- ✱ Neumann $0 \overset{!}{=} \partial_\sigma X^A(\tau,\sigma)\Big|_{\sigma=0,\pi}$

  $\Rightarrow \; p_L^A = p_R^A \;\;\&\;\; \alpha_n^A = \tilde{\alpha}_n^A \quad \forall\; 0 \le A \le p$

- ✱ Dirichlet : $c^I, d^I$ as above

  $\Rightarrow \; x^I = c^I \;\;\&\;\; p_L^I = -p_R^I = \dfrac{d^I - c^I}{2\pi\alpha'}$

  $\& \;\; \alpha_n^I = -\tilde{\alpha}_n^I \quad \forall\; p+1 \le I \le D-1$

- classical $m^2$-formula from $T_{\alpha\beta} = 0$ (zero mode)

$$m^2 = (p_L^0 + p_R^0)^2 - \sum_{i=1}^{p} (p_L^i + p_R^i)^2$$

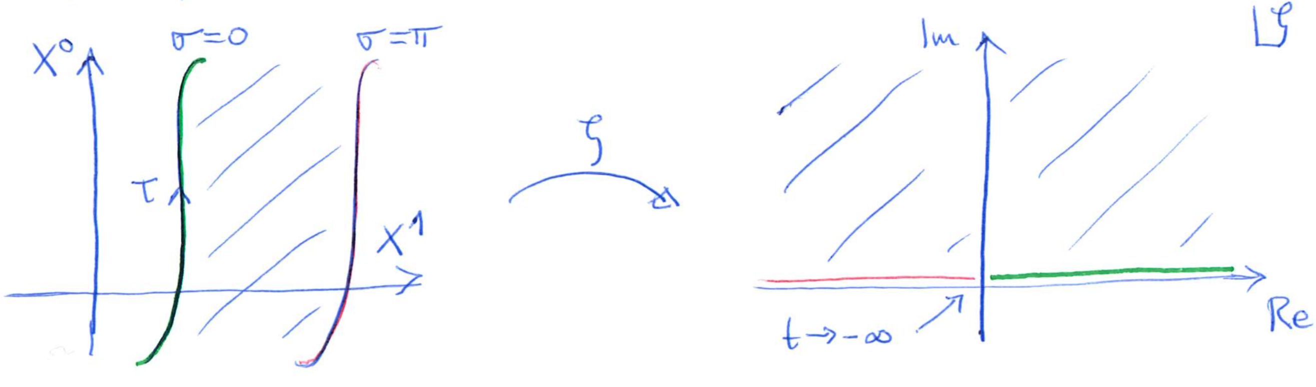
factor 1/4 relative to 4/α' of closed strings

$$= \underbrace{\frac{1}{2\pi\alpha'} \sum_{I=p+1}^{D-1} (c^I - d^I)^2}_{\text{linear mass contribution from spatial stretching}} + \underbrace{\frac{1}{\alpha'} \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n}_{\text{both Dirichlet \& Neumann dir. oscillators}}$$

---

## 3.2) Boundary CFT & open-string spectrum

$\zeta(z) = e^z$ cylinder → plane (with $z = t + i\sigma$ & $\tau = -it$) maps open-string worldsheet → upper half plane $\mathbb{H}$



- bdy as extra structure $\Rightarrow$ conformal symmetry $z \to \zeta(z)$ broken to $\zeta$ preserving the boundary $z \in \mathbb{R}$

* generated by $T(z)$ & $\overline{T}(\bar{z})$ restricted to $\mathbb{H}$

& subject to gluing condition $T(z) = \overline{T}(\bar{z})$ $\forall z \in \mathbb{R}$

* one copy $L^{op}_{m\in\mathbb{Z}}$ of Vir modes (instead of $L_m$ & $\bar{L}_m$)

$$L^{op}_m = \oint_{B_R(0)} \frac{dz}{2\pi i} T^{op}(z)\, z^{m+1}$$

exceeds $\mathbb{H}$, i.e. domain of $T(z)$

$$T^{op}(z) = \begin{cases} T(z) & : \operatorname{Im} z \geq 0 \\ \overline{T}(\bar{z}) & : \operatorname{Im} z < 0 \end{cases}$$

continuous by gluing condition

* by $T(z) \sim (\partial_z X)^2$ & $\overline{T}(\bar{z}) \sim (\partial_{\bar{z}} X)^2$

$\exists$ 2 solutions to gluing $T = \overline{T}$ $\forall z \in \mathbb{R}$

$$\partial_z X_\mu (z \in \mathbb{R}) = \begin{cases} +\partial_{\bar{z}} X_\mu (z \in \mathbb{R}) & : \text{Neumann } \partial_\sigma \sim \partial_z - \partial_{\bar{z}} = 0 \\ -\partial_{\bar{z}} X_\mu (z \in \mathbb{R}) & : \text{Dirichlet } \partial_\tau \sim \partial_z + \partial_{\bar{z}} = 0 \end{cases}$$

* can again combine $\partial_z X_\mu$ & $\partial_{\bar{z}} X_\mu$ for $z \in \mathbb{H}$ to single field $\partial_z X^{op}$ on $\mathbb{C} \cup \{\infty\}$ with one copy of oscillators $\alpha^{op}_n$

- open-string spectrum from vertex op's $V^{op}_{phys}$ integrated over the bdry, $\int_{\subseteq\mathbb{R}} dz\, V^{op}_{phys}(z)$
  * diff × Weyl invariant if $V^{op}_{phys}$ is conformal primary of $h=1$ w.r.t $T^{op}$
  * want eigenstate for momentum $\alpha_0^{A=0,1,\dots,p}$ in Neumann directions $\Rightarrow$ ansatz $V^{op}_{phys}(z) = :\phi^{op}_{phys}(z)\, e^{ip\cdot X^{op}(z)}:$

$p \in \mathbb{R}^{1,p}$

- * $\phi^{op}_{phys}$ built from $\partial_t^{n\geq 1} X^{op}_{\mu} \leftrightarrow \alpha^{op}_{\mu,-n}$ derivative along bdy

  with $\mu$ in Neumann or Dirichlet direction

  * since $:e^{ip\cdot X^{op}}:$ contribute $\alpha' p^2$ to $h$

    $1 \overset{!}{=} \alpha' p^2 + h_\phi$ with $h_\phi \in \mathbb{N}_0$ & $p^2 = -m^2$

    $\Rightarrow m^2 = \frac{1}{\alpha'}(h_\phi - 1) \in \frac{1}{\alpha'}\{-1, 0, 1, 2, \dots\}$

  * again, ~~simplest state~~ tachyon @ $\phi^{op}_{phys} \to 1$ ~~is tachyon~~

    $V_T^{op}(z) = :e^{ip\cdot X^{op}(z)}:$ , $p^2 = \frac{1}{\alpha'}$

  * only $\partial_t$, no distinction $\partial_z \leftrightarrow \partial_{\bar z}$ in $\phi^{op}_{phys}$

    $\Rightarrow$ open-string polarizations are halves of closed string

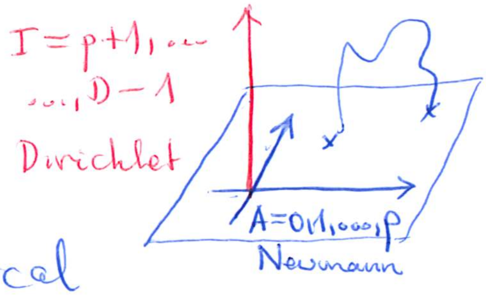
## 3.3) D branes & non-abelian gauge theory

- $(p+1)$-dim hypersurfaces $\subset \mathbb{R}^{1,D-1}$ defined by

  $X^I |_{\sigma=0,\pi} = c^I, d^I$ are called Dp branes $(p+1 \leq I \leq D-1)$

  * break Lorentz symm

    $SO(1,D-1) \to SO(1,p) \times SO(D-1-p)$



  * viewed as coming from dynamical object in $SO(1,D-1)$-inv. fundamental theory

  * tension of D-branes diverges in our perturbative description of strings, i.e. they appear as ∞ heavy

- massless open-string excitations $h_\phi = 1 \Rightarrow \phi_{phys} \in \{\partial X_A, \partial X_I\}$

  * suppressing superscript "op" in $\phi_{phys}$, X, T

- * Neumann polarization $\zeta^{A=0,1,\dots,p}$ in

  $$V_\zeta(z) = \zeta^A :\partial X_A e^{ip\cdot X}: \quad , \quad p^2=0$$

  $\Rightarrow$ OPE with $T(z) \sim \frac{\zeta^A p_A}{(z-w)^3} :e^{ip\cdot X}:$

  $\Rightarrow$ need transversality $\zeta^A p_A = 0$

- * spurious state from $\partial :e^{ip\cdot X}: = ip_A :\partial X^A e^{ip\cdot X}:$

  $\Rightarrow$ spurious polarization $\zeta_A \to p_A$

- * spin 1 polarization with $\zeta\cdot p = 0$ and gauge freedom $\delta\zeta_A = p_A$ $\Rightarrow$ $V_\zeta$ describes gauge field on Dp brane with (p-1) d.o.f.

- * Dirichlet polarizations $\varphi^{I=p+1,\dots,D-1}$

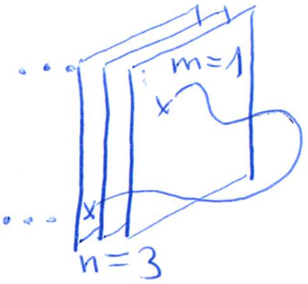
  $$V_\varphi(z) = \varphi^I :\partial X_I e^{ip\cdot X}: \quad , \quad p^2=0$$

  no transversality / gauge freedom

- * total of (D-p-1) massless scalars $\varphi^I$ interpreted as transverse fluctuation of Dp brane

• multiple Dp branes

- * N coincident branes: the 2 endpts m, n have N × N choices to be associated with brane $m=1,\dots,N$ & $n=1,\dots,N$



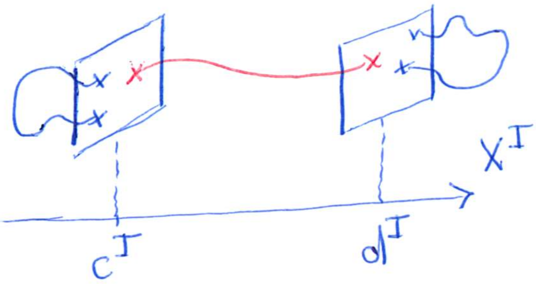
- * entire open-string spectrum $(V_T, V_\zeta, \dots) \to N\times N$ matrices

  $$|\text{phys}\rangle_{\text{open}} \to |\text{phys}; (m,n)\rangle = \sum_{a=1}^{N^2} (T^a)_m{}^n\, |\text{phys}; a\rangle$$

- changed basis to $N^2$ generators $T^a$ of $U(N)$

  $N \times N$ "Chan Paton factors" with $U(N)$ adjoint index $a = \overset{1,2,\dots,N^2}{\smile}$

- separating Dp branes breaks $U(N) \to U(1)^N$



- at $N=2$ for instance,

  off-diag states with

  $T^a \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

  acquire mass shift

  $$m^2(c,d) = \frac{1}{(2\pi\alpha')^2} \sum_{I=p+1}^{D-1} (c^I - d^I)^2 + \frac{h_\phi - 1}{\alpha'}$$

- geometric Higgs effect: off-diag gauge bosons $V_\xi$ become massive "$W^\pm$" & diagonal scalars $\sim \begin{pmatrix} \varphi_{11} & 0 \\ 0 & \varphi_{22} \end{pmatrix}$ acquire VEVs from Dp brane positions

- $D=10$ superstrings with D3 branes:

  4 dim gauge boson + 6 scalars + fermions

  $\Rightarrow$ max. SUSY $\mathcal{N}=4$ SYM @ gauge grp $U(N)$

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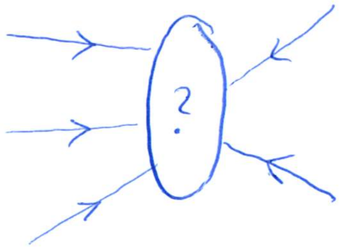
## 4) Bosonic string amplitudes

### 4.1) Topological expansion



- Similar to QFT amplitudes, path-integrate over all histories (Feynman diagrams $\Rightarrow$ string worldsheets)

- fundamental difference to point-particle interactions

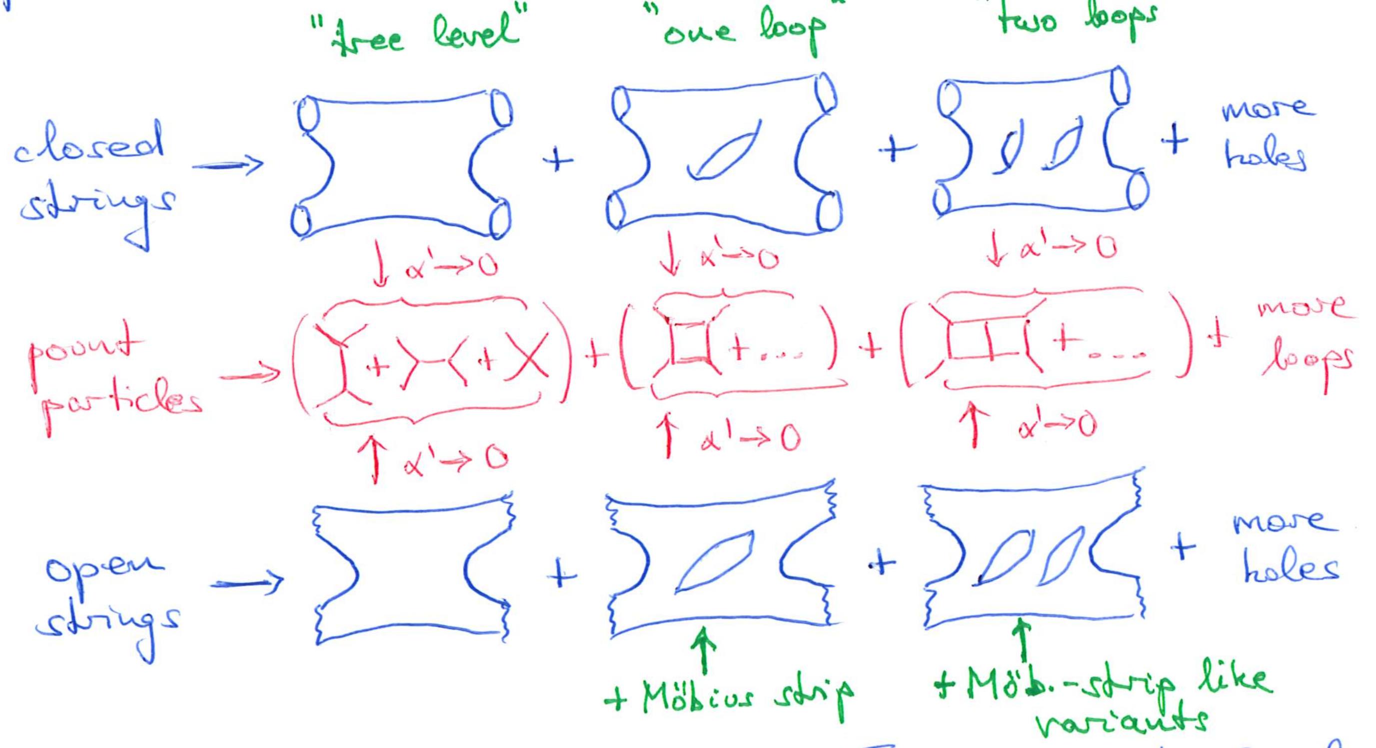
"smears out" interaction

short-distance regime
⇒ UV divergences

exponentially soft in UV

$*$ interacting string locally looks like free string
⇒ determined by $S_P$, no "by-hand" addition of $L_{int}$

- roughly 3 lines of motivation for string amplitudes
  - $*$ $\alpha' \to 0$ ⇒ new angle on perturbative gauge thy / gravity
  - $*$ interplay with number thy / algebraic geometry
  - $*$ testing / exploiting string dualities order by order in $\alpha'$
- perturbative expansion in diff × Weyl-inequiv. worldsheets



$*$ single world-sheet @ genus $g$ $\Big\} \xrightarrow{\alpha' \to 0} \Big\{$ all Feynman diag's @ $g$ loop (e.g. $(2n-5)!!$ $\phi^3$-diag's @ n-pt tree level)

- how to weigh different ~~Top.~~ worldsheets

  ↳ extend Polyakov action by topolog. term

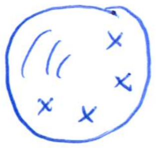
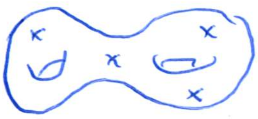
determined by dilaton VEV

$$S_{bos}[X,h] = S_p[X,h] + \lambda \cdot \chi$$ ← Euler characteristics

$$\chi = \frac{1}{4\pi}\int d^2\sigma \underbrace{\sqrt{-\det h}\; R}_{\text{total derivative in 2 dim}} = 2-2g$$

Ricci scalar on worldsheet

- after diff × Weyl mapping ∞ long tubes/strips to punctures

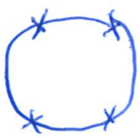
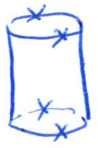
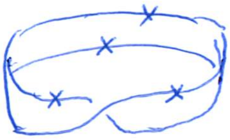
closed strings → $e^{-2\lambda}\int_{M_{0;4}}$  + $\int_{M_{1;4}}$  + $e^{2\lambda}\int_{M_{2;4}}$  + ...

* $M_{g;n}$ = moduli-space of compact genus-g Riemann surfaces with n marked pts

* integrand : $\langle V_{\Phi_1}(z_1) \dots V_{\Phi_n}(z_n)\rangle_{\Sigma_g}$ on given worldsheet topology

also ghosts & more, see next

* open-string analogue

$e^{-\lambda}\int$  $+ \int \Big($  +  $\Big) + e^{+\lambda}\int \dots$

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## 4.2) Path integrals & correlators : ⟨X...⟩ @ genus 0

[Towards mathematical expressions for above cartoons]

Define n-pt correlation functions for $\phi = \phi(X, \partial_z^n X, \partial_{\bar z}^n X)$ via

$$\langle \phi_1(z_1)\dots\phi_n(z_n)\rangle = \frac{1}{Z}\int \mathcal{D}[X]\, \phi_1(z_1)\dots\phi_n(z_n)\, e^{-S_{CG}[X]}$$

$$Z = \int \mathcal{D}[X]\, e^{-S_{CG}[X]} \quad , \quad S_{CG}[X] = \frac{1}{\pi\alpha'}\int d^2z\; \partial_z X_\mu\, \partial_{\bar z} X^\mu$$

[applies to arbitrary worldsheet topologies and will here be applied to sphere & disk]

- 2 pt function of $X^\mu$ :

total derivatives on path integrals vanish

$$0 = \int \mathcal{D}[X] \, \frac{\delta}{\delta X_\mu(z)} \left( X^\nu(w) \, e^{-S_{CG}[X]} \right)$$

$$= \int \mathcal{D}[X] \left( \eta^{\mu\nu} \delta^2(z-w) + \frac{2}{\pi\alpha'} \partial_z \partial_{\bar z} X^\mu(z) X^\nu(w) \right) e^{-S_{CG}[X]}$$

$$\Rightarrow \partial_z \partial_{\bar z} \langle X^\mu(z) X^\nu(w) \rangle = -\frac{\alpha'}{2} \pi \eta^{\mu\nu} \delta^2(z-w)$$

$$\Rightarrow \langle X^\mu(z) X^\nu(w) \rangle = -\frac{\alpha'}{2} \eta^{\mu\nu} \log|z-w|^2 =: \underbrace{G_{cl}(z,w)}_{\text{Green fct. on the sphere}} \eta^{\mu\nu}$$

- n-pt fct of $:e^{ip\cdot X}:$

$$\left\langle \prod_{j=1}^n :e^{ip_j\cdot X(z_j)}: \right\rangle = \frac{1}{Z} \int \mathcal{D}[X] \exp\Big( i \sum_{j=1}^n p_j \cdot X(z_j) \cancel{+\int \frac{d^2z}{\pi\alpha'}}$$

$$+ \int \frac{d^2 z}{\pi\alpha'} X_\mu(z) \partial_z \partial_{\bar z} X^\mu(z) \Big)$$

$$= \overset{\text{Gaussian integral}}{\cdots\cdots} = \overbrace{\delta^D\Big(\sum_{j=1}^n p_j\Big)}^{\text{from } \mathcal{D}[X] \to d^D x \ \& \ e^{ip_j\cdot X(z_j)} \to e^{ip_j\cdot x}} \exp\Big( -\frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^n p_i \cdot p_j \, G_{cl}(z_i,z_j) \Big)$$

$$= \delta^D\Big(\sum_{j=1}^n p_j\Big) \prod_{1\leq i<j}^n |z_i - z_j|^{\alpha' p_i\cdot p_j} \quad \text{"Koba-Nielsen factor"}$$

- open-string counterpart (Green fct on disk)

  * same Laplace eq. $\partial_z \partial_{\bar z} G_{op}^{N,D}(z,w) = -\frac{\alpha'}{2} \pi \delta^2(z-w)$

  but different domain ($z,w \in \mathbb{H}$) & bdy. conditions

$$(\partial_z - \partial_{\bar z}) G_{op}^N(z,w) = 0 = (\partial_z + \partial_{\bar z}) G_{op}^D(z,w) \quad \forall z \in \mathbb{R}$$

  * solve via method of image charges, say

$$G_{op}^N(z,w) = G_{cl}(z,w) + G_{cl}(z,\bar w)$$

$$= -\frac{\alpha'}{2} \left( \log|z-w|^2 + \log|z-\bar w|^2 \right)$$

* doubles exponent in

$$\left\langle \prod_{j=1}^{n} :e^{i p_j \cdot X^{\mathrm{op}}(z_j)}: \right\rangle = \delta^D\left(\sum_{j=1}^{n} p_j\right) \prod_{1\leq i<j}^{n} |z_i - z_j|^{2\alpha' p_i \cdot p_j}$$

- at genus ≥ 1, Green fct's involve $\theta$-fct's

[beyond the scope of these lectures]

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## 4.3) Ghosts at different genera

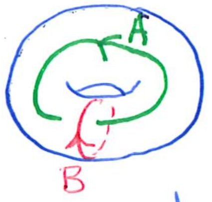
Fixing diff × Weyl symm of $S_P[X,h]$ via path integrals

$$Z \to Z_P^{(g)} = \int \frac{\mathcal{D}[X]\,\mathcal{D}[h]}{\mathrm{vol}(\mathrm{diff}\times\mathrm{Weyl})} e^{-S_P[X,h]}$$

formally compensate infinite overcount

- cannot just cancel $\int \mathcal{D}[h] \leftrightarrow \mathrm{vol}^{-1}(\mathrm{diff}\times\mathrm{Weyl})$:
  * Jacobian $\Delta_{FP}[h]$ from change of field variables $\mathcal{D}[h] \leftrightarrow \mathcal{D}[(\mathrm{diff}\times\mathrm{Weyl})\text{-parameters}]$
  * can only locally gauge fix $h_{\alpha\beta} \to \eta_{\alpha\beta}$ ; at genus $g \geq 1$, $\int \mathcal{D}[h]$ leaves finite-dim integral over complex-structure moduli $\tau$ [shapes of surface]
  * $$Z_P^{(g=1)} = \int d^2\tau \int \mathcal{D}[X]\,\Delta_{FP}[h(\tau)]\, e^{-S_P[X,h(\tau)]}$$

    $\tau$ is ratio of $\dfrac{B\text{-cyc length}}{A\text{-cyc length}}$



  * $Z_P^{(g\geq 2)}$ instead has $3g-3$ cplx-structure moduli $\tau_j$
- $g=0$ : no $\tau_j$-integrals / moduli of disk / sphere, only Jacobian $\Delta_{FP}[h] \Rightarrow$ Grassmann-odd $(b,c)$ ghost system

$$\Delta_{FP}[h] = \int \mathcal{D}[b]\,\mathcal{D}[c]\, e^{-S_{gh}[b,c,h]}$$

$$S_{gh}[b,c,h] = \frac{1}{2\pi}\int d^2\sigma \sqrt{-\det h}\; b_{\alpha\beta} \nabla^\alpha c^\beta$$

* in conformal gauge

$$Z_p^{(g=0)} = \int \mathcal{D}[X]\,\mathcal{D}[b]\,\mathcal{D}[c]\, e^{-S_{CG}[X]-S_{gh}[b,c]}$$

$$S_{gh}[b,c] = \frac{1}{\pi}\int d^2z \left( b\, \partial_{\bar z} c + \bar b\, \partial_z \bar c \right)$$

where $(b, \bar b) = (b_{zz}, b_{\bar z \bar z})$ & $(c, \bar c) = (c^z, c^{\bar z})$

* $S_{gh}$ defines CFT with (zz-cpt) $T_{gh}(z) = \overset{2}{\cancel{1}} :(\partial_z c) b: + :c \partial_z b:$

$h_b = 2$, $h_c = -1$, $c_{gh} = -26$

obtained from OPE $c(z) b(w) \sim \frac{1}{z-w} + \ldots$

* in $g=0$ amplitudes, need $(z_{ij} = z_i - z_j)$

$\langle c(z_1)\, c(z_2)\, c(z_3) \rangle = z_{12}\, z_{13}\, z_{23}$ $\left(\begin{matrix}\text{same @ } c \to \bar c \\ \text{and } z_i \to \bar z_i\end{matrix}\right)$

- $(b,c)$-system at general genus

* $\int \mathcal{D}[b]\,\mathcal{D}[c] \supset$ g-dependent zero-mode integrals

* $\#(\underset{\nwarrow}{N_c}) - \#(\underset{\nearrow}{N_b}) = 3-3g$ (by Riemann Roch thm, see sec 6.2 BLT)

numbers of resp. zero modes

* $\langle \ldots \rangle_g = 0$ unless ghost charges add up to $\#(b) - \#(c) = 3g-3$, cancelling "backgrd. charge"

* at $g=0$, consistent with above $\langle ccc \rangle$

- critical dimension vs. critical central charge

  * total central charge ⇒ conformal anomaly

  $$\langle T^{\alpha}{}_{\alpha} \rangle = -\frac{1}{12} c_{tot} R$$

  [if uncancelled, can no longer decouple negative-norm states]

  * with $c_{b,c} = -26$ and $c_X = D$, bosonic string has $c_{b,c} + c_X = D - 26 \overset{!}{=} 0$

  $\Rightarrow$ critical dimension $D_{crit} = 26$

  * additional CFT sectors of central charge $c_{add}$

  $$0 \overset{!}{=} c_{b,c} + c_X + c_{add} = D - 26 + c_{add}$$

  admit lower crit. dimensions $D_{crit} = 26 - c_{add}$

  * similar reasoning for superstring

  $$\Rightarrow D_{crit} = 10 - \frac{2}{3} c_{add}$$

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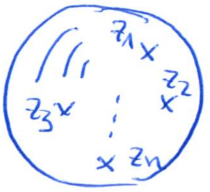
## 4.4) Closed-string tree amplitudes: first look

n-pt tree amplitudes $\longleftrightarrow \int_{M_{0,n}}$



ext. states $\Phi_i$ with $p_i$ & $V_{\Phi_i}(z_i)$

$$M^{tree}_{cl}(\{\Phi_i, p_i\}; \alpha') = \int_{M_{0,n}} \frac{d^2z_1 \ldots d^2z_n}{\text{vol}\, SL_2(\mathbb{C})} \langle V_{\Phi_1}(z_1) \ldots V_{\Phi_n}(z_n) \rangle$$

- formal $(\text{vol}\, SL_2(\mathbb{C}))^{-1} \longleftrightarrow$ mod out by those conformal trf. well defined on $S^2$ (6 generators)

  $$z \rightarrow \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$$

- can always find a, b, c, d to map any $(z_i, z_j, z_k) \to (0, 1, \infty)$
- Jacobian $|\langle c(z_i)\, c(z_j)\, c(z_k)\rangle|^2 = |z_{ij}\, z_{ik}\, z_{jk}|^2$ in passing from $d^2z_i\, d^2z_j\, d^2z_k \to d^6(a,b,c,d)$
- trade $3 \times \int d^2z\, V_\Phi(z)$ for $c(z)\,\bar{c}(\bar{z})\, V_\Phi(z)$ ["unintegrated vertex"] saturating (b, c) backgrd charge at $g=0$
- as a result ($\forall\, i, j, k \in \{1, 2, \ldots, n\}$)

$$M^{\text{tree}}_{\text{cl}}(\{\Phi_i, p_i\}; \alpha') = \int_{\mathbb{C}^{n-3} \setminus \{z_i = z_j\}} \Big\langle \prod_{a \neq i,j,k}^{n} d^2z_a\, V_{\Phi_a}(z_a)$$

$$c\bar{c}\, V_{\Phi_i}(z_i)\; c\bar{c}\, V_{\Phi_j}(z_j)\; c\bar{c}\, V_{\Phi_k}(z_k) \Big\rangle \Big|_{\substack{z_i \to 0 \\ z_j \to 1 \\ z_k \to \infty}}$$

- integration var's are conf. invariant cross ratios $z_a = \frac{z_{ai}\, z_{jk}}{z_{ji}\, z_{ak}} \Big|_{\substack{z_i \to 0 \\ z_j \to 1 \\ z_k \to \infty}}$

• 3 tachyon example ($V_{T_i}(z_i) = :e^{ip_i \cdot X(z_i)}:$ @ $p_i^2 = \frac{4}{\alpha'}$)

- insert $\alpha' p_1 \cdot p_2 = \frac{\alpha'}{2}\left[(p_1+p_2)^2 - p_1^2 - p_2\right]$ & $p_1 + p_2 = -p_3$ into

$$\Big\langle \prod_{j=1}^{3} :e^{ip_j \cdot X(z_j)}: \Big\rangle = \delta^D(p_1+p_2+p_3)\left(|z_{12}|^{\alpha' p_1 \cdot p_2} \times \text{cyc}(1,2,3)\right)$$

$$= \frac{\delta^D(p_1+p_2+p_3)}{|z_{12}\, z_{13}\, z_{23}|^2}$$

- cancels $z_i$-dependence of $\langle \prod_{j=1}^{3} c\bar{c}(z_j)\rangle$

$$\Rightarrow M^{\text{tree}}_{\text{cl}}(\{T_1, T_2, T_3; p_1, p_2, p_3\}; \alpha') = \delta^D(p_1+p_2+p_3)$$

- drop ubiquitous $\delta^D\big(\sum_{i=1}^{n} p_i\big)$ henceforth

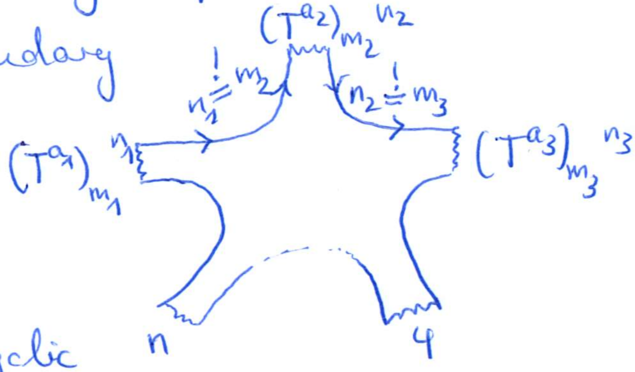
## 4.5) Open-string tree amplitudes

- color decomposition: all $T^{a_j}$-dependence in traces

$$M_{op}^{tree}(\{\Phi_j, p_j, a_j\};\alpha') = \sum_{\rho \in S_{n-1}} \mathrm{Tr}(T^{a_{\rho(1)}} T^{a_{\rho(2)}} \ldots T^{a_{\rho(n-1)}} T^{a_n}) \times A^{tree}(\Phi_{\rho(1)}, \Phi_{\rho(2)}, \ldots, \Phi_{\rho(n-1)}, \Phi_n; \alpha')$$

* permutations $\rho$ of $1, 2, \ldots, n-1$

$\Longleftrightarrow$ cyclically inequiv. orderings of external states on disk boundary

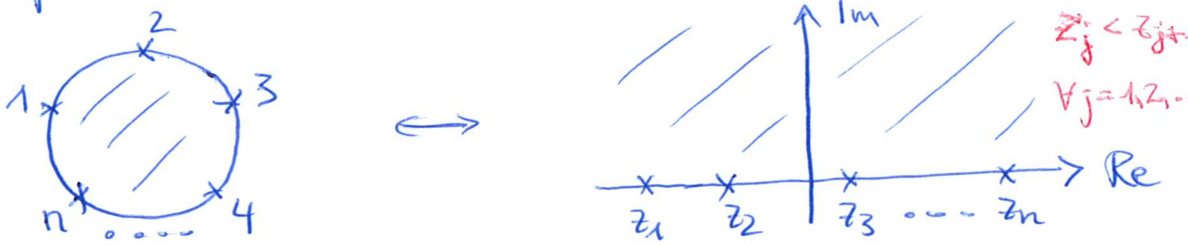


* "endpt-consistency" of neighbors $\Rightarrow \delta_{n_1}^{m_2} \delta_{n_2}^{m_3} \ldots$

* "partial amplitude" only cyclic

$$A^{tree}(\Phi_1, \ldots, \Phi_n; \alpha') = A^{tree}(\Phi_n, \Phi_1, \ldots, \Phi_{n-1}; \alpha')$$

* integration domain of $\int dz\, V_\Phi(z)$ tailored to $\rho$



$z_j < z_{j+1}$

$\forall j = 1, 2, \ldots, n-1$

$$A^{tree}(\Phi_1, \ldots, \Phi_n; \alpha') = \int_{-\infty < z_1 < z_2 < \ldots < z_n < +\infty} \frac{dz_1 \ldots dz_n}{\mathrm{vol}\, SL_2(\mathbb{R})} \left\langle \prod_{a=1}^{n} V_{\Phi_a}(z_a) \right\rangle$$

* inverse vol $SL_2(\mathbb{R})$: fix $(z_i, z_j, z_n) \to (0, 1, \infty)$ & insert c's

$$\Rightarrow A^{tree}(\ldots \underbrace{\Phi_i \ldots \Phi_j}_{} \ldots \Phi_n; \alpha') \sim \int_{-\infty < z_1 < z_2 < \ldots < z_n < \infty} \left\langle \prod_{a \neq i,j}^{n-1} dz_a V_{\Phi_a}(z_a) \, c V_{\Phi_i}(z_i)\, c V_{\Phi_j}(z_j)\, c V_{\Phi_n}(z_n) \right\rangle \Bigg|_{\substack{z_i \to 0 \\ z_j \to 1 \\ z_n \to \infty}}$$

need $i < j$ for compatibility with $z_l < z_{l+1}$

- 3 gluon example $V_{\xi_j}(z_j) = \xi_j^{A_j} :\partial X_{A_j} e^{ip_j \cdot X(z_j)}:$

  @ $p_j^2 = 0 = \xi_j \cdot p_j$ & $p_1 + p_2 + p_3 = 0 \Rightarrow p_i \cdot p_j = 0$

  * $A^{\text{tree}}(\xi_1, \xi_2, \xi_3; \alpha') \sim \langle c V_{\xi_1}(z_1)\, c V_{\xi_2}(z_2)\, c V_{\xi_3}(z_3) \rangle$

$$= z_{12} z_{13} z_{23}\, \xi_1^{A_1} \xi_2^{A_2} \xi_3^{A_3} \Big\langle \prod_{j=1}^{3} :\partial X_{A_j} e^{ip_j \cdot X(z_j)}: \Big\rangle$$

$$= i\, z_{12} z_{13} z_{23}\, \xi_1^{A_1} \xi_2^{A_2} \xi_3^{A_3} \Big\{ (2\alpha')^3 \frac{p^2_{A_1} z_{23}}{z_{12} z_{13}} \frac{p^3_{A_2} z_{31}}{z_{23} z_{21}} \frac{p^1_{A_3} z_{12}}{z_{31} z_{32}}$$

$$+ (2\alpha')^2 \Big[ \frac{\eta_{A_1 A_2}}{z_{12}^2} \frac{p^2_{A_3} z_{12}}{z_{31} z_{32}} + \text{cyc}(1,2,3) \Big] \Big\}$$

$$= i (2\alpha')^2 \Big\{ \Big[ \underbrace{(\xi_1 \cdot \xi_2)(\xi_3 \cdot p_2) + \text{cyc}(1,2,3)}_{\text{from YM 3pt vertex } \operatorname{Tr}(A^2 \partial A)} \Big] + \underbrace{2\alpha' (\xi_1 \cdot p_2)(\xi_2 \cdot p_3)(\xi_3 \cdot p_1)}_{\text{from } \alpha' \operatorname{Tr}(F^3)} \Big\}$$

  * missing normalization factor $g_{YM}/(2\alpha')^2$

  * color dependence $\operatorname{Tr}([T^{a_1}, T^{a_2}] T^{a_3}) \sim f^{a_1 a_2 a_3}$

- 4 tachyon example

  * with $p_j^2 = \frac{1}{\alpha'}$, rewrite $2\alpha' p_i \cdot p_j = s_{ij} - 2$
  with dim' less Mandelstam inv's $s_{ij} = \alpha'(p_i + p_j)^2$

  * by $s_{14} + s_{24} + s_{34} = 4$, fixing $(z_1, z_3, z_4) \to (0, 1, \infty)$

$$\Rightarrow \Big\langle V_{T_2}(z_2) \prod_{j=1,3,4} c V_{T_j}(z_j) \Big\rangle = z_{13} z_{14} z_{34} \prod_{1 \le i < j}^{4} |z_{ij}|^{2\alpha' p_i \cdot p_j}$$

$$\longrightarrow - |z_2|^{s_{12}-2} |1 - z_2|^{s_{23}-2}$$

  * integrate unfixed $z_2$ over $(z_1, z_3) = (0, 1)$

$$\Rightarrow A^{\text{tree}}(T_1, T_2, T_3, T_4; \alpha') \sim \int_0^1 dz_2\, z_2^{s_{12}-2} (1 - z_2)^{s_{23}-2}$$

  * famous Veneziano amplitude '68 → beta-/ $\Gamma$-fcts

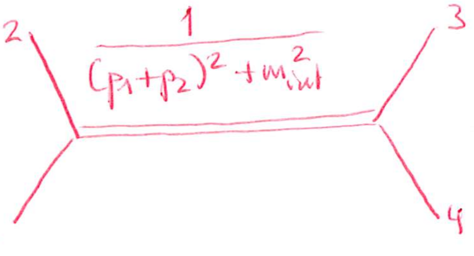
$$A^{\text{tree}}(T_1, T_2, T_3, T_4; \alpha') \sim \frac{\Gamma(s_{12}-1)\Gamma(s_{23}-1)}{\Gamma(s_{12}+s_{23}-2)}$$

  [birth of string theory]

* pole expansion via $(1-z_2)^{s_{23}-2} = \sum_{n=0}^{\infty} (-z_2)^n \binom{s_{23}-2}{n}$

$$\Rightarrow A^{\rm tree}(T_1,T_2,T_3,T_4;\alpha') = \sum_{n=0}^{\infty} \frac{(-)^n \binom{s_{23}-2}{n}}{s_{12}+n-1}$$

$$= \underset{\substack{\nearrow \\ \text{tachyon}}}{\frac{1}{s_{12}-1}} + \underset{\substack{\uparrow \\ \text{massless}}}{\frac{2-s_{23}}{s_{12}}} + \underbrace{\frac{(s_{23}-2)(s_{23}-3)}{2(s_{12}+1)}}_{\substack{\uparrow \\ \infty \text{ massive tower}}} + \dots$$



* poles in $s_{12} = 1, 0, -1, -2, \dots$

$\Leftrightarrow (p_1+p_2)^2 = \frac{1}{\alpha'}, 0, -\frac{1}{\alpha'}, -\frac{2}{\alpha'}; \dots$

matches $-m^2_{\rm int}$ with

$m^2_{\rm int} \in \{$open-string spectrum$\}$

* "duality": $\infty$ many $s_{12}$-channel poles already incorporate $\infty$ many $s_{23}$-channel poles

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## 4.6) Revisiting closed-string amplitudes

At n points: $M^{\rm tree}_{cl} \sim \sum_{\rho,\sigma} f_{\rho,\sigma}(s_{ij}) A^{\rm tree}(\rho) \bar{A}^{\rm tree}(\sigma)$ ← essentially products of $\sin(\frac{\pi\alpha'}{2} p_i \cdot p_j)$

→ KLT relations [Kawai-Lewellen-Tye '86]

- at 3pt, $A^{\rm tree}(\zeta_1,\zeta_2,\zeta_3) = A_{YM} + 4\alpha' A_{F^3}$

$$\Rightarrow M^{\rm tree}_{cl}(\{\zeta_1,\zeta_2,\zeta_3\};\alpha') = A^{\rm tree}(\zeta_1,\zeta_2,\zeta_3;\tfrac{\alpha'}{4}) \bar{A}^{\rm tree}(\zeta_1,\zeta_2,\zeta_3;\tfrac{\alpha'}{4})$$

$$= \underbrace{A_{YM}\bar{A}_{YM}} + \alpha' \underbrace{(A_{YM}\bar{A}_{F^3} + A_{F^3}\bar{A}_{YM})} + (\alpha')^2 \underbrace{A_{F^3}\bar{A}_{F^3}}$$

3pt vertex from $\mathcal{L}_{EH} = \sqrt{-\det G}\, R$ @ $G_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ & $\kappa^2 = 32\pi^2 G_N$

$\alpha' R^2$ correction to $\mathcal{L}_{EH}$

$\alpha'^2 R^3$ correction to $\mathcal{L}_{EH}$

* graviton / B-field / dilaton pol's from $\zeta^{\mu\nu} = \zeta^{\mu}\bar{\zeta}^{\nu}$

* same double copy at $(\zeta_1,\zeta_2,\zeta_3) \to$ any $(\Phi_1,\Phi_2,\Phi_3)$

- pedestrian computation extremely tedious already @ $\alpha' \to 0$

$$\mathcal{L}_{EH} = h\partial^2 h + \underbrace{\kappa h^2 \partial^2 h}_{} + \underbrace{\kappa^2 h^3 \partial^2 h}_{} + \begin{pmatrix} \infty \text{ more} \\ \text{vertices} \end{pmatrix}$$

171 - term    2850 - term 

- at $\geq 4$ pt, KLT relations have pi-dependent kernel

$$M^{\text{tree}}_{cl}(\{\Phi_1,\Phi_2,\Phi_3,\Phi_4\};\alpha') \sim \sin\left(\frac{\pi\alpha'}{2} p_1 \cdot p_2\right)$$

$$\times A^{\text{tree}}(\underbrace{\Phi_1,\Phi_2}_{\text{flip}},\Phi_3,\Phi_4;\tfrac{\alpha'}{4}) \overline{A}^{\text{tree}}(\underbrace{\Phi_2,\Phi_1}_{\text{flip}},\Phi_3,\Phi_4;\tfrac{\alpha'}{4})$$

* 4pt tachyon amplitudes

$$A^{\text{tree}}(T_1,T_2,T_3,T_4;\alpha') \sim \frac{\Gamma(s_{12}-1)\Gamma(s_{23}-1)}{\Gamma(2-s_{13})}$$

$$\Rightarrow M^{\text{tree}}_{cl}(\{T_1,T_2,T_3,T_4\};\alpha') \sim \frac{1}{\pi}\int_{\mathbb{C}} d^2 z\, |z|^{\frac{s_{12}}{2}-4} |1-z|^{\frac{s_{23}}{2}-4}$$

$$= \frac{\Gamma\left(\frac{s_{12}}{4}-1\right)\Gamma\left(\frac{s_{23}}{4}-1\right)\Gamma\left(\frac{s_{13}}{4}-1\right)}{\Gamma\left(2-\frac{s_{12}}{4}\right)\Gamma\left(2-\frac{s_{23}}{4}\right)\Gamma\left(2-\frac{s_{13}}{4}\right)}$$

poles @ $s_{ij} = 4, 0, -4, \ldots$ $\leftrightarrow$ closed-string spectrum

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## 5) RNS formulation of superstrings

### 5.1) Worldsheet variables of type IIA/B superstrings

$$S_{RNS}[X,\psi,\overline{\psi}] = \frac{1}{2\pi}\int d^2 z \left\{ \frac{2}{\alpha'} \partial_z X_\mu \partial_{\bar z} X^\mu + \psi_\mu \partial_{\bar z} \psi^\mu + \overline{\psi}_\mu \partial_z \overline{\psi}^\mu \right\}$$

- obtained from 2-dim supergravity in "superconformal gauge" (e.g. sec 8.2 of Uppsala lecture notes) with metric $h_{\alpha\beta} \to \eta_{\alpha\beta}$, gravitino $\chi_\alpha^{i=1,2} \to 0$, worldsheet fermion $\Psi_\mu^{i=1,2} = (\psi_\mu, \overline{\psi}_\mu)$ and $i=1,2$ Dirac spinor index on worldsheet

[in 2 dim, Dirac spinors have 2 cpts.]

- residual gauge freedom: "superconformal transformations": diffeo's & SUSY that can be undone via (super-)Weyl
- superconformal algebra generated by

$$T(z) = -\frac{1}{\alpha'} :\partial_z X_\mu \partial_z X^\mu: + \underbrace{\frac{1}{2} :(\partial_z \psi_\mu)\psi^\mu:}_{\text{central charge } 1/2 \text{ per } \psi_\mu \text{ cpt}}$$

$$\Rightarrow \text{total of } c_{X,\psi} = \frac{3D}{2}$$

$$G(z) = \frac{1}{\sqrt{2\alpha'}} :i\partial_z X_\mu \psi^\mu:$$

- OPE $\psi^\mu(z)\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{z-w} + \dots$

  $\Rightarrow$ $\psi^\mu$ is conformal primary of $h = 1/2$

  $\Rightarrow$ SUSY transformations

$$G(z)\,\partial_w X^\mu(w) \sim -\frac{i}{2}\sqrt{\frac{\alpha'}{2}}\left\{\frac{\psi^\mu(w)}{(z-w)^2} + \frac{\partial_w \psi^\mu(w)}{z-w} + \dots\right\}$$

$$G(z)\,\psi^\mu(w) \sim \frac{i}{\sqrt{2\alpha'}}\,\frac{\partial_w X^\mu(w)}{z-w} + \dots$$

---

## 5.2) Physical spacetime bosons of open superstring

$|\text{phys}_{\text{bos}}\rangle \longleftrightarrow$ superconformal primary $(\Phi_-^{NS}, \Phi_+^{NS})$ of $h(\Phi_-^{NS}) = 1/2$ & $h(\Phi_+^{NS}) = 1$ w.r.t. $T(z)$ such that

$$G(z)\,\Phi_-^{NS}(w) \sim \frac{\Phi_+^{NS}(w)}{2(z-w)} + \dots$$

$$G(z)\,\Phi_+^{NS}(w) \sim \frac{\Phi_-^{NS}(w)}{2(z-w)^2} + \frac{\partial_w \Phi_-^{NS}(w)}{z-w} + \dots$$

- decompose $\Phi_-^{NS}(z) = :\underbrace{\square(z)}_{\partial_z^{n\geq 1} X^\mu \text{ and } \partial_z^{n\geq 0}\psi^\mu} e^{ip\cdot X(z)}:$

later justified by absence of branch cuts

* impose $\square(z)$ to have odd $\underbrace{\#(\partial_z^{n\geq 0}\psi)}_{\text{worldsheet fermion number}}$ "GSO projection"

$\Rightarrow$ admissible $h_\Xi \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$

$\Rightarrow$ $m^2 = \frac{1}{\alpha'}(h_\Xi - \frac{1}{2}) \in \frac{4}{\alpha'}\{0, 1, 2, \dots\}$ tachyon free

* $m^2 = 0$ from $\Xi(z) = \xi^\mu \psi_\mu(z)$ @ $h_\Xi = \frac{1}{2}$

$\Rightarrow$ $\Phi_-^{NS}(z) = \xi^\mu \psi_\mu(z) :e^{ip\cdot X(z)}:$

* superconformal primary condition requires $\xi \cdot p = 0$:

$$G(z)\,\Phi_-^{NS}(w) \sim \frac{\xi \cdot p}{\sqrt{2\alpha'}\,(z-w)^2} :e^{ip\cdot X(w)}: + \underbrace{\mathcal{O}((z-w)^{-1})}$$

* associated $\Phi_+^{NS}(w)$ from $\#(z-w)^{-1}$

$$\Phi_+^{NS}(z) \sim \xi^\mu :\left(i\partial_z X_\mu(z) + 2\alpha'(p\cdot\psi)\psi_\mu(z)\right) e^{ip\cdot X(z)}:$$

$\hookrightarrow$ total derivative in $z$ if $\xi^\mu \to p^\mu$ [spacetime gauge freedom]

* get gauge bosons @ $m^2 = 0$, with 8 d.o.f @ D9 branes

- actual vertex operators: $\Phi_+^{NS}(z) = V^{(0)}(z)$ ok by $h(\Phi_+^{NS}) = 1$, but $\Phi_-^{NS}(z)$ lacks 1/2 unit of $h$

* need SUSY partner of $(b, c)$ ghosts:

Grassmann-even $(\beta, \bar\beta)$ & $(\gamma, \bar\gamma)$ worldsheet spinors

$$S_{sgh}[\beta, \gamma] = \frac{1}{\pi}\int d^2z \left(\beta\, \partial_{\bar z}\gamma + \bar\beta\, \partial_z \bar\gamma\right)$$

* another CFT sector @ $h_\beta = \frac{3}{2}$, $h_\gamma = -\frac{1}{2}$, $c_{\beta,\gamma} = 11$

[sec 13.1 in BLT on first-order systems]

$\Rightarrow$ $c_{total} = c_X + c_\psi + c_{b,c} + c_{\beta,\gamma} = \frac{3D}{2} - 15$

* missing $\delta h = \frac{1}{2}$ of $\Phi_-$ from change of variables $(\beta, \gamma) \longrightarrow$ fermions $(\eta, \xi)$ & chiral boson $\phi$

$\beta(z) = :e^{-\phi(z)}: \partial_z \xi(z)$ , $\gamma(z) = :e^{+\phi(z)}: \eta(z)$

with $\eta(z)\,\xi(w) \sim \frac{1}{z-w} + \dots$, $\phi(z)\,\phi(w) \sim -\log(z-w) + \dots$

* unusual $T_\phi(z) = -\frac{1}{2} :\partial_z\phi\, \partial_z\phi: - \partial_z^2 \phi$

⇒ $h(:e^{q\phi}:) = -\frac{1}{2} q^2 - q$

⇒ $q = -1$ unique realization of $h(:e^{-\phi}:) = \frac{1}{2}$

⇒ vertex operator $V^{(-1)}(z) = \Phi_-^{NS}(z) :e^{-\phi(z)}:$
in "ghost picture −1"

* 2 representatives $|{}^{\text{phys}}_{\text{bos}}\rangle \leftrightarrow \int d^2z \, V^{(0)}(z)$ or $V^{(-1)}(z)$
to be chosen that $\langle \dots \rangle_{\beta,\gamma}$ has total ghost picture $2g-2$ & cancels $(\beta,\gamma)$ background charge

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## 5.3) Physical spacetime fermions (open superstring)

Spinors $\psi^\mu$ only defined up to ±, sign

⇒ 2 admissible mode expansions on plane

$\psi^\mu(z) = \sum_{r \in \mathbb{Z}+1/2} \psi^\mu_r \, z^{-r-1/2}$ : NS sector, $z \to e^{2\pi i} z$ periodic

$\psi^\mu(z) = \sum_{r \in \mathbb{Z}} \psi^\mu_r \, z^{-r-1/2}$ : R sector, $z \to e^{2\pi i} z$ antiperiodic

* Vir algebra of $L^\psi_m = \frac{1}{2} \oint_{B_R(0)} \frac{dz}{2\pi i} z^{m+1} :(\partial_z \psi_\mu)\psi^\mu(z):$
only matches form of $[L^{bos}_m, L^{bos}_n]$
when shifting $L^\psi_0|_R$ by $\frac{D}{16}$ ; $L^\psi_0|_R = \frac{1}{2} \sum_{r \in \mathbb{Z}} r :\psi^\mu_{-r} \psi_{r\mu}: + \frac{D}{16}$

* R vacuum is therefore generated by conformal primary $S(z)$ of weight ~~$D$~~ $h = \frac{D}{16}$

* by $[L_0, \psi_0^\mu] = 0$ & Clifford algebra $\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu}$, $S(z)$ must be spacetime spinor of $SO(1, D-1)$

* in $D=10$ with Dirac-spinor index $A = 1, 2, \ldots, 32$

$$|A\rangle_R = \lim_{z \to 0} S_A(z) |0\rangle_{NS} \text{ with } \underbrace{\text{"spin field" } S_A}_{\text{open \& closes branch cut for } \psi^\mu} @ \, h = \frac{5}{8}$$

- vertex operators involving $S_A \leftrightarrow$ spacetime fermions

* also need spin-field for $\beta, \gamma$ system:

by $:e^{q_1 \phi(z)}::e^{q_2 \phi(w)}: \sim (z-w)^{-q_1 q_2} :e^{(q_1+q_2)\phi(w)}: + \ldots$

get branch cut for $:e^{-\phi(z)}: \ni V^{(-1)}$ via $:e^{\pm \phi(z)/2}:$

* $\geq 2$ ghost pictures per $|{}^{\text{phys}}_{\text{ferm}}\rangle \longleftrightarrow \int d^2 z \, V^{(\pm 1/2)}$

where $V^{(\pm 1/2)}(z) = \Phi_\pm^R(z) :e^{\pm \phi(z)/2}:$

* in the same way as $|\Phi_+^{NS}\rangle = G_{-1/2} |\Phi_-^{NS}\rangle$, relate $|\Phi_+^R\rangle = G_{-1} |\Phi_-^R\rangle$ via worldsheet SUSY, i.e. $\Phi_+^R(w) = \oint_{B_\epsilon(w)} \frac{dz}{2\pi i} G(z) (z-w)^{-1/2} \Phi_-^R(w)$

* by $[L_1, G_{-1}] = \frac{3}{2} G_0$, only get $h=1$ primary ~~$\Phi_+^R$~~ $V^{(+1/2)}$ if

$$G_0 |\Phi_-^R\rangle = 0 \implies \oint_{B_\epsilon(w)} \frac{dz}{2\pi i} G(z) (z-w)^{1/2} \Phi_-^R(w) = 0$$

* since $V^{(-1/2)}(z) \sim \underbrace{S_A(z) :e^{-\phi/2(z)}}_{h=1} \underbrace{e^{ip \cdot X(z)}:}_{h = \alpha' p^2} \underbrace{(\partial^{m \geq 1} X, \partial^{m \geq 0} \psi)}_{h \in \mathbb{N}_0}$

admissible masses $m^2 \in \frac{1}{\alpha'} \{0, 1, 2, \ldots\}$ (only integer modes $\psi_{\mathbb{Z}}$)

- massless fermions → SO(1,9) spinor wavefunction $\chi^A$

$$V_\chi^{(-1/2)}(z) = \chi^A S_A(z) :e^{-\phi/2(z)} e^{ip\cdot X(z)}:$$

* mutual branch cuts of $\chi^A$ are Dirac spinors

$$:e^{-\phi/2(z)}: S_A(z) :e^{-\phi/2(w)}: S_B(w)$$

charge conjugation matrix

$$\sim \frac{C_{AB} :e^{-\phi(w)}:}{(z-w)^{3/2}} + \mathcal{O}((z-w)^{-1})$$

property of SO(1,9), no scalar in tensor prod of 2 Weyl spinors

⇒ need $\chi^A C_{AB} \chi^B = 0$

⇒ pick $\chi$ to be Weyl spinor $\chi^{\alpha=1,\dots 16}$ where $C_{\alpha\beta}=0$

* GSO projection in R sector ∀ $m^2$:

$S_{\alpha=1,\dots,16}$ left-handed (LH) Weyl spinor,

2 inequiv. options $\bar{S}_\beta$ ⟺ LH or $\bar{S}^{\mathring{\beta}=\mathring{1},\dots,\mathring{16}}$ ⟺ RH

$S_\alpha \otimes \bar{S}_\beta$ ⟺ type IIB & $S_\alpha \otimes \bar{S}^{\mathring{\beta}}$ ⟺ type IIA

chiral thy non-chiral thy

* back to $m^2=0$ open string: $G_0$ condition imposes

$\chi^\alpha \gamma^\mu_{\alpha\beta} p_\mu = 0$ massless Dirac eq.

⇒ $\chi^\alpha$ have 8 d.o.f. just like $\xi^\mu$ on D9 brane

⇒ $\chi^\alpha$ are gauginos of 10 dim $N=1$ SYM

or 4 dim $N=4$ SYM

- spacetime SUSY (32 supercharges) of type IIA/B obscured in RNS formulation, but ∃ worldsheet rep's of supercharges with ghost picture, e.g.

$$Q_\alpha^{(-1/2)} = \sqrt{2}\,(\alpha')^{-1/4} \oint_{B_\epsilon(0)} \frac{dz}{2\pi i} S_\alpha(z) :\cancel{e^{ip\cdot X(z)}}: :e^{-\phi(z)/2}:$$