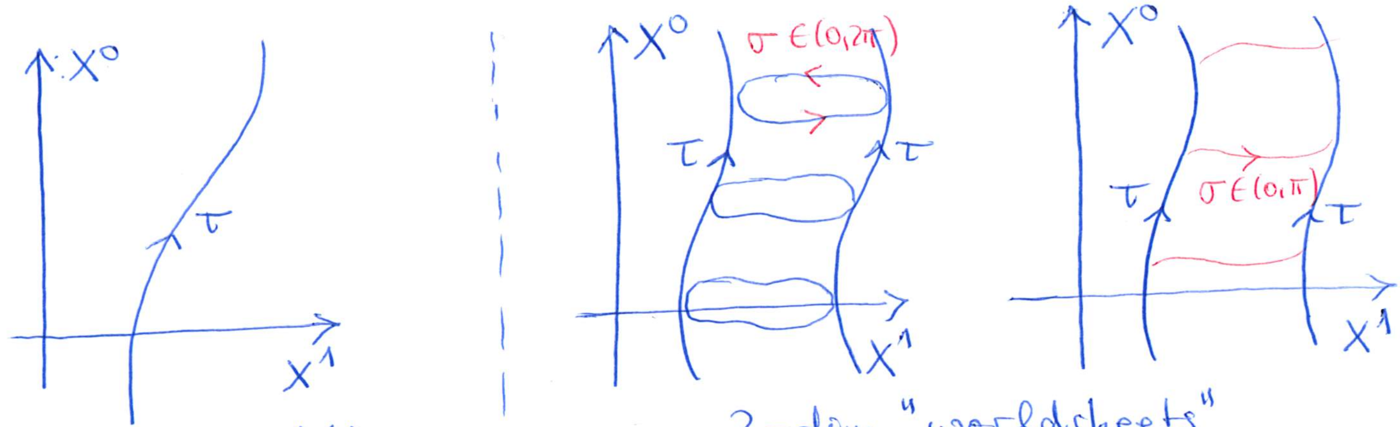
1) Basics of closed bosonic strings

- relativistic point particles vs. strings propagating in $\mathbb{R}^{1,D-1}$ with $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,\ldots,+1)$, $\mu,\nu = 0,1,\ldots,D-1$



1-dim worldline (no thickness) proper time τ

2-dim "worldsheets"

closed strings $\leftrightarrow$ open strings

$\sigma \in (0,2\pi)$ $\qquad$ $\sigma \in (0,\pi)$

1.1) Nambu Goto action

point-particle motion minimizes worldline length

$$S_{WL}[X] = -m \int_{\tau_i \to -\infty}^{\tau_f \to +\infty} d\tau \sqrt{-\eta_{\mu\nu} \dot{X}^\mu(\tau) \dot{X}^\nu(\tau)} \quad , \quad \dot{X}^\mu = \frac{d}{d\tau} X^\mu$$

independent on parametrization, i.e. invariant under "worldline diffeomorphism" $\tau \to \tilde{\tau}(\tau)$

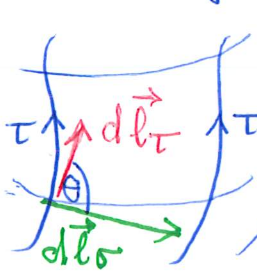
- by analogy: strings minimize worldsheet area

$$S_{NG}[X] = -T \int dA \qquad \text{"Nambu-Goto action"}$$

* area element dA: warmup in Euclidean $\vec{X} \in \mathbb{R}^D$ first

$$dA_E = |d\vec{\ell}_\tau| \cdot |d\vec{\ell}_\sigma| \cdot \sin\theta$$

$$= \sqrt{d\vec{\ell}_\tau^{\,2}\, d\vec{\ell}_\sigma^{\,2} - (d\vec{\ell}_\tau \cdot d\vec{\ell}_\sigma)^2}$$



using $\cos\theta = \dfrac{\vec{dl_\tau}\cdot\vec{dl_\sigma}}{|\vec{dl_\tau}|\cdot|\vec{dl_\sigma}|}$ and $\sin\theta = \sqrt{1-\cos^2\theta}$

and $\vec{dl_\tau} = \dfrac{\partial\vec{X}}{\partial\tau}\, d\tau$ & $\vec{dl_\sigma} = \dfrac{\partial\vec{X}}{\partial\sigma}\, d\sigma$

* in Minkowskian, extra (-1) inside $\sqrt{\cdots}$ and set

$\vec{dl_\tau} \to \dot{X}^\mu d\tau$, $\vec{dl_\sigma} \to X'^\mu d\sigma$, where $X'^\mu = \partial_\sigma X^\mu$

$$\Rightarrow dA = \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}\, d\tau\, d\sigma$$

* notation $\sigma^{\alpha=0,1} = (\tau,\sigma)$ and $d^2\sigma = d\tau\, d\sigma$

* relate area element to "induced metric"

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu} \quad (\text{pullback of } \eta_{\mu\nu})$$

$$\Rightarrow S_{NG}[X] = -T\int d^2\sigma\,\sqrt{-\det\gamma}$$

* invariant under reparametrizations or "worldsheet diffeomorphisms" $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma^\beta)$

* prefactor T is tension $= \dfrac{\text{potential energy}}{\text{spatial length}}$

[I planned to go through the derivation around (1.16) of Tong 0908.0333 during the Q & A]

convention $T = \dfrac{1}{2\pi\alpha'}$ with $\alpha' = l_{string}^2$ "Regge slope" and fundamental string length scale l_{string}

* later: $l_{string} \to 0$ (point-particle limit) or $\alpha' p^2 \to 0$ in dim'less combinations with momenta p^μ recover gauge / gravity interactions from open/closed strings

1.2) Polyakov action

[quadratic in X ⟹ much easier to quantize than $\sqrt{\ldots\dot{X}\ldots X'\ldots}$ in S_{NG}]

Alternative worldsheet action with independent (not above "induced") metric $h_{\alpha\beta}$ (matrix inverse $h^{\alpha\beta}$)

$$S_P[X,h] = \frac{-1}{4\pi\alpha'}\int d^2\sigma\, \sqrt{-\det h}\; h^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X$$

here and below: shorthand for $\eta_{\mu\nu}$ contraction

- on support of e.o.m. $\dfrac{\delta S_P[X,h]}{\delta h^{\alpha\beta}} = 0$, S_P reduces to S_{NG} ⟹ classically equivalent
  - * step 1: after 3–5 lines, $\frac{\delta}{\delta h^{\alpha\beta}}$ variation simplifies to

$$T_{\alpha\beta} := \frac{4\pi}{\sqrt{-\det h}}\,\frac{\delta S_P[X,h]}{\delta h^{\alpha\beta}}$$

$$= -\frac{1}{\alpha'}\left(\partial_\alpha X\cdot\partial_\beta X - \frac{1}{2}h_{\alpha\beta}\left(h^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X\right)\right)$$

which defines energy-momentum tensor $T_{\alpha\beta}$ on the worldsheet for later reference

  - * step 2: present solution $h^{class}_{\alpha\beta}$ of $T_{\alpha\beta}=0$ as

$$h^{class}_{\alpha\beta} = F(\sigma)\,\gamma_{\alpha\beta} \quad \text{with} \quad F(\sigma) = \frac{2}{h^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X}$$

  - * step 3: plug back & enjoy dropout of $F(\sigma)$

$$S_P[X, h\to h^{class}] = \frac{-1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{-F^2\det\gamma}\;\frac{1}{F}\gamma^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X$$

$$= \frac{-1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{-\det\gamma}\;\underbrace{\gamma^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X}_{=\gamma^{\alpha\beta}\gamma_{\alpha\beta}=2} = S_{NG}[X]$$

- symmetries of Polyakov action
  - * diffeomorphisms $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma^\beta)$ (local symm)

$$\tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma) \quad \& \quad \tilde{h}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta}h_{\gamma\delta}(\sigma)$$

* Weyl transformations (local)

$$h_{\alpha\beta}(\sigma) \to \Omega^2(\sigma) h_{\alpha\beta}(\sigma) \quad \& \quad X^\mu(\sigma) \text{ invariant}$$

* Poincaré invariance (global, $\Lambda^\mu{}_\nu$ & c^μ σ-indep.)

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu \quad , \quad \Lambda \in SO(1, D-1)$$

* imposing these symmetries rules out most extra terms, e.g.

potential — cosmolog. constant

$$S_V = \int d^2\sigma \sqrt{-\det h}\, V(X) \quad \& \quad S_\mu = \mu \int d^2\sigma \sqrt{-\det h}$$

1.3) Classical equations of motions & constraints

Can use 2+1 d.o.f of diff × Weyl symmetry to locally eliminate 3 d.o.f of $h_{\alpha\beta} = h_{\beta\alpha}$

- conformal gauge: use 2 d.o.f $\sigma^{\alpha=0,1} \to \tilde{\sigma}^{\alpha=0,1}(\sigma^\beta)$

  $h_{\alpha\beta} \to \Omega^2\, \eta_{\alpha\beta}$ where $\eta_{\alpha\beta} = \mathrm{diag}(-1,1)$

  * PDE's on required $\tilde{\sigma}^\alpha(\sigma^\beta)$ locally have solutions
  * simplifies Polyakov action to

  $$S_P[X, h \to \Omega^2 \eta] = \frac{-1}{4\pi\alpha'} \int d^2\sigma\, \partial_\alpha X \cdot \partial^\alpha X =: S_{CG}[X]$$

  * e.o.m. for X^μ is free wave eq. in 2 dim

  $$\frac{\delta S_{CG}[X]}{\delta X_\mu} \sim \partial_\alpha \partial^\alpha X^\mu = 0$$

  * Weyl invariance can further enforce

  $$\Omega^2 \to 1 \Rightarrow h_{\alpha\beta} = \eta_{\alpha\beta}$$

- classical d.o.f / mode expansion
  - * 2 dim wave operator factorizes in lightcone coordinates $\sigma^{\pm} = \tau \pm \sigma \Rightarrow \partial_{\pm} = \frac{\partial}{\partial \sigma^{\pm}} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$

    $\Rightarrow 0 = \partial_\alpha \partial^\alpha X^\mu = -4\, \partial_+ \partial_- X^\mu$

  - * parametrize $\sigma \cong \sigma + 2\pi$ - periodic solutions via

    "left- & right-movers"

    $$X^\mu(\tau, \sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$$

    $$X_L^\mu(\sigma^+) = \frac{1}{2} x^\mu + \frac{\alpha'}{2} p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\sigma^+}$$

    $$X_R^\mu(\sigma^-) = \underbrace{\frac{1}{2} x^\mu + \frac{\alpha'}{2} p^\mu \sigma^-}_{} + \underbrace{i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-in\sigma^-}}_{}$$

    adds up to $x^\mu + \alpha' p^\mu \tau \Rightarrow$ interpret x^μ & p^μ as center of mass position & momentum

    weird-looking conventions for later convenience

- classical constraints: still have to impose

  $\frac{\delta S_P}{\delta h^{\alpha\beta}} = 0$, setting $h_{\alpha\beta} \to \Omega^2 \eta_{\alpha\beta}$ <u>after</u> variation

  $$0 = T_{\alpha\beta}\Big|_{h_{\alpha\beta}\to\Omega^2\eta_{\alpha\beta}} = \frac{4\pi}{\sqrt{-\det h}} \frac{\delta S_P[X, h]}{\delta h^{\alpha\beta}}\Big|_{h_{\alpha\beta}\to\Omega^2\eta_{\alpha\beta}}$$

  $$= -\frac{1}{\alpha'}\left(\partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X\right)$$

  - * 2 independent constraints by $T_{\alpha\beta} = T_{\beta\alpha}$ & $\underbrace{\eta^{\alpha\beta} T_{\alpha\beta} = 0}$

    e.g. $-\frac{\alpha'}{2}(T_{00} \pm T_{01}) = (\partial_{\pm} X)^2 \overset{!}{=} 0$

    $= \alpha' \cancel{\#} T_{\pm\pm}(-1)$

    $T_{+-} = 0$

* simplified mode expansion @ $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu$

$$\partial_+ X^\mu = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-in\sigma^+}, \quad \partial_- X^\mu = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n^\mu e^{-in\sigma^-}$$

* by $T_{\alpha\beta} = 0$, Fourier modes of $(\partial_\pm X)^2$ have to vanish classically,

$L_n = \tilde{L}_n = 0$ where $L_n = \frac{1}{2} \sum_{k\in\mathbb{Z}} \alpha_{n-k} \circ \alpha_k$

& same @ $(L_n, \alpha_m^\mu) \to (\tilde{L}_n, \tilde{\alpha}_m^\mu)$

* classical mass-shell condition from $L_0 = 0$

$$p^2 = -m^2_{class} \quad @ \quad m^2_{class} = \frac{4}{\alpha'} \sum_{n=1}^{\infty} \alpha_n \circ \alpha_{-n} = \frac{4}{\alpha'} \sum_{n=1}^{\infty} \tilde{\alpha}_n \circ \tilde{\alpha}_{-n}$$

* spoiler: will later obtain m^2-formula in quantized theory with "offset" $-\frac{4}{\alpha'}$

$$m^2 \to \frac{4}{\alpha'} \left(-1 + \sum_{n=1}^{\infty} \text{"operator version of } \alpha_{-n} \circ \alpha_n\text{"} \right)$$

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2) Two-dimensional conformal field theory

will quantize closed bosonic strings using

- infinite-dimensional symmetry
- power of complex analysis

2.1) Infinite-dimensional symmetry from Polyakov action

diff × Weyl invariance of $S_P[X, h]$

$\Rightarrow$ can fix $h_{\alpha\beta} \to \eta_{\alpha\beta}$ locally

- still, $\exists$ residual gauge freedom:

  diffeo's $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma^\beta)$ that can be undone via Weyl

$$\tilde{h}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta} h_{\gamma\delta}(\sigma) \stackrel{!}{=} \Omega^{-2}(\sigma)\, h_{\alpha\beta}(\sigma) \qquad (\text{CT})$$

  i.e. $h_{\alpha\beta} = \eta_{\alpha\beta} \xrightarrow[\text{(CT)}]{\text{diffeo}} \Omega^{-2}\eta_{\alpha\beta} \xrightarrow{\text{Weyl}} \eta_{\alpha\beta}$

  * diffeo's (CT) rescaling $h_{\alpha\beta}$ are called "conformal trf."

  * in $d>2$ dimensions, conformal group has $\frac{1}{2}(d+1)(d+2)$ generators, isomorphic to $SO(2,d)$

  * in $d=2$, conformal group becomes ∞-dimensional

$$ds^2 = -\underbrace{d\sigma^+ d\sigma^-}_{}\quad \text{with } \sigma^\pm = \tau \pm \sigma$$

  simply rescales under any pair of 1-var diffeo's $\underbrace{\sigma^+ \to f(\sigma^+),\ \sigma^- \to g(\sigma^-)}_{}$

  $\exists\ \infty$ many, but set of measure zero in 2-var diffeo

- pass to complex coord's

  * Euclidean time $\tau = -it$ on worldsheet

$$\Rightarrow \left\{ \begin{aligned} z &= i\sigma^+ = t + i\sigma \\ \bar{z} &= i\sigma^- = t - i\sigma \end{aligned} \right\} \Rightarrow ds^2 = dz\, d\bar{z}$$

  * conformal transformations become (anti-) meromorphic maps

  $z \to f(z,\cancel{\bar{z}})$ & $\bar{z} \to g(\cancel{z},\bar{z}) \Rightarrow$ rescales ds^2

- conformal algebra

  * organize ∞ of infinitesimal meromorphic maps via $z \to z + \epsilon z^{n+1}$, $n \in \mathbb{Z}$, $|\epsilon| \ll 1$

* associated generators $\ell_n = -z^{n+1}\partial_z$ & $\bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$ form 2 decoupled Witt algebras $[\ell_m, \ell_n] = (m-n)\ell_{m+n}$ ($m, n \in \mathbb{Z}$ & also $[\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n}$)

* subalgebra $\{\ell_{\pm 1}, \ell_0, \bar{\ell}_{\pm 1}, \bar{\ell}_0\} \cong sl_2(\mathbb{C})$ globally well-defined on Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$

* exponentiating $sl_2(\mathbb{C})$ with 6 generators
  $\Rightarrow SO(2,2) = SO(2,d)|_{d=2}$
  $\Rightarrow$ Möbius transformations $z \to \frac{az+b}{cz+d}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$

- upshot: ∞-dim conformal symmetry from residual gauge freedom of $S_p[X,h]$ for closed strings

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2.2) Conserved currents of 2-dim CFT

- consider infinitesimal diffeo $\sigma^\alpha \to \sigma^\alpha + \eta^\alpha(\sigma)$

  $\Rightarrow h_{\alpha\beta} \to h_{\alpha\beta} - 2\partial_{(\alpha}\eta_{\beta)} + \mathcal{O}(\eta^2)$

  $\Rightarrow$ general action ($\neq S_p$) varies by

  $$\delta S[h, \ldots] = \int d^2\sigma \, \frac{\delta S[h, \ldots]}{\delta h_{\alpha\beta}(\sigma)} \delta h_{\alpha\beta}(\sigma)$$
  $$= \frac{-1}{2\pi} \int d^2\sigma \sqrt{-\det h} \, T^{\alpha\beta} \partial_\alpha \eta_\beta$$

  * for conformal trf $\partial_{(\alpha}\eta_{\beta)} \sim h_{\alpha\beta}\,\omega^2(\sigma)$

    $\Rightarrow \delta S[h, \ldots] \sim \int d^2\sigma \sqrt{-\det h} \, T^{\alpha\beta} h_{\alpha\beta}\,\omega^2$

    $\Rightarrow$ in conformally invariant theories, $T_{\alpha\beta}h^{\alpha\beta} = 0$

    (must be true pointwise since $\omega^2(\sigma)$ was arbitrary)

- in translationally invariant theory $\partial_\alpha T^{\alpha\beta} = 0$ @ $h_{\alpha\beta} = \eta_{\alpha\beta}$

$$\delta S[h, \ldots] \longrightarrow -\frac{1}{2\pi} \int d^2\sigma \, \partial_\alpha (T^{\alpha\beta} \eta_\beta)$$

$\Rightarrow$ conserved Noether current of inf. diffeo $\eta_\beta(\sigma)$ is

$$J^\alpha_{(\eta)} \sim T^{\alpha\beta} \eta_\alpha$$

* in cplx coordinates where $ds^2 = dz \, d\bar{z}$

$$J^z_{(\eta)} \sim \frac{1}{4} (T^{zz} \eta_{\bar{z}} + T^{z\bar{z}} \eta_z) = T_{\bar{z}\bar{z}} \eta_{\bar{z}} + T_{\bar{z}z} \eta_z$$

* in conformal theories, $T_{\bar{z}z} = T^{z\bar{z}} = 0$ by tracelessness

$\Rightarrow \partial_z T_{\bar{z}\bar{z}} = 0 = \partial_{\bar{z}} T_{zz}$ by $\partial_\alpha T^{\alpha\beta} = 0$

$\Rightarrow$ ~~holo~~ (anti-) meromorphic $T_{zz} = T(z)$ & $T_{\bar{z}\bar{z}} = \bar{T}(\bar{z})$

* Noether currents for conformal trf $\eta_z = \eta(z)$ & $\eta_{\bar{z}} = \bar{\eta}(\bar{z})$

$J_{(\eta)}(z) = T(z) \eta(z)$ meromorphic $(= J^{\bar{z}}_{(\eta)}$ above$)$

$\bar{J}_{(\eta)}(\bar{z}) = \bar{T}(\bar{z}) \bar{\eta}(\bar{z})$ antimero' $(= J^z_{(\eta)}$ above$)$

* see later for associated conserved charges

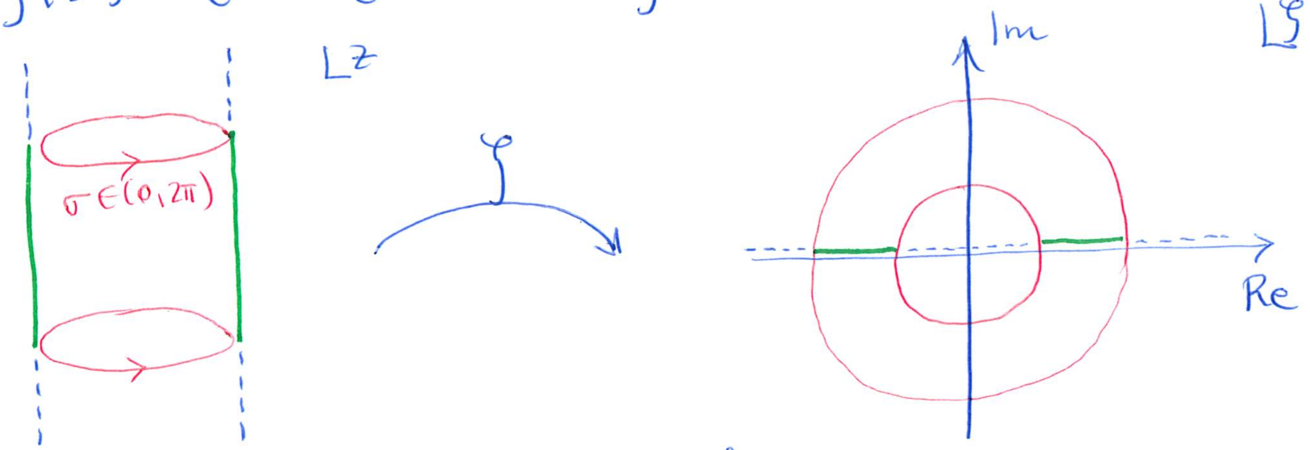
2.3) Cylinder vs. plane & contour integrals

Pick convenient coord's to

- automatize periodicity $\sigma \cong \sigma + 2\pi$ of closed strings
- map infinite past / future $t \to \pm\infty$ to points

- starting from $z = t + i\sigma$ & $\bar{z} = t - i\sigma$ apply (meromorphic) exponential map

$$\xi(z) = e^{z} = e^{t+i\sigma} \quad \& \quad \bar{\xi}(\bar{z}) = e^{\bar{z}} = e^{t-i\sigma}$$



$*$ infinite past at origin $\lim_{t \to -\infty} \xi(z) = 0$

$*$ infinite future $\rightarrow$ point "∞" on $S^2 = \mathbb{C} \cup \{\infty\}$

- mode expansion of free boson

$$\partial_+ X^\mu(\sigma^+) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu \, e^{-in\sigma^+} \quad \& \quad (\sigma^+, \alpha_n^\mu) \to (\sigma^-, \tilde{\alpha}_n^\mu)$$

$*$ upon $\partial_+ = i\partial_z$ & $\partial_- = i\partial_{\bar{z}}$

$$\partial_z X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu \, e^{-nz} \quad \& \quad (z, \alpha_n^\mu) \to (\bar{z}, \tilde{\alpha}_n^\mu)$$

$*$ upon $\partial_z = \xi \partial_\xi$

$$\partial_\xi X^\mu(\xi) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu \, \xi^{-n-1} \quad \& \quad (\xi, \alpha_n^\mu) \to (\bar{\xi}, \tilde{\alpha}_n^\mu)$$

$*$ contour-integral rep of Fourier / Laurent modes

$$\alpha_n^\mu = i\sqrt{\frac{2}{\alpha'}} \oint_{B_R(0)} \frac{d\xi}{2\pi i} \, \xi^n \, \partial_\xi X^\mu(\xi)$$

with $B_R(z) = S^1$ of radius R around z

using Cauchy $\displaystyle\oint_{B_R(0)} \frac{d\xi}{2\pi i} \, \xi^m = \begin{cases} 1 & : m = -1 \quad \forall R > 0 \\ 0 & : \text{otherwise} \end{cases}$

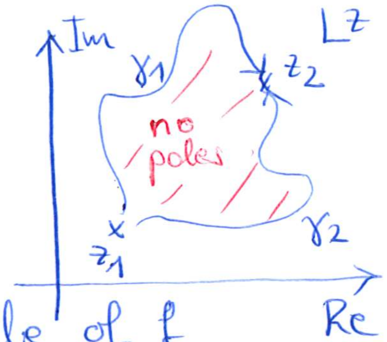
- similar contour integrals for conserved charges of conformal symmetry (η meromorphic)

* constant-time slice $\sigma \in (0, 2\pi)$ on worldsheet
  $\rightarrow$ circle $B_R(0)$ in ξ-plane @ fixed $R = e^t$
* conserved charges ($\Rightarrow$ indep. on R)

$$Q_{(\eta)} = \oint_{B_R(0)} \frac{d\xi}{2\pi i} J_{(\eta)}(\xi)$$

* rename $\xi \rightarrow z$ henceforth (will always be on plane)

- besides Cauchy, will apply homotopy invariance of $\int dz\, f(z)$ for $f(z)$ meromorphic



$$\int_{\gamma_1} dz\, f(z) = \int_{\gamma_2} dz\, f(z)$$

if $\partial_{\bar{z}} f(z) = 0$ & $\gamma_1 \circ \gamma_2^{-1}$ encloses no pole of f

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2.4) Conformal primary fields & OPEs

Generalize transformation of $\partial_z X^\mu \rightarrow \partial_\xi X^\mu$ ("vector") to general powers of Jacobian in $\partial_z = \frac{\partial \xi}{\partial z} \partial_\xi$ ("tensors")

- definition: $\phi_{h,\bar{h}}(z,\bar{z})$ is conformal primary of (conformal) weight $(h,\bar{h})$ if $\left\{ \begin{array}{l} z \rightarrow \xi(z) \\ \bar{z} \rightarrow \bar{\xi}(\bar{z}) \end{array} \right\}$ map

$$\phi_{h,\bar{h}}(z,\bar{z}) \longrightarrow \phi'_{h,\bar{h}}(\xi,\bar{\xi}) = \left(\frac{\partial \xi}{\partial z}\right)^{-h} \left(\frac{\partial \bar{\xi}}{\partial \bar{z}}\right)^{-\bar{h}} \phi_{h,\bar{h}}(z,\bar{z})$$

* examples $\partial_z X^\mu \Leftrightarrow (h,\bar{h}) = (1,0)$, $\partial_{\bar{z}} X^\mu \Leftrightarrow (h,\bar{h}) = (0,1)$
* like contravariant tensor with $h + \bar{h}$ indices

$$A_{\overbrace{zz\ldots z}^{h}\overbrace{\bar{z}\bar{z}\ldots\bar{z}}^{\bar{h}}}$$

- infinitesimal version for $\xi(z) = z + \eta(z)$ & $\bar{\xi}(\bar{z}) = \bar{z}$

$$\delta_\eta \phi_{h,\bar{h}}(z) := \phi'_{h,\bar{h}}(z) - \phi_{h,\bar{h}}(z)$$

$$= -\eta(z)\,\partial_\xi \phi'_{h,\bar{h}}(\xi) + \left[\left(\frac{\partial \xi}{\partial z}\right)^{-h} - 1\right]\phi_{h,\bar{h}}(z) + \mathcal{O}(\eta^2)$$

$$= -\eta(z)\,\partial_z \phi_{h,\bar{h}}(z) - h\,\phi_{h,\bar{h}}(z)\,\partial_z \eta(z) + \mathcal{O}(\eta^2)$$

- in any QFT: δ_η must be generated by $[Q_{(\eta)}, \circ]$ with associated charge (dropping $\bar{h}, \bar{z}$)

$$\delta_\eta \phi_h(z) \overset{!}{=} -[Q_{(\eta)}, \phi_h(z)] \quad \text{where}$$

$$Q_{(\eta)} = \oint_{B_R(0)} \frac{dz}{2\pi i}\, T(z)\,\eta(z)$$

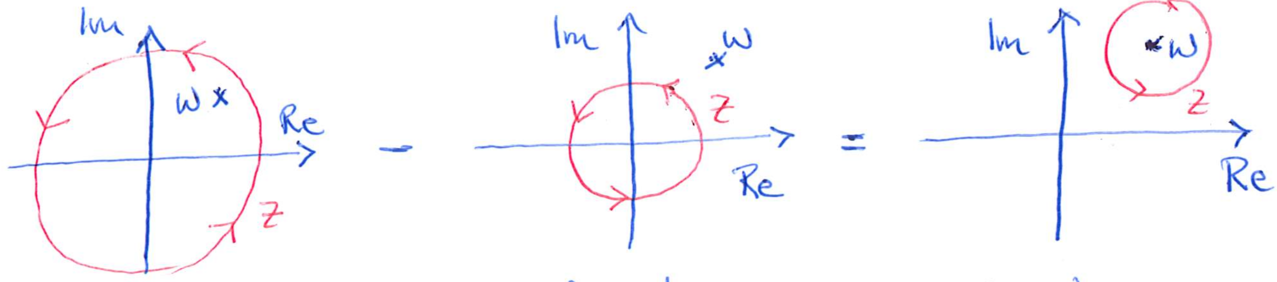
  * evaluate commutators $[T(z)\eta(z), \phi_h(w)]$ by imposing time-ordering on cylinder
    $\rightarrow$ radial ordering on plane

$$\left.\begin{aligned} &+T(z)\eta(z)\phi_h(w) && \text{has } |z| > |w| \\ &-\phi_h(w)T(z)\eta(z) && \text{has } |z| < |w| \end{aligned}\right\} \text{late times on left}$$

  * translate into radii of integration contours

$$[Q_{(\eta)}, \phi_h(w)] = \oint_{|z|>|w|} \frac{dz}{2\pi i}\, T(z)\,\eta(z)\,\phi_h(w) - \oint_{|z|<|w|} \frac{dz}{2\pi i}\, \phi_h(w)\,T(z)\,\eta(z)$$

* by holomorphicity in $z \neq w$, use homotopy invariance



$$\Rightarrow \left[Q_{(\eta)}, \phi_h(w) \right] = \oint_{B_R(w)} \frac{dz}{2\pi i} T(z)\eta(z)\phi_h(w)$$

* impose this to ~~reproduce~~ above $-\delta_\eta \phi_h(w)$

  from $\oint_{B_R(w)} \frac{dz}{2\pi i} (z-w)^m = \delta_{m,-1}$

  get $\eta(w)\partial_w \phi_h(w)$ by setting $T(z)\phi_h(w) \to \frac{\partial_w \phi_h(w)}{z-w}$

  get $h\phi_h(w)\partial_w \eta(w)$ —"— $T(z)\phi_h(w) \to \frac{h\phi_h(w)}{(z-w)^2}$

* in summary, need to impose ($\forall$ primaries $\phi_{h,\bar{h}}$)

  $$T(z)\phi_{h,\bar{h}}(w) \sim \frac{h\phi_{h,\bar{h}}(w)}{(z-w)^2} + \frac{\partial_w \phi_{h,\bar{h}}(w)}{z-w} + \ldots \quad \text{(OPE)}$$

  ↳ simplest ~~expansion~~ example of "operator product expansion"

• general OPEs:

  * 2 operators @ $z, w \in \mathbb{C}$ look like single operator (here: $\phi_h(w), \partial_w \phi_h(w)$) @ w if $z \to w$

  * only valid within correlation fct's (see later) where radial ordering applies

  * only track singular terms in $(z-w)$

2.5) OPEs vs. commutators

For quantization of bos. strings with $\partial_z X^\mu(z) \sim \sum_{n\in\mathbb{Z}} \alpha_n^\mu z^{-n-1}$, info on $[\alpha_m^\mu, \alpha_n^\nu]$ contained in OPE's of $\partial_z X(z)$

- as a $(h,\bar{h}) = (1,0)$ primary, $\partial_z X^\mu$ has 2pt function

$$\langle \partial_z X^\mu(z)\, \partial_w X^\nu(w) \rangle = -\frac{\alpha'}{2} \frac{\eta^{\mu\nu}}{(z-w)^2}$$

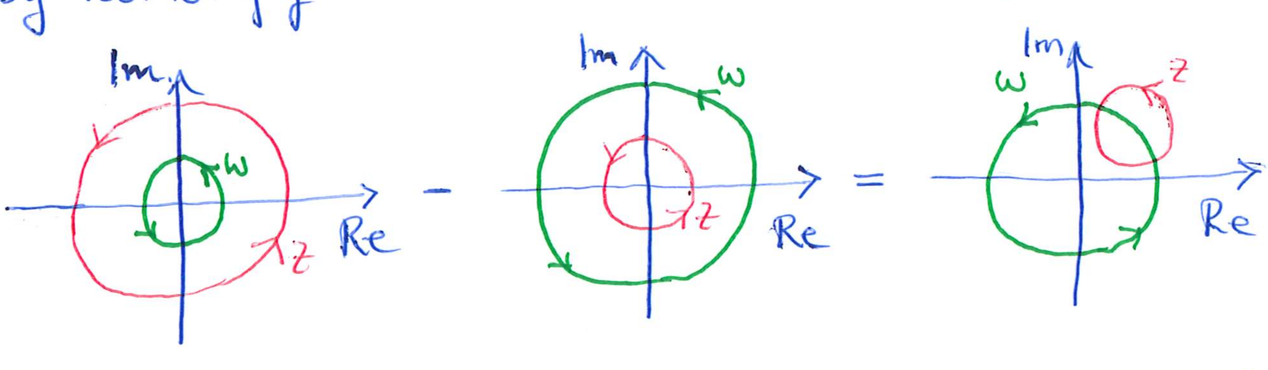
  * z, w dependence completely determined by $z \to \xi(z)$ properties
  * see later for path-integral definition / derivation
  * read off OPE from singularity

$$\partial_z X^\mu(z)\, \partial_w X^\nu(w) \sim -\frac{\alpha'}{2} \frac{\eta^{\mu\nu}}{(z-w)^2} + \dots$$

- reconstruct commutators of α_n^μ
  * recall $\alpha_n^\mu = i\sqrt{\frac{2}{\alpha'}} \oint_{B_R(0)} \frac{dz}{2\pi i} z^n \partial_z X^\mu(z)$
  * imposing radial ordering

$$[\alpha_m^\mu, \alpha_n^\nu] = -\frac{2}{\alpha'} \left(\oint\oint_{\substack{B_R(0)\text{'s @}\\ |z|>|w|}} - \oint\oint_{\substack{B_R(0)\text{'s @}\\ |z|<|w|}} \right) \frac{dz}{2\pi i} z^m \partial_z X^\mu(z) \times \frac{dw}{2\pi i} w^n \partial_w X^\nu(w)$$

  * by homotopy inv.



* integrate using $2\times$ Cauchy

$$\omega^m + m(z-\omega)\,\omega^{m-1} + \dots$$

$$[\alpha^\mu_m, \alpha^\nu_n] = -\frac{2}{\alpha'} \oint_{B_R(0)} \frac{dw}{2\pi i} w^n \oint_{B_\epsilon(w)} \frac{dz}{2\pi i} z^m \underbrace{\partial_z X^\mu(z)\, \partial_w X^\nu(w)}_{-\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2}+\dots}$$

$$= m\,\eta^{\mu\nu} \oint_{B_R(0)} \frac{dw}{2\pi i} w^{m+n-1} = m\,\eta^{\mu\nu}\,\delta_{m+n,0}$$

* reverses earlier logic where we started from $[Q_{(\eta)}, \phi_h(w)]$ and inferred $T(z)\,\phi_h(w)$ OPE

* same result as canonical quantization

2.6) Wick theorem, Virasoro algebra and momentum eigenstates

- several applications of OPE methods in this section
- distributivity of OPE: write $:\ldots:$ in

$$T(z) = -\frac{1}{\alpha'} :\partial_z X^\mu(z)\, \partial_z X_\mu(z):$$

to explicitly exclude OPEs between enclosed factors (you obviously want to avoid $\frac{1}{z-z} \to \frac{1}{0}$)

* apply "Leibniz rule" for

$$T(z)\,\partial_w X^\nu(w) \sim -\frac{1}{\alpha'} \partial_z X^\mu(z)\, \underbrace{\mathrm{OPE}\big(\partial_z X_\mu(z)\, \partial_w X^\nu(w)\big)}_{-\frac{\alpha'}{2}\frac{\delta_\mu^\nu}{(z-w)^2}}$$

$$\partial_w X^\mu(w) + (z-w)\,\partial_w^2 X^\mu(w) + \dots$$

$$-\frac{1}{\alpha'} \partial_z X_\mu(z)\, \mathrm{OPE}\big(\partial_z X^\mu(z)\, \partial_w X^\nu(w)\big)$$

$$= \frac{\partial_w X^\nu(w)}{(z-w)^2} + \frac{\partial_w^2 X^\nu(w)}{z-w} + \dots$$

as expected for $(h, \bar h) = (1, 0)$ primary $\partial_w X^\nu(w)$

- Wick theorem for OPE of general normal ordered products: sum over all (partial) pair contractions

e.g. $:A(z)B(z)::C(w)D(w):$ to be Taylor-expanded around $z=w$

$$\sim A(z) D(w)\ \mathrm{OPE}\big(B(z)C(w)\big) + (A \leftrightarrow B) + (C \leftrightarrow D)$$

$$+ \mathrm{OPE}\big(A(z)D(w)\big)\,\mathrm{OPE}\big(B(z)C(w)\big) + (A \leftrightarrow B)$$

* apply to 2 copies of $T \sim :\partial_z X^\mu \partial_z X_\mu:$

$$T(z)T(w) \sim \frac{D/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \dots$$

$\Rightarrow$ $T(z)$ is <u>not</u> a conformal primary,

e.g. map $\varsigma(z) = e^z$ cylinder $\to$ plane

picks up zero-pt energy $T_{cyl}(z) = \varsigma^2 T_{plane}(\varsigma) - \frac{D}{24}$

which you may otherwise get from $\sum_{n=1}^{\infty} n = \frac{-1}{12}$

* mode expansion on plane

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

* in more general CFTs, $\#(z-w)^{-4} = \frac{c}{2}$ in $T(z)T(w)$ with "central charge" c (here: $c_X = D$)

- Virasoro algebra from $T(z)T(w)$ OPE

* use $L_n = \oint_{B_R(0)} \frac{dz}{2\pi i} z^{n+1} T(z)$ and contour deformations as for $[\alpha^\mu_m, \alpha^\nu_n]$

$$\Rightarrow [L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m+n,0}$$

* term $\sim c$ is "central extension" of Witt algebra of $l_m = -z^{m+1}\partial_z$

- another Wick application : plane waves

$$u_p(z) = :e^{ip\cdot X(z)}: = \sum_{n=0}^{\infty} \frac{i^n}{n!} p^{\mu_1} \dots p^{\mu_n} :X_{\mu_1}(z) \dots X_{\mu_n}(z):$$

* eigenstate of momentum operator

$$\partial_z X^\mu(z)\, u_p(w) \sim -\frac{i\alpha'}{2} \frac{p^\mu}{z-w} u_p(w) + \dots \quad \text{(by Wick)}$$

$$\Rightarrow \sqrt{\frac{2}{\alpha'}} [\alpha_0^\mu, u_p(w)] = \frac{2i}{\alpha'} \oint_{B_R(w)} \frac{dz}{2\pi i} \partial_z X^\mu(z)\, u_p(w)$$

$$= p^\mu u_p(w)$$

* u_p are conf. primaries @ $(h, \bar{h}) = \left(\frac{\alpha' p^2}{4}, \frac{\alpha' p^2}{4}\right)$ since

$$T(z)\, u_p(w) \sim \frac{\alpha' p^2 u_p(w)}{4(z-w)^2} + \frac{\partial_w u_p(w)}{z-w} + \dots$$

2.7) Closed-string spectrum & vertex operators

- state-operator correspondence : each state $|\psi\rangle$ in CFT has corresponding field $\phi_\psi(z)$ such that

$$|\psi\rangle = \lim_{z\to 0} \phi_\psi(z)\, |0\rangle$$

* intuition: asymptotic state since $z \to 0$ is $t \to -\infty$

* vacuum state $|0\rangle$ subject to $L_{n\geq -1}|0\rangle = 0$

* state-correspondent of $T(z)$

$$L_{-2}|0\rangle = \lim_{z\to 0} T(z)|0\rangle$$

nothing diverges

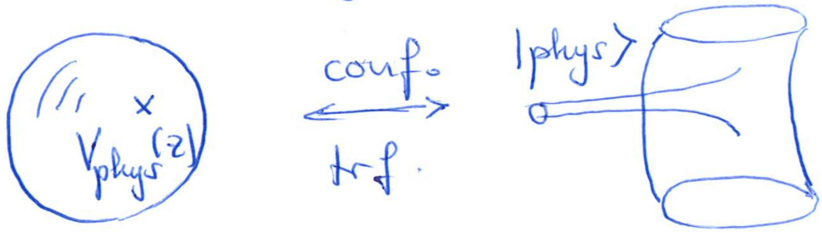
* similarly : $\alpha_{-1}^\mu |0\rangle = i\sqrt{\frac{2}{\alpha'}} \lim_{z\to 0} \partial_z X^\mu(z)\, |0\rangle$

where $\alpha_{n\geq 0}^\mu |0\rangle = 0$

* refinement : $|0;p\rangle = \lim_{z\to 0} :e^{ip\cdot X(z)}: |0\rangle$

such that $\alpha_0^\mu |0;p\rangle = \sqrt{\frac{\alpha'}{2}} p^\mu |0;p\rangle$ & $\alpha_{n\geq 1}^\mu |0;p\rangle = 0$

- $\forall$ physical state $|phys\rangle$ of bos. string,
  field correspondent $V_{phys}(z)$ is called "vertex operator"
  $$|phys\rangle = \lim_{z\to 0} V_{phys}(z)\,|0\rangle$$
  * intuition: $V_{phys}(z)$ inserts asymptotic physical state on string worldsheet



  punctures replacing infinite tubes (also if $z \neq 0$)

  * fixing preferred z incompatible with diff $\times$ Weyl
    $\Rightarrow$ need to integrate vertex op's over worldsheet
  * obtain diff $\times$ Weyl invariant $\int d^2z\, V_{phys}(z)$ if
    $V_{phys} \leftrightarrow$ conformal primary of $(h,\bar h) = (1,1)$
    $\Rightarrow$ transforms opposite to $d^2z = \frac{\partial z}{\partial \xi}\frac{\partial \bar z}{\partial \bar \xi}\, d^2\xi$

- to create α_0^μ eigenstate,
  $$V_{phys}(z) = :\phi_{phys}(z)\, e^{ip\cdot X(z)}:$$
  * only differentiated X^μ in ϕ_{phys} with
    $\alpha^\mu_{-n} \longleftrightarrow \partial_z^n X^\mu$ contributing $(n,0)$ to $(h,\bar h)$
  * since $:e^{ip\cdot X}:$ contribute $\alpha' p^2/4$ to h & $\bar h$ level matching
    $1 \overset{!}{=} \frac{\alpha' p^2}{4} + h_\phi$ with $h_\phi \in \mathbb{N}_0$ $= h_\phi$ & $p^2 = -m^2$
    $\Rightarrow m^2 = \frac{4}{\alpha'}(h_\phi - 1) \in \frac{4}{\alpha'}\{-1, 0, 1, 2, \dots\}$

* lowest m^2 @ $\phi_{phys} \to 1$

$V_T(z) = :e^{ip\cdot X(z)}:$, $p^2 = \frac{4}{\alpha'}$

"tachyon" T, only state @ $m^2 < 0$

[signals instability of vacuum we chose to expand around & absent in superstring spectra]

2.8) Physical polarizations

- focus on $m^2 = 0 \iff h_\phi = 1 = \bar{h}_\phi$
  * ansatz $\phi_\zeta(z) = \zeta_{\mu\nu}\, \partial_z X^\mu(z)\, \partial_{\bar{z}} X^\nu(\bar{z})$

    with rk-2 polarization tensor $\zeta_{\mu\nu}$ of $SO(1, D-1)$
  * not all $\zeta_{\mu\nu}$ make $V_\zeta(z) := :\phi_\zeta(z)\, e^{ip\cdot X(z)}:$ primary

    $$T(z)\, V_\zeta(w) \sim -2i\alpha' \underbrace{\frac{p^\mu \zeta_{\mu\nu}}{(z-w)^3}}_{\text{obstruction to primary unless}} :\partial_{\bar{z}} X^\nu e^{ip\cdot X}: + \mathcal{O}((z-w)^{-2})$$

    $\begin{cases} p^\mu \zeta_{\mu\nu} \overset{!}{=} 0 & \text{from } T(z) \\ \zeta_{\mu\nu} p^\nu \overset{!}{=} 0 & \text{from } \bar{T}(\bar{z}) \end{cases}$
  * some of $(D-1)^2$ solutions to $p^\mu \zeta_{\mu\nu} = 0 = \zeta_{\mu\nu} p^\nu$ decouple:

    total derivatives $\leftrightarrow$ "spurious states"

    $V_{phys}(z) = \partial_z W(z)$ or $\partial_{\bar{z}} \bar{W}(z) \Rightarrow \int d^2z\, V_{phys}(z) = 0$
  * spurious $m^2 = 0$ states from $W(z) = \bar{\xi}_\nu :\partial_{\bar{z}} X^\nu e^{ip\cdot X}:$

    $\Rightarrow \partial_z W(z) = i p_\mu \bar{\xi}_\nu :\partial_z X^\mu \partial_{\bar{z}} X^\nu e^{ip\cdot X}:$

    $\Rightarrow$ both of $\zeta_{\mu\nu} \to p_\mu \bar{\xi}_\nu$ & $\zeta_{\mu\nu} \to \xi_\mu p_\nu$

    are spurious (need $\xi\cdot p = 0 = \bar{\xi}\cdot p$)

* subtracting above $W(\bar{z})$ & cc

$\Rightarrow (D-2)^2$ solutions to $\begin{cases} p^\mu \zeta_{\mu\nu} = 0 = \zeta_{\mu\nu} p^\nu : \text{physical} \\ \zeta_{\mu\nu} \neq p_\mu \bar{\xi}_\nu \text{ or } \xi_\mu p_\nu : \text{non-spur.} \end{cases}$

- physics of massless states

  * decompose $\zeta_{\mu\nu}$ into Lorentz-irreps

$$\zeta_{\mu\nu} = \underbrace{\zeta_{\langle\mu\nu\rangle}}_{\substack{\text{symm. \& traceless} \\ \to \text{graviton}}} + \underbrace{\zeta_{[\mu\nu]}}_{\substack{\text{antisymm} \\ \to B\text{-field}}} + \underbrace{\eta^\perp_{\mu\nu} \, \mathrm{tr}_\perp(\zeta)}_{\substack{\text{trace} \\ \to \text{dilaton}}}$$

  * spurious states shift graviton

$$\delta\zeta_{\langle\mu\nu\rangle} = p_\mu \xi_\nu + p_\nu \xi_\mu \quad @ \quad p\cdot\xi = 0$$

$\Rightarrow$ linearized spacetime diffeomorphism!

spin 2 with diffeo' symmetry $\xrightarrow{\text{uniquely}}$ graviton

  * dilaton sets string coupling through its VEV

- massive states

  * recall that $h_\phi \in \mathbb{N}_0$ in $m^2 = \frac{4}{\alpha'}(h_\phi - 1)$

    counts both $\# \partial_z$ and $\# \partial_{\bar{z}}$ in ϕ_{phys}

  * first $m^2 > 0$ @ $h_\phi = 2$ with $p^2 = -\frac{4}{\alpha'}$ and

$$\phi_{phys} \leftrightarrow :\!\left(\partial_z X^\mu \partial_z X^\nu \oplus \partial_z^2 X^\mu\right)\left(\partial_{\bar{z}} X^\lambda \partial_{\bar{z}} X^\rho \oplus \partial_{\bar{z}}^2 X^\rho\right) e^{ip\cdot X}\!:$$

polarization tensors with 2-4 indices constrained by primary constraints from $T(z)$ & $\bar{T}(\bar{z})$

* state correspondents : tachyon $\Leftrightarrow |0;p\rangle = \lim_{z\to 0} :e^{ip\cdot X(z)}: |0\rangle$

$$h_\phi = 1 \quad \longleftrightarrow \quad \alpha^\mu_{-1} \otimes \tilde{\alpha}^\nu_{-1} |0;p\rangle$$

$$h_\phi = 2 \quad \longleftrightarrow \quad \left(\alpha^\mu_{-1}\alpha^\nu_{-1} \oplus \alpha^\mu_{-2}\right) \otimes \left(\tilde{\alpha}^\lambda_{-1}\tilde{\alpha}^\rho_{-1} \oplus \tilde{\alpha}^\lambda_{-2}\right) |0;p\rangle$$

$$\vdots$$

$h_\phi = N \quad \longleftrightarrow \quad$ up to spin $2N$ from $\prod_{j=1}^{N} \alpha^{\mu_j}_{-1}\tilde{\alpha}^{\nu_j}_{-1}|0;p\rangle$

$$\Rightarrow \quad J_{max} = \frac{\alpha'}{2} m^2 + 2$$

α' as (Regge-) slope in spin$-m^2-$plot

* conditions on polarizations ($\xi_{\mu\nu}$ & massive ones) from ~~the~~ primary constraint on $V_{phys}(z)$ removes negative-norm states !

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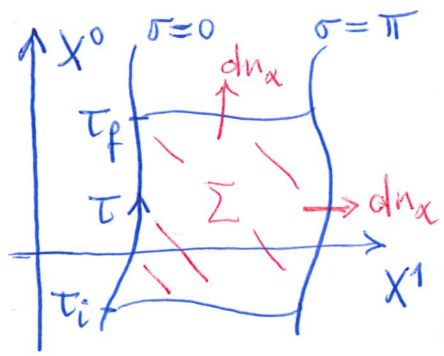
3) Open bosonic strings & D branes

3.1) Dirichlet & Neumann bdy conditions

open strings $\longleftrightarrow$ worldsheets Σ with non-periodic spatial coord $\sigma \in (0,\pi)$



• how to avoid bdy terms in variation of Polyakov action ?

$$\delta S_{CG}[X] = \frac{-1}{4\pi\alpha'} \int_\Sigma d^2\sigma \; 2\, \partial_\alpha(\delta X^\mu)\, \partial^\alpha X_\mu$$

$$= \frac{1}{2\pi\alpha'} \Big\{ \underbrace{\int_\Sigma d^2\sigma \, \delta X^\mu \partial_\alpha \partial^\alpha X_\mu}_{\text{see closed strings}} - \underbrace{\int_{\partial\Sigma} dn_\alpha \, \delta X^\mu \partial^\alpha X_\mu}_{\text{see below}} \Big\}$$

* tentative bdy term

$$\int_{\partial\Sigma} dn_\alpha\, \delta X^\mu \partial^\alpha X_\mu = \underbrace{-\int_0^\pi d\sigma\, \delta X^\mu \partial_\tau X_\mu \Big|_{\tau=\tau_i}^{\tau=\tau_f}}_{=0\ @\ \tau_i,\tau_f \to \pm\infty} + \int_{\tau_i}^{\tau_f} d\tau \boxed{\delta X^\mu \partial_\sigma X_\mu \Big|_{\sigma=\pi}^{\sigma=0}}$$

need to impose its vanishing

* @ both of $\sigma = 0$ & π, $\exists$ 2 choices of bdy conditions to make either δX^μ or $\partial_\sigma X_\mu$ vanish

* Neumann conditions (endpt's move freely)

$$\partial_\sigma X^A \Big|_{\sigma=0,\pi} = 0 \quad \text{for some } 0 \leq A \leq p$$

in $p = 1, 2, \ldots D-1$ spatial directions

$p \in \mathbb{N}_0$ does not refer to momentum here

* Dirichlet conditions (endpt's fixed in space)

$$\delta X^I \Big|_{\sigma=0,\pi} = 0 \quad \text{for some } p+1 \leq I \leq D-1$$

$$\Rightarrow X^I(\tau, \sigma=0) = c^I \ \& \ X^I(\tau,\sigma=\pi) = d^I \text{ for some } c^I, d^I$$

- mode expansions / classical solutions to $\partial_\alpha \partial^\alpha X^\mu = 0$
  * replace $\sigma \cong \sigma + 2\pi$ by Dirichlet / Neumann cond's
  * ansatz $X^\mu(\tau,\sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$ as before with

$$X_L^\mu(\sigma^+) = \frac{1}{2} x^\mu + \alpha' p_L^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\sigma^+}$$

$$X_R^\mu(\sigma^-) = \frac{1}{2} x^\mu + \underbrace{\alpha' p_R^\mu \sigma^-}_{\text{factor of 2 \& possibly } p_L \neq p_R} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-in\sigma^-}$$

as before