Entanglement

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Divide a total system into disjoint subsystems A and B.
Eg, partition space into two subregions, and consider the Hilbert space of states w/ support on only A, $\mathcal{H}_A$, and on only B, $\mathcal{H}_B$

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \qquad \text{Basis: } |i\rangle_A |j\rangle_B \qquad {}_A\langle i|j\rangle_A = \delta_{ij}$$

$$ {}_A\langle i|j\rangle_B = 0 \quad \forall i,j$$

Operator $\mathcal{O}$ that acts only on A and is blind to B:

$$\mathcal{O}(|i\rangle_A |j\rangle_B) = (\mathcal{O}|i\rangle_A)\, |j\rangle_B$$

Consider now a state $|\psi\rangle \in \mathcal{H}$, and measure such operator in it:

$$\langle\psi|\mathcal{O}|\psi\rangle = \mathrm{Tr}(|\psi\rangle\langle\psi|\mathcal{O}) = \mathrm{Tr}_A \mathrm{Tr}_B (|\psi\rangle\langle\psi|\mathcal{O})$$

$$= \mathrm{Tr}_A(\mathrm{Tr}_B |\psi\rangle\langle\psi|\mathcal{O}) = \mathrm{Tr}_A((\mathrm{Tr}_B|\psi\rangle\langle\psi|)\mathcal{O})$$

$\hookrightarrow \mathcal{O}$ acts trivially on B

$$= \mathrm{Tr}_A(\rho_A \mathcal{O})$$

$$\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$$

: if we only have access to A but not to B, we describe the system with a mixed density matrix ρ_A.

Entanglement entropy $\quad S_A = -\mathrm{Tr}_A(\rho_A \log \rho_A)$: measures the amount of correlations between A and B in the state $|\psi\rangle$

When the total state is pure (as in the case above) one can easily prove that

$$S_A = S_B$$

If $|\psi\rangle = |\phi\rangle_A |\chi\rangle_B$ then $S_A = S_B = 0$: no entanglement

More generally, any pure state can be written in the Schmidt decomposition

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$$|\psi\rangle = \sum_i \sqrt{p_i}\, |i\rangle_A |\tilde{i}\rangle_B \qquad \text{with } p_i \geq 0 \text{ and } \begin{array}{l}|i\rangle_A \\ |\tilde{i}\rangle_B\end{array} \text{orthonormal}$$

The Schmidt number is the number of non-zero p_i.

If the Schmidt number $= 1 \Rightarrow$ no entanglement $S_A = S_B = 0$

" " " " $> 1 \Rightarrow \exists$ entanglement, $S_A, S_B \neq 0$

If all $p_i \neq 0$ the system is fully entangled

If all p_i are equal, the system is maximally entangled

Examples:

- Bell states in a two-qubit system are maximally entangled

$$|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A |0\rangle_B \pm |1\rangle_A |1\rangle_B\right)$$

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A |1\rangle_B \pm |1\rangle_A |0\rangle_B\right)$$

$$S_A = S_B = \log 2 \quad \text{for all Bell states}$$

- In a system made of two copies of the same system

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \qquad \mathcal{H}_B \simeq \mathcal{H}_A$$

The <u>Thermofield double state</u> is

$$|TFD\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\beta E_n/2} |n\rangle_A |\tilde{n}\rangle_B \qquad H|n\rangle = E_n |n\rangle$$

$$p_n = \frac{e^{-\beta E_n}}{Z}$$

$$S_A = \text{Thermal entropy at Temperature } T = 1/\beta$$

The TFD state is fully entangled

Maximally entangled when $\beta \to 0$ $T \to \infty$ (not really possible in ∞-dimensional Hilbert spaces, eg QFT)