu \delta g_{\rho\nu} \right)$$

We want to recover the Einstein equations by imposing Dirichlet bc's on the metric, $\delta g_{\mu\nu}\big|_{\partial\mathcal{M}} = 0$

(actually we only need to require that $\delta h_{ab}=0$: we can always redefine coordinates such that $\delta h_{ab}=0$ implies $\delta g_{ab}\big|_{\partial\mathcal{M}}=0$)

But this boundary condition doesn't eliminate the boundary term.

Let's analyze this term in more detail:

$$\int_{\mathcal{M}} \sqrt{-g}\, \nabla_\mu v^\mu = \int_{\partial\mathcal{M}} \sqrt{h}\, n_\mu v^\mu$$

$$n^\mu v_\mu = n^\mu g^{\nu\rho} \left(\nabla_\rho \delta g_{\mu\nu} - \nabla_\mu \delta g_{\rho\nu} \right) = \left(n^\mu h^{\nu\rho} \pm \underbrace{n^\mu n^\nu n^\rho \right) \nabla_{[\rho} \delta g_{\mu]\nu}}_{=0 \text{ by symmetry}}$$

$$= n^\mu h^{\nu\rho} \left(\underbrace{\nabla_\rho \delta g_{\mu\nu}}_{\substack{=0 \text{ since} \\ h^{\nu\rho}\nabla_\rho \delta g_{\mu\nu}=0}} - \nabla_\mu \delta g_{\rho\nu} \right) = -\, n^\mu h^{\nu\rho} \nabla_\mu \delta g_{\nu\rho}$$

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We have a derivative of a normal variation of the metric, projected onto $\partial\mathcal{M}$.
This reminds us of the extrinsic curvature:

$$K_{\mu\nu} = h_\mu^\rho \nabla_\rho n_\nu \qquad K = h_\mu^\nu \nabla_\nu n^\mu = h_\mu^\nu \partial_\nu n^\mu + h_\mu^\nu \Gamma^\mu_{\nu\rho} n^\rho$$

If we fix $g_{\mu\nu}$ at $\partial\mathcal{M}$, then $h_{\mu\nu}$ and n_μ are fixed. But even if we only fix $h_{\mu\nu}$, the variations of n_μ can be gauged away w/ a diffeo.

Then

$$\delta K = h_\mu^\nu \delta\Gamma^\mu_{\nu\rho} n^\rho = \frac{1}{2} n^\rho \overbrace{h_\mu^\nu g^{\mu\sigma}}^{h^{\nu\sigma}} (\nabla_\nu \delta g_{\rho\sigma} + \nabla_\rho \delta g_{\nu\sigma} - \nabla_\sigma \delta g_{\nu\rho})$$

$$= \frac{1}{2} n^\rho h^{\nu\sigma} \left(\nabla_{[\nu} \delta g_{\sigma]\rho} + \nabla_\rho \delta g_{\nu\sigma} \right)$$

$$= \frac{1}{2} n^\rho h^{\nu\sigma} \nabla_\rho \delta g_{\nu\sigma}$$

$$= -\frac{1}{2} n^\mu v_\mu$$

$$\Rightarrow \delta I_G = \int_{\mathcal{M}} \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} - 2 \int_{\partial\mathcal{M}} \sqrt{h}\, \delta K \qquad \text{when } \delta g_{\mu\nu}\big|_{\partial\mathcal{M}} = 0$$

In order to eliminate the last term and have a well defined variational problem w/ Dirichlet b.c.'s, we fix the action by adding to it an extra boundary term:

$$\boxed{I_G = \frac{1}{16\pi G} \int_{\mathcal{M}} \sqrt{-g}\, R + \frac{1}{8\pi G} \int_{\partial\mathcal{M}} \sqrt{h}\, K}$$

$\downarrow$ Einstein-Hilbert $\qquad$ $\downarrow$ Gibbons-Hawking-York

The GHY term is crucial whenever the gravitational action (instead of the gravitational equations) is used. For instance, when

- defining asymptotic conserved energies and momenta
- computing the action, for instance, for black hole thermodynamics, or for holographic complexity
- doing any semiclassical quantum gravity analysis
- boundaries are important

Why did we have to introduce it? For a scalar or vector field, we didn't need that. The reason is that the Lagrangian has R, which has 2^{nd} derivatives of the metric (since Riemann $\sim \partial\Gamma + \Gamma\Gamma$ and $\partial\Gamma \sim \partial^2 g$) whereas usually Lagrangians only have first derivatives: $\partial_\mu \phi \partial_\nu \phi$
$(\partial_\mu A_\nu)(\partial^\mu A^\nu)$

Consider as a toy model

$$I = -\frac{1}{2}\int dt\, q\ddot{q} \qquad q \to q + \delta q \quad \dot{q} \to \dot{q} + \delta\dot{q} \quad \ddot{q} \to \ddot{q} + \delta\ddot{q}$$

$$\delta I = -\int_I dt\, \ddot{q}\,\delta q + \frac{1}{2}\int_I dt \frac{d}{dt}\left(-q\,\delta\dot{q} + \dot{q}\,\delta q\right)$$

$$= -\int_I dt\, \ddot{q}\,\delta q + \frac{1}{2}\left(-q\,\delta\dot{q} + \dot{q}\,\delta q\right)\Big|_{\partial I}$$

and if we impose $\delta q\big|_{\partial I} = 0$

Then we don't cancel the term $\delta\dot{q}\big|_{\partial I}$

We fix it by adding an extra bdy term:

$$\tilde{I} = I + \overbrace{\frac{1}{2} q\dot{q}\Big|_{\partial I}}^{GHY} = -\frac{1}{2}\int dt\, q\ddot{q} + \frac{1}{2}\int dt \frac{d}{dt}(q\dot{q}) = \frac{1}{2}\int dt\, \dot{q}^2$$

: so we recover the usual action, w/ only first derivatives

$$\delta\tilde{I} = -\int_I \ddot{q}\,\delta q + \dot{q}\,\delta q\Big|_{\partial I}$$

$\hookrightarrow \delta q\big|_{\partial I} = 0$

$$\frac{\delta \tilde{I}}{\delta q} = -\ddot{q}$$

: we obtain the correct eom w/ Dirichlet b.c.s