Introduction to the AdS/CFT correspondence

Alberto Zaffaroni[1]

Università di Milano-Bicocca and INFN, sezione di Milano-Bicocca, Piazza della Scienza, 3; I-20126 Milano, Italy

Abstract

The aim of these notes is to give a basic introduction to the AdS/CFT correspondence without assuming a previous exposure to string theory and D-branes. Some minimal knowledge of supersymmetry is required in the second part of the lectures where explicit realizations of the AdS/CFT correspondence are discussed. These notes arise from a Ph. D. course given at EPFL in Lausanne in 2008 for the Troisième cycle de la physique en Suisse romande and last (partially) updated in 2023 for the LACES school in GGI, Florence. They are still full of typos so use them at your own risk.

[1] alberto.zaffaroni@mib.infn.it

Contents

1 Introduction

One of the finest achievements of string theory in the last decade is the AdS/CFT correspondence and the use of holography to investigate strongly coupled quantum field theories. One crucial aspect of the correspondence is the possibility of computing quantum effects in a strongly coupled field theory using a classical gravitational theory. This has deep consequences that go far beyond string theory. Originally introduced to study the quantum behaviour of scale invariant theories, the correspondence has been extended to non conformal theories, where it gives an explanation for confinement and chiral symmetry breaking. It has also been used to study non-equilibrium phenomena in strongly coupled plasmas, and, more recently, applied to condensed matter systems. The correspondence also naturally implements the 't Hooft large N expansion, thus providing a verification of many ideas about gauge theories at large N.

These lectures grow from a Ph. D. course given at the EPFL in Lausanne in 2008 to a public with minimal previous exposure to string theory and D-brane. The conceptual framework of the correspondence is so simple that it can be discussed without entering deeply in the string realm. We shall introduce the relevant ingredients of string theory and D-branes using an effective theory language. In particular, the first half of these lectures, where the abstract duality between CFTs and gravitational theories in AdS is defined and discussed, uses nothing else than standard quantum field theory and General Relativity. We shall need a little bit of string theory, in particular D-branes, to provide explicit examples of dual pairs.

These lectures provide a very elementary introduction to the correspondence. In particular, we do not cover in details the existing non conformal supersymmetric solutions, the correspondence for conformal theories with less supersymmetry or in dimension different than four, and the inclusion of flavors. The reader may find more details on the basics of the correspondence in the original articles [1–4] and in very good reviews and books [5–8, 15]. Due to the large numbers of papers in the field, we refer to the list of references in [5] and the other reviews. References to specific papers are given only for specific results and are not exhaustive. A survey of more recent developments can be found in [9–14].

These lectures are organized as follows. In the rest of this long introduction, we introduce the two basic players of the correspondence, the conformal field theories and the gravitational theories with AdS vacuum. We also give a brief review of the large N limit in quantum field theory. General references for this background material are [16, 17]. The reader with some previous knowledge of these subjects is encouraged to start from the main part of these lectures, beginning with section 2, and to return to the introduction for reference.

In section 2, we discuss the conceptual content of the AdS/CFT correspondence without resorting to explicit realizations. We shall explain how to construct consistent correlation functions of a local quantum field theory from the equations of motion of a classical gravitational theory in AdS. We shall also extend the correspondence to the non-conformal case and discuss the holographic description of confinement.

In section 3 and 4 we discuss explicit realizations of the AdS/CFT correspondence, focusing on the best understood example, the duality between $\mathcal{N} = 4$ SYM and $AdS_5 \times S^5$. Section 3 requires a minimal knowledge of supersymmetry, at the level of $\mathcal{N} = 1$ multiplets and superfields. All the necessary ingredients of string theory and D-branes are introduced and discussed at the effective action level. Obviously, an idea of what string theory is will help, but no specific knowledge is required.

1.1 Conformal theories

Theories without scales or dimensionful parameters are classically scale invariant. A simple example is the scalar field with only quartic interaction

$$S = \int dx^4 \left((\partial\phi)^2 + \frac{\lambda}{4!}\phi^4 \right) . \tag{1.1}$$

The action is invariant if we simultaneously rescale the space-time coordinates (scale transformation) and the field with a specific weight

$$\phi(x) \rightarrow \lambda^\Delta \phi(\lambda x) \tag{1.2}$$

Δ is called the scaling dimension of the field and here it coincides with the canonical dimension $\Delta = 1$. The same theory would not be invariant if we add a mass term, as the reader may easily check. Another example of classically scale invariant theory is Yang-Mills coupled to massless fermions and scalars. In all these theories, scale invariance is broken by quantum corrections, but we shall see soon examples of quantum field theories with exact scale invariance.

1.1.1 The conformal group

Invariance under scale transformations typically implies invariance under the bigger group of conformal transformations.

A conformal transformation in a D-dimensional space-time is a change of coordinates that rescales the line element,

$$\begin{aligned} \text{dilatation}: \quad & x_\mu \rightarrow \lambda x_\mu \qquad (dx)^2 \rightarrow \lambda^2 (dx)^2 \\ \text{conformal transformation}: \quad & x_\mu \rightarrow x'_\mu \qquad (dx)^2 \rightarrow (dx')^2 = \Omega^2(x)(dx)^2 \end{aligned} \tag{1.3}$$

where $\Omega(x)$ is an arbitrary function of the coordinates. Clearly, scale transformations are a particular case of conformal transformations with constant $\Omega = \lambda$. Conformal transformations rescale lengths but preserve the angles between vectors. At the infinitesimal level $x'_\mu = x_\mu + v_\mu(x)$, $\Omega(x) = 1 + \omega(x)/2$ we easily derive the condition

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \omega(x)\eta_{\mu\nu} \tag{1.4}$$

Taking a trace we obtain $D\omega = 2\partial^\mu v_\mu$ and, substituting this expression in the previous formula, we obtain an equation identifying conformal transformations at the infinitesimal level

$$\partial_\mu v_\nu + \partial_\nu v_\mu - \frac{2}{D}(\partial^\tau v_\tau)\eta_{\mu\nu} = 0\,. \tag{1.5}$$

It is well known that in two dimensions there are infinite solutions of this equation (given, after Euclidean continuation, by all possible holomorphic functions on a plane) and the conformal group is infinite dimensional. For $D \neq 2$ the number of solutions is smaller and given by at most quadratic functions $v_\mu(x)$. The general solution is indeed

$$\begin{aligned}
\delta x_\mu \;=\; & \quad a_\mu && \quad P_\mu \\
& \quad \omega_{\mu\nu}x_\nu && \quad J_{\mu\nu} \qquad (\omega_{\mu\nu} = -\omega_{\nu\mu}) \\
& \quad \lambda x_\mu && \quad D \\
& \quad (b_\mu x^2 - 2x_\mu(bx)) && \quad K_\mu\,.
\end{aligned} \tag{1.6}$$

We recognize, in the first two lines, translations (whose generator will be denoted by P_μ) and Lorentz transformations (generated by $J_{\mu\nu}$); they are obviously conformal transformations (with $\Omega = 1$) since they leave the line element invariant. The third line corresponds to the dilatation (generated by D). The only novelty is the special conformal transformation (generated by K_μ) given by the fourth line. The corresponding finite transformation is

$$x_\mu \to \frac{x_\mu + c_\mu x^2}{1 + 2cx + (cx)^2} \tag{1.7}$$

Altogether we have

$$D + \frac{D(D-1)}{2} + 1 + D \;=\; \frac{(D+1)(D+2)}{2} \tag{1.8}$$

generators. In fact, one can check that the group is isomorphic to $SO(2,D)$ (for an algebraic proof see below). There is an extra discrete symmetry that acts as a conformal transformation,

$$\begin{aligned}
x_\mu &\to \frac{x_\mu}{x^2} \\
(dx)^2 &\to \frac{(dx)^2}{x^4}\,.
\end{aligned} \tag{1.9}$$

Adding this discrete transformation we obtain the full conformal group $O(2,D)$.

The important point is that, under mild conditions, a scale invariant theory is also conformal invariant. We can easily construct currents associated with the conformal transformations,

$$J_\mu = T_{\mu\nu}\, \delta x^\nu \tag{1.10}$$

This expression, with some subtleties and redefinitions, can be derived from Noether's theorem [16]. Conservation of the currents corresponding to translations requires conservation of the stress energy tensor $\partial^\mu T_{\mu\nu} = 0$ and conservation of the currents corresponding to Lorentz transformations is then automatic if $T_{\mu\nu}$ is symmetric. The current for dilation $J_\mu = T_{\mu\nu} x^\nu$ is now conserved if

$$\partial^\mu (T_{\mu\nu} x^\nu) = T^\nu_\nu \equiv 0 \tag{1.11}$$

We see that the condition for scale invariance is the tracelessness of the stress energy tensor. Now, we easily see that in a Poincaré and scale invariant theory (with a symmetric traceless conserved stress energy tensor) the conformal currents are automatically conserved,

$$\partial^\mu (T_{\mu\nu} v^\nu) = \partial^\mu T_{\mu\nu} v^\nu + T_{\mu\nu} \partial^\mu v_\nu = \frac{1}{2} T^{\mu\nu} \left(\partial_\mu v_\nu + \partial_\nu v_\mu\right) = \frac{1}{D} \partial^\tau v_\tau T^\mu_\mu \equiv 0 \tag{1.12}$$

The conditions on trace and symmetry properties of the stress energy tensor can be easily realized in most reasonable classical and quantum field theories. Although exotic counterexamples exist, we can safely assume that a scale invariant theory enjoys the full conformal invariance.

In the presence of supersymmetry, the conformal group is enhanced to a supergroup [1] obtained from $O(2, D)$ by adding the supercharges Q^a and the R-symmetry that rotates them. We also need to add the so-called conformal supercharges S^a. These are required to close the superconformal algebra $[K, Q] \sim S$. We shall not use explicitly the algebra of the superconformal group; the reader can find more details in the Appendix.

1.1.2 Conformal quantum field theories

In a quantum theory, conformal invariance is broken by the introduction of a renormalization scale. The Renormalization Group (RG) and the Callan-Symanzik equation can be seen as anomalous Ward identity for dilatations. For example, in a pure Yang-Mills theory, which is classically scale invariant, the gauge coupling runs with

[1] The superconformal group in four dimensions is usually denoted $SU(2, 2|N)$ where N is the number of supersymmetries; $SU(2, 2) \sim SO(4, 2)$.

the energy scale, a dimensionful parameter is introduced by dimensional transmutation, and the quantum stress energy tensor is not traceless anymore,

$$\begin{aligned} \mu \frac{d}{d\mu} g &= \beta(g) \to g(\mu), \Lambda_{QCD} \\ T^{\mu}_{\mu} &\sim \beta(g) F^2_{\mu\nu} \end{aligned} \tag{1.13}$$

In a more general theory with gauge fields, fermions and scalars, all dimensionless couplings run with the energy scale. In the following we will denote generically with g the set of couplings of a theory. The classical dimension d of a field will be corrected by the anomalous dimension

$$\Delta = d + \gamma(g), \qquad \gamma = \frac{1}{2} \mu \frac{d}{d\mu} \ln Z \tag{1.14}$$

Conformally invariant quantum field theories can be obtained as

- Fixed points of the RG. At points where the beta function vanishes $\beta(g^*) = 0$ the stress energy tensor becomes traceless, the RG equation becomes the Ward identity for dilatations, with a quantum dimension for the fields given by $\Delta = d + \gamma(g^*)$. We can even start with massive theories in the UV and let them flow

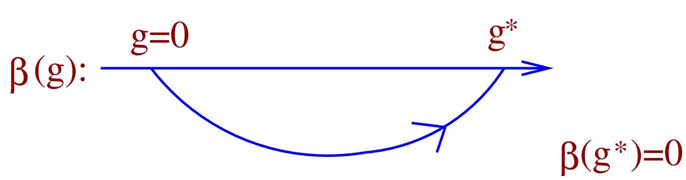

Figure 1: A standard textbook picture for the beta function behaviour near a fixed point.

  in the IR. Under certain circumstances, at low energies, we can find IR fixed points.

- Finite theories. Suppose that we have a theory with no divergences at all. In this case $\beta(g) = 0$ for all values of g and there is no RG flow. The theory is conformal also at the quantum level. Since g can have an arbitrary value we have a line (or manifold if there is more than one g) of fixed points. The standard example in this class of theories is $\mathcal{N} = 4$ SYM. As we shall discuss extensively in section 3.1.5, the theory has non-abelian gauge fields, transforming under a group G, coupled to four Weyl fermions and six real scalars, all in the adjoint representation of G. The standard textbook formula for the one loop beta function is

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}c(A) - \frac{2}{3}\sum c(\text{weyl}) - \frac{1}{6}\sum c(\text{scalar})\right) \tag{1.15}$$

where c denotes the second Casimir of the representation of gauge fields, fermions and scalars. Since all the fields transform in the adjoint representation, all the Casimir are equal and we see that the fermions and scalars balance the negative contribution of the gauge fields

$$-\frac{11}{3} + \frac{2}{3}4 + \frac{1}{6}6 \;=\; 0 \tag{1.16}$$

and the one-loop beta function is zero. It can be checked that the theory is finite at three-loops and it is believed to be finite at all orders. It is customary to combine coupling constant and theta angle in a complex parameter

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \tag{1.17}$$

The theory is finite (and therefore conformal) for all values of τ, $\beta(\tau) = \gamma(\tau) = 0$ and we have a complex line of fixed points. The conformal group is enhanced to $SU(2,2|4)$.

1.1.3 Constraints from conformal invariance

In a conformally invariant theory we have an unitary action of the conformal group on the Hilbert space. The generators P, J, D, K will be represented by hermitian operators. It is a tedious exercise to check that the generators P, J, D, K close the following algebra ($\eta_{\mu\nu} = \text{diag}(-1, 1, \cdots, 1)$)

$$\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= i\eta_{\mu\rho}J_{\nu\sigma} \pm \text{permutation} \\
[J_{\mu\nu}, P_\rho] &= i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right) \\
[J_{\mu\nu}, K_\rho] &= i\left(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu\right) \\
[J_{\mu\nu}, D] &= 0 \\
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= -2iJ_{\mu\nu} - 2i\eta_{\mu\nu}D
\end{aligned} \tag{1.18}$$

where, as familiar from quantum field theory courses, the first line is the algebra of the Lorentz group $SO(1,D-1)$, the next three lines state that D is a scalar and P_μ, K_μ are vectors, the next two lines state that P_μ and K_μ are ladder operators for D, increasing and decreasing its eigenvalue, respectively. The last equation states that P and K close on a Lorentz transformation and a dilatation. We can assemble all generators in

$$J_{MN} = \begin{pmatrix} J_{\mu\nu} & \frac{K_\mu - P_\mu}{2} & -\frac{K_\mu + P_\mu}{2} \\ -\frac{K_\mu - P_\mu}{2} & 0 & D \\ \frac{K_\mu + P_\mu}{2} & -D & 0 \end{pmatrix} \qquad M,N = 1,...,D+2 \tag{1.19}$$

and check that the antisymmetric J_{MN} is a rotation is a $D+2$ dimensional space with signature $(2,D)$ ($\eta_{MN} = \mathrm{diag}(-1,1,\cdots,1,-1)$)

$$[J_{MN}, J_{RS}] = i\eta_{MR} J_{NS} \pm \text{permutation}\,. \tag{1.20}$$

We thus recover algebraically the group $SO(2,D)$.

Particles are usually identified by mass and Lorentz quantum numbers, corresponding to the Casimirs of the Poincaré group. When conformal invariance is present, the mass operator $P_\mu P^\mu$ does not commute anymore with other generators, for example D. Mass and energy can be in fact rescaled by a conformal transformation. If a representation of the conformal group contains a state with given energy, it will contain states with arbitrary energy from zero to infinity obtained by applying dilatations. For this reason the entire formalism of S matrix does not make sense for conformal theories. We need to find different ways of labeling states. In a conformal theory we consider fields with good transformation properties under dilatations. If we set $\lambda = e^\alpha$, $e^{i\alpha D}$ will generates a dilatation. The quantum version of equation (1.2) is $[D,\phi(x)] = i(\Delta + x_\mu \partial^\mu)\phi(x)$ and identifies fields of conformal dimension Δ. We shall be interested in gauge theories and, in this case, the physical objects are gauge invariant operators with given conformal dimension. We can also restrict to fields or operators annihilated (at $x=0$) by the lowering operator K_μ; these are called *primary* operators; the others, obtained by applying P_μ and other generators repeatedly, are called descendants. The reader is referred to the appendix for more details. Primary operators are classified according to the dimension Δ and the Lorentz quantum numbers.

Note that there is another possibility of finding good quantum numbers for the conformal group. D and $J_{\mu\nu}$ correspond to the non-compact subgroup $SO(1,1) \times SO(1,3)$ of $SO(2,4)$. Sometimes it is more convenient to use the maximal compact subgroup $SO(2)\times SU(2)\times SU(2)\subset SO(2,4)$. States are still labeled by three numbers (Δ, j_1, j_2), now viewed as eigenvalues of the Cartan generators of $SO(2)\times SU(2)\times SU(2)$. The $SO(2)$ generator is $H=(P_0+K_0)/2$ and it is called the conformal energy. It would seem that its eigenvalues are integer. However, quantum theories strictly realize representations of the covering space of $SO(2,4)$, obtained by unwinding $SO(2)$ and Δ can assume continuous real values. The physical interpretation of the quantum numbers under the maximal compact subgroup is more evident in the Euclidean version of the theory, since $\mathbb{R}^4$ can be mapped by a conformal transformation to $S^3\times\mathbb{R}$. In this new description of the theory H becomes an Hamiltonian corresponding to time translation and $SU(2)\times SU(2)=SO(4)$ gives quantum numbers of an expansion on S^3. This type of conformal transformation is familiar from two dimensions and corresponds to a radial quantization of the theory.

Conformal invariance gives many constraints on a quantum field theory:

- The Ward identities for the conformal group give constraints on the Green functions. One can always find a basis of primary operators $O_i(x)$, with fixed scale dimension Δ_i. The set of (O_i,Δ_i) gives the spectrum of the CFT. One-, two- and three-point functions are completely fixed by conformal invariance. For example, one-point functions are zero, while two-point functions equal

$$\langle O_i(x)O_j(y)\rangle = \frac{A\delta_{ij}}{|x-y|^{2\Delta_i}} \tag{1.21}$$

  The coordinates dependence of 3-point functions is also fixed

$$\langle O_i(x_i)O_j(x_j)O_k(x_k)\rangle = \frac{\lambda_{ijk}}{|x_i-x_j|^{\Delta_i+\Delta_j-\Delta_k}|x_j-x_k|^{\Delta_j+\Delta_k-\Delta_i}|x_k-x_i|^{\Delta_k+\Delta_i-\Delta_j}}\,. \tag{1.22}$$

- Unitarity of the theory gives bounds restricting the possible dimensions of primary fields. We have inequalities that depend on the Lorentz quantum number $\Delta\geq f(j_1,j_2)$, which are discussed in the Appendix. Three cases will be particularly important for us

  - The dimension of a four-dimensional scalar field must be greater than one, $\Delta\geq 1$, and the saturation of the bound, $\Delta=1$, implies that the operator obeys free field equations.
  - For a vector field O_μ , $\Delta\geq 3$ and the bound is saturated if and only if the operator is a conserved current $\partial^\mu O_\mu=0$. Analogously, a spin 2 symmetric operator $O_{\mu\nu}$ has $\Delta\geq 4$, and $\Delta=4$ corresponds to conservation

$\partial^\mu O_{\mu\nu} = 0$. In particular, conserved currents have canonical dimension and are not renormalized.

– In supersymmetric theories the bounds relate dimension to spin and R symmetry quantum numbers. A typical case in 4d $\mathcal{N} = 1$ supersymmetric theories is the scalar bound $\Delta \geq \frac{3}{2}R$, relating dimension to R-charge, which is saturated by chiral operators. In this case, the saturation of the bound implies that the operator is annihilated by some combinations of the supercharges. This will be further discussed in section 3.3.1

In all cases, the saturation of the bound corresponds to a shortening of the (super) conformal multiplet and some non-renormalization property, which are discussed in the Appendix.

1.2 AdS space

We introduce now the gravitational side of the story.

AdS_5 is the maximally symmetric solution of the Einstein equations in five dimensions with cosmological constant. From

$$\begin{aligned} S &= \frac{1}{16\pi G_5}\int dx^5 \sqrt{|g|}\,(\mathcal{R} - \Lambda) \\ \mathcal{R}_{\mu\nu} - \frac{g_{\mu\nu}}{2}\mathcal{R} &= -\frac{\Lambda}{2} g_{\mu\nu} \end{aligned} \tag{1.23}$$

we have $\mathcal{R} = \frac{5}{3}\Lambda$ and therefore the Ricci tensor is proportional to the metric

$$\mathcal{R}_{\mu\nu} = \frac{\Lambda}{3} g_{\mu\nu} \tag{1.24}$$

This equation tells us that the solution is an Einstein space. If we further require that

$$\mathcal{R}_{\mu\nu\tau\rho} = \frac{\Lambda}{12}\left(g_{\mu\tau}g_{\nu\rho} - g_{\mu\rho}g_{\nu\tau}\right) \tag{1.25}$$

we have a maximally symmetric space. In Euclidean signature, the maximally symmetric solution with positive cosmological constant (and therefore positive curvature) is the sphere S^5 with isometry $SO(6)$ and the one with negative curvature is the hyperboloid H^5 with isometry $SO(1,5)$. In Minkowskian signature, the maximally symmetric solution with $\Lambda > 0$ is called de-Sitter space (dS_5) and the one with $\Lambda < 0$ is called Anti-de-Sitter (AdS_5). All of these spaces can be realized as the set of solutions of a quadratic equation in a six dimensional flat space with suitable signature

$\mathbb{R}^{d,6-d}$ [2]. Let us focus on AdS_5. We define it as the set of solutions of

$$x_0^2 + x_5^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = R^2\,, \qquad \frac{1}{R^2} = -\frac{\Lambda}{12} \tag{1.26}$$

in a flat $\mathbb{R}^{2,4}$ with line element $ds^2 = -dx_0^2 - dx_5^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$. It is obvious from this defining equation, that AdS_5 has isometry group $O(2,4)$, identical to the conformal group in four dimensions.

A set of coordinates is given by

$$\begin{aligned} x_0 &= R\cosh\rho\cos\tau \\ x_5 &= R\cosh\rho\sin\tau \\ x_i &= R\sinh\rho\hat{x}_i\,, \qquad \sum_{i=1}^{4}\hat{x}_i^2 = 1 \end{aligned} \tag{1.27}$$

and the metric reads

$$ds^2 = R^2\left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3\right) \tag{1.28}$$

where Ω_3 is the line element of a three-sphere. It is easy to verify that $\rho \in \mathbb{R}^+$ and $\tau \in [0, 2\pi]$ cover the Minkowskian hyperboloid exactly once, and for this reason these coordinates are called *global*. Note that time is periodic and therefore we have close time-like curves. To avoid this we can take the universal cover where $\tau \in \mathbb{R}$: we shall always refer to AdS_5 as this universal cover.

We can find a second set of coordinates given by a four dimensional Lorentz vector x_μ and a fifth coordinate $u > 0$ by a redefinition

$$\begin{aligned} x_0 &= \frac{1}{2u}\left(1 + u^2(R^2 + \vec{x}^2 - t^2)\right) \\ x_5 &= R\,u\,t \\ x_{1,2,3} &= R\,u\,x_{1,2,3} \\ x_4 &= \frac{1}{2u}\left(1 - u^2(R^2 - \vec{x}^2 + t^2)\right) \end{aligned} \tag{1.29}$$

which brings the metric to the form

$$ds^2 = R^2\left(\frac{du^2}{u^2} + u^2(dx_\mu dx^\mu)\right) \tag{1.30}$$

We see that the metric has slices isomorphic to four-dimensional Minkowski spacetime, and for this reason these coordinates are called *Poincaré* coordinates. The

[2] For example, the sphere S^5, is defined by a quadratic equation $x_0^2 + x_1^2 + ... + x_5^2 = R^2$ in $\mathbb{R}^6$ and the hyperboloid H^5 by $x_0^2 - x_1^2 - ... - x_5^2 = R^2$ in $\mathbb{R}^{1,5}$ and are spaces with Euclidean signature.

four dimensional space-time is foliated over u which runs from zero to infinity. The Minkowski metric is multiplied by a warp factor u^2, whose meaning is that an observer living on a Minkowski slice sees all lengths rescaled by a factor of u according to its position in the fifth dimension. The plane $u = \infty$ is referred as the boundary of AdS_5. Note however that for $u \to \infty$ the metric ds^2 blows up. Mathematically $u = \infty$ is a conformal boundary (strictly speaking, it is the conformally equivalent metric $d\tilde{s}^2 = ds^2/u^2$ to have a boundary $\mathbb{R}^{1,3}$ at $u = \infty$). The plane $u = 0$ is instead a horizon: the killing vector $\frac{\partial}{\partial t}$ has zero norm at $u = 0$. These coordinates are convenient since they contain a Minkowski slice, and we shall use them in most of our applications. However, they cover only half of the hyperboloid; $u = 0$ does not correspond to a singularity and the metric can be extended after the horizon (using for example global coordinates).

There are other forms of the metric in Poincaré coordinates that are commonly used. They all differ by a redefinition of the fifth coordinate u. For example with $u = 1/z = e^r$ we have

$$ds^2 = R^2 \left(\frac{dz^2 + dx_\mu dx^\mu}{z^2} \right) = R^2 \left(dr^2 + e^{2r} dx_\mu dx^\mu \right) \tag{1.31}$$

The boundary is now at $z = 0$ and $r = \infty$ and the horizon at $z = \infty$ and $r = -\infty$.

As we have already said, the isometry group of AdS_5 is $SO(2, 4)$ which is the same as the conformal group in four dimensions. The AdS/CFT correspondence exploits deeply this fact. It is interesting to compare closely the realization of the two groups. Since the full group is not always manifest in explicit realizations or choices of coordinates, we shall look at particular subgroups,

- The subgroup $SO(2) \times SO(4)$ is manifest when we use global coordinates. In field theory, it is useful for studying quantization on $S^3 \times \mathbb{R}$. We see an explicit copy of S^3 in the metric and a time τ. $SO(2)$ corresponds to the Hamiltonian in field theory and it is time translation in AdS_5. Notice that in global coordinates the Killing vector $\frac{\partial}{\partial \tau}$ is never vanishing and everywhere defined. Both in field theory and gravity we take time $\tau \in \mathbb{R}$ and we consider the universal cover of $SO(2, 4)$.

- The subgroup $SO(1, 1) \times SO(1, 3)$ is manifest in Poincaré coordinates. The Minkowski slice in the metric with isometry $SO(1, 3)$ can be associated with the four-dimensional space-time where we quantize our field theory. $SO(1, 1)$ is the dilatation in field theory and it is realized as $(u, x_\mu) \to (\lambda u, x_\mu/\lambda)$.

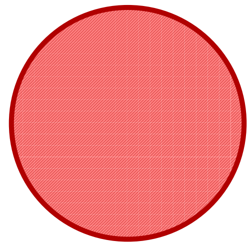

Figure 2: The Euclidean picture of AdS_5 as a five-dimensional ball.

To conclude this brief excursus on the geometry of AdS_5 let us consider the Euclidean continuation of the metric. This is important because in field theory we shall often perform a Wick rotation to Euclidean signature. We can do this by sending $x_5 \to -ix_5$, or, in each set of coordinates, $\tau \to -i\tau$ and $t \to -it$. The resulting metric is

$$R^2 \left(\cosh^2 \rho \, d\tau_E^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_3\right) = R^2 \left(\frac{du^2}{u^2} + u^2(dt_E^2 + d\vec{x}^2)\right) \tag{1.32}$$

The $u = \infty$ boundary plane $R^{1,3}$ of the Minkowskian version is replaced by $\mathbb{R}^4$. On the other hand, the $u = 0$ plane, which was a plane of null vectors in the Minkowski version, now shrinks to a point.

It is sometimes convenient to compactify the boundary of our flat four-dimensional space to S^4 by adding the point $u = 0$ to the boundary $\mathbb{R}^4$. One can show that the space is diffeomorphic to a five-dimensional ball in $\mathbb{R}^5$ with metric

$$\mathbb{R}^5 : \qquad y_1^2 + \cdots + y_5^5 \leq R^2, \qquad ds^2 = \frac{dy^2}{(R^2 - |y|^2)^2}$$

Exercise: It is probably instructive for the reader to check these statements in details for AdS_2, obtained by the previous formulae by neglecting $\vec{x}$. In Poincaré coordinates, by defining $z = t_E + \frac{i}{u}$ we have

$$R^2 \left(\frac{du^2}{u^2} + u^2 dt_E^2\right) = R^2 \frac{dz d\bar{z}}{(\mathrm{Im} z)^2} \tag{1.33}$$

and we recognize a familiar hyperbolic metric on the upper half-plane. The boundary is the real axis and we can include the point at infinity. With a conformal transformation we can map the half-plane to a disk. The metric will diverge at the circle bounding the disk.

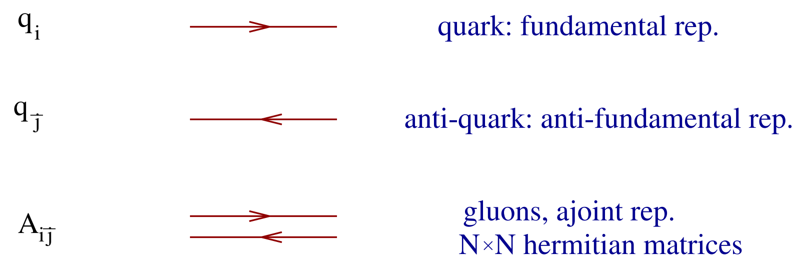

Figure 3: Double-line notation for objects transforming in the fundamental (quarks), anti-fundamentals (anti-quark) and adjoint representation (gluons) of the gauge group.

1.3 The large N limit for gauge theories

An $U(N)$ Yang-Mills gauge theory can be simplified in the limit where the number of colors N is large. t'Hooft first proposed to send $N \to \infty$ and do a systematic expansion in $1/N$. The large N expansion has proved to be useful for various reasons:

- It is a systematic expansion.
- It provides a weakly coupled Lagrangian for mesons and glueballs and it explains $U(1)$ anomalies.
- It simplifies the perturbative computation.
- Some QCD models in two dimensions become solvable in the large N limit.

Let us discuss the large N expansion for an $U(N)$ gauge theory

$$L = \mathrm{Tr}\left(F_{\mu\nu}^2 + L_{\mathrm{matter}}\right) \tag{1.34}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g_{YM}[A_\mu, A_\nu]$ and L_{matter} is the matter Lagrangian, which will include fundamental and adjoint fields in our applications. There is a convenient pictorial representation of Feynman graphs in terms of a double line notation, described in Figure 3. Fundamental and anti-fundamental fields can be written as q_i and $q_{\bar{j}}$, respectively, where $i, \bar{i} = 1, \cdots, N$ and the bar distinguishes indices transforming in the anti-fundamental representation. Adjoint fields of $U(N)$ can be written as hermitian matrices $A_{i\bar{j}}$ and thought as formal products of a fundamental and anti-fundamental representation. We shall use a Feynman graph notation where oriented lines are associated with indices i and $\bar{j}$ and not with fields. In this way, the propagator for an adjoint field can be then naturally written as a double line.

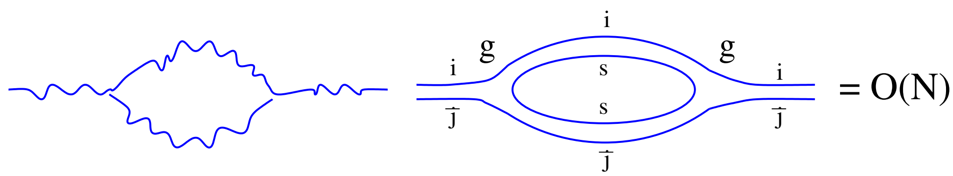

Figure 4: A graph contributing to the gluon self-energy.

The Feynman rules follow straightforwardly from the Lagrangian and can be easily understood by looking at explicit examples. Consider the case of the self-energy of a gluon, pictured in Figure 4. The indices at the beginning and end of a line have been identified, since the kinetic term in the Lagrangian is diagonal on each component of q_i and $A_{i\bar{j}}$. Similarly, indices in the vertices have been contracted according to matrix multiplication $\mathrm{Tr}A^3 = A_{i\bar{j}}A_{j\bar{p}}A_{p\bar{i}}$. We see that the only free index is the internal one s, which may take N different values. The self-energy diverges as $O(N)$. Many other graphs diverge as well. It seems that $N \to \infty$ is not a sensible limit. However, the self-energy contains powers of the coupling constant and it is of order $O(g_{YM}^2 N)$. If we take the t'Hooft limit

$$\begin{aligned} N &\to \infty, \\ g_{YM} &\to 0 \qquad\qquad x = g_{YM}^2 N \ \text{ fixed} \end{aligned} \tag{1.35}$$

the self-energy remains finite. The same happens to all other graphs. We shall now see that the t'Hooft limit makes sense for the entire perturbative expansion.

It is better to redefine fields and bring all dependence on g_{YM} in front of the Lagrangian,

$$L = \frac{1}{g_{YM}^2}\mathrm{Tr}\left(F_{\mu\nu}^2 + ...\right) = \frac{N}{x}\mathrm{Tr}\left(F_{\mu\nu}^2 + ...\right) \tag{1.36}$$

This convention will be used in the rest of these notes. The propagators now bring a factor of x/N and all type of vertices a factor of N/x. Let us first restrict to a theory with only adjoint fields. Two simple examples of graphs contributing to the free energy are reported in Figure 5. Let us focus on the contribution given by the two diagrams in double line notation. The first graph is planar, meaning that it can be drawn on a plane. More formally, it can be seen as a triangulation of a sphere, as indicated in the Figure. The second graph instead is not planar; if we insist to draw it on a plane some of its (double) lines will intersect in points which do not correspond to vertices of the graph. The best we can do is to consider it as drawn on a torus. Every graph can be drawn without intersecting lines on a Riemann surfaces of Euler characteristic $2-2g = F+V-E$ where F is the number of faces of the graph, E is

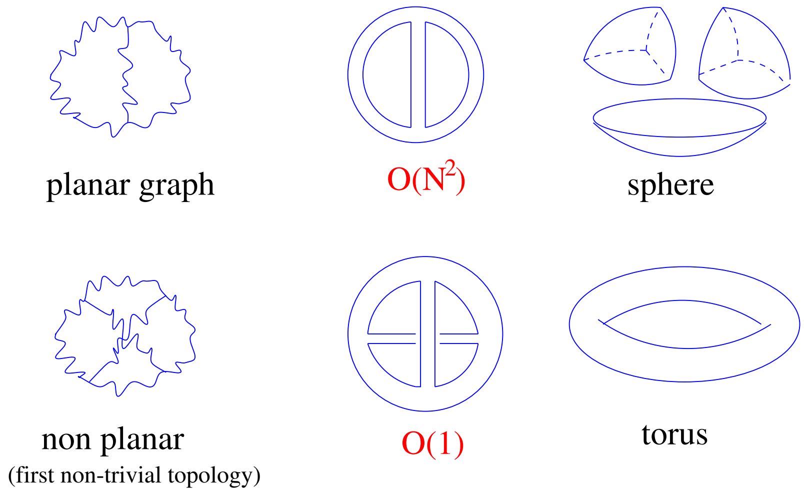

Figure 5: Planar and non-planar graphs and their relation with Riemann surfaces

the number of edges and V the number of vertices. g is the genus, or the number of holes, of the Riemann surface. We see from the examples that graphs with different topology have different powers of N. We can derive a general formula, taking into account that we have a factor of x/N for each propagator (E), a factor of N/x for each vertex (V) and a factor of N for each loop (F),

$$x^{E-V} N^{F+V-E} = O(N^{2-2g}) \tag{1.37}$$

We see that the t'Hooft expansion organizes graphs according to their topology. The expansion of the free energy in powers of $1/N$ is particularly simple

$$F = \sum_{g=0}^{\infty} N^{2-2g} f_g(x) \tag{1.38}$$

One may worry that the planar graphs give a contribution of order $O(N^2)$ to the free-energy, which seems to diverge. However, this is of the same order of the Lagrangian itself evaluated on a generic configuration

$$\frac{1}{g_{YM}^2} \mathrm{Tr} F^2 = \frac{N}{x} \mathrm{Tr} F^2 \sim O(N^2) \tag{1.39}$$

since the trace of a matrix is of typical order N. So the leading term in the free energy correctly reproduces the behavior of the Lagrangian. The subleading terms are suppressed by powers of $1/N^2$.

We could repeat a similar analysis for Green functions. In these notes we shall be interested in composite operators rather than in elementary fields. These can be written as traces or product of traces of elementary fields. Particularly important for us are the single trace operators, for example $\mathrm{Tr}F_{\mu\nu}^2$. We normalize single traces of products of adjoint fields with a further factor of N

$$O = \frac{1}{N}\mathrm{Tr}(\phi\phi\phi....) \tag{1.40}$$

in such a way that they are of order $O(1)$ on a generic field configuration. Connected correlation functions of O have then a $1/N^2$ expansion according to the topology of the graph starting with a leading term of order $O(1)$ given by the planar graphs.

The large N expansion considerably simplifies perturbation theory. For $N \to \infty$ only the planar graphs (f_0) survive. This has been used to solve some two-dimensional models in the planar limit. However no similar solvable model exists in dimension greater than two. The reader should be alerted that f_0 contains an infinite number of graphs which should be re-summed. The perturbative expansion simplifies in the planar limit but it is not solved in general.

Let us finish this discussion with few observations:

- In most of these notes we are interested in finite theories, where g_{YM} does not run and it is a dimensionless parameter. For theories like QCD, N is the only dimensionless parameter: the coupling constant runs with the scale $g(\mu)$ and it is better traded for the Renormalization Group invariant quantity Λ_{QCD}.

- As we see each graph corresponds to a Riemann surface and it is classified by the genus. The t'Hooft expansion is similar to the word-sheet expansion of string theory. The formula for the free-energy (1.39) is similar to the loop expansion of a string with coupling $g_s = e^{\phi} = 1/N$.

- We only discussed the case of adjoint fields, which will be the one of most interest for us. Fundamental fields introduce some novelty. It is easy to see that loops of fundamental fields have further powers of $1/N$ with respect to loops of adjoint fields, since a single line instead of a double line is used. In particular, fundamental fields are suppressed in the planar limit and they enter in the perturbative expansion with odd powers of N. The topology of the graphs is also changed: double lines allow to draw the graphs on closed surfaces, single lines necessarily introduce boundaries. We then obtain an expansion in Riemann surfaces with boundaries. The power of N of a graph is still given by the Euler characteristic which now can be odd. The expansion of a theory with only adjoints is in powers of $1/N^2$, the expansion of a theory with also

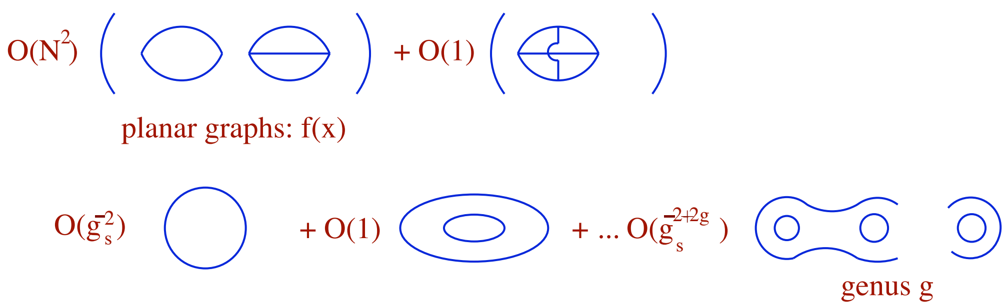

Figure 6: Emergence of a string structure from the Feynman graph expansion in the large N limit.

fundamentals is in powers of $1/N$. Pursuing the analogy with string theory, we see that fundamental fields are associated with open strings. In the context of the AdS/CFT correspondence a theory with adjoints is associated with a closed string background while fundamentals require the introduction of open string (typically in the form of D-branes which introduce the required boundaries).

- Finally a technical remark. In the large N limit $SU(N) \sim U(N)$. However this is only true in the planar limit. The gluon propagator for $SU(N)$ is not diagonal, $< A_{i\bar{j}} A_{p\bar{q}} >\sim \delta_{i\bar{q}}\delta_{p\bar{j}} - (1/N)\delta_{i\bar{j}}\delta_{p\bar{q}}$ and the $1/N$ term mixes with the subleading terms of the $1/N$ expansion.

2 The AdS/CFT correspondence

Up to now we saw two apparently different ways of realizing theories with $O(4,2)$ symmetry

- Conformal field theories in four dimensions
- Relativistic gravitational theories in AdS_5

Since the two objects live in space-times with different dimensions it is difficult to imagine a relation between them. However, holography has always been a favorite principle when dealing with gravity. For several reasons, it is thought that in quantum gravity the number of degrees of freedom of a region of space-time grows with the area of the region and not the volume. Here we shall see another form of holography, related to the fact that the boundary of AdS_5 is Minkowski space-time. All dynamics in AdS_5 can be reformulated as a boundary effect and it is captured by a four-dimensional local field theory. We shall set up a correspondence between CFT in four dimensions and gravitational theories in AdS_5.

The study of this correspondence is the purpose of the rest of these notes. In this Section, we start with the general construction without referring to explicit realizations of the correspondence which will be analyzed in Sections 3 and 4. More precisely, now we shall construct, for every five-dimensional gravitational theory with AdS_5 vacuum, a set of correlation functions for operators which satisfy all constraints required by a four dimensional local field theory with scale invariance. In this Section we will mostly work at the level of effective action in 5 dimensions. We deal with a theory that at low energy reduce to General Relativity coupled to gauge and matter fields. We assume the existence of a suitable quantum gravity UV completion of our five dimensional theory although this will not be really used in this Section. We shall enlighten some general aspects of the correspondence which do not refer to explicit realizations. We shall also make clear that AdS_5 is not really necessary. All we need is a space with the topological structure of AdS_5 and a conformal boundary.

We will focus on four-dimensional field theories but what we discuss can be straightforwardly extended to the correspondence between a CFT_d and a gravitational theory in AdS_{d+1}.

2.1 Formulation of the correspondence

To define the correspondence we need a map between the observables in the two theories and a prescription for comparing physical quantities and amplitudes.

We refer to the fields in five dimensions as bulk fields. We assume that there is a quantum gravity theory (string theory, higher spin theory, ...) that describes their interaction. We also assume that, at low energy, the interaction is described by an effective action

$$S_{AdS_5}(g_{\mu\nu}, A_\mu, \phi, ...) \tag{2.1}$$

with AdS_5 vacuum. We included the metric, gauge fields and scalars. In most applications this effective action corresponds to some supergravity with also tensor fields and fermions. We assume a potential for the scalar field with a negative value at the minimum thus creating a negative cosmological constant for the AdS_5 vacuum.

We refer to the CFT fields as boundary fields. We call L_{CFT} the four-dimensional Lagrangian. Recall that most often the elementary fields are not observables in the quantum theory, due to gauge invariance. The spectrum is specified by a complete set of gauge invariant primary operators in the CFT.

A basic point of the correspondence is the following statement: a field h in AdS is associated with an operator in the CFT with the same quantum numbers and they know about each other via boundary couplings. More precisely, from the four-dimensional point of view, every operator O can be associated to a source h

$$L_{CFT} + \int d^4 x h O \tag{2.2}$$

For the moment $h(x)$ is a four-dimensional background field that is introduced in order to compute correlation functions for the operator O. As usual,

$$e^{W(h)} = \left\langle e^{\int hO} \right\rangle_{QFT} \tag{2.3}$$

define $W(h)$ as the functional generator for connected correlation functions of O

$$\langle O....O \rangle_c = \frac{\delta^n W}{\delta h^n}\Big|_{h=0} \tag{2.4}$$

We can now think of the source $h(x)$ as the boundary value of a five dimensional field $h(x, x_5)$.

The fundamental statement of the AdS/CFT correspondence is now:

$$Z_{\mathrm{CFT}}[h(x)] = e^{W(h)} = \left\langle e^{\int hO} \right\rangle_{\mathrm{QFT}} \equiv Z_{\mathrm{QG}}\Big|_{\text{boundary value}\, h(x)} \tag{2.5}$$

where on the left hand side we have a functional depending on a d-dimensional configuration $h(x)$ and on the right hand side we have the quantum gravity path integral over all configurations $h(x_5, x)$ that reduce to $h(x)$ at the boundary. Since the knowledge of $W(h)$ for all possible sources of composite operators determines completely the CFT, the previous formula states the required equivalence between the CFT and the five-dimensional theory.

The prescription is actually useful when the gravity theory is weakly coupled. In this case, the quantum gravity partition function can be evaluated in a saddle point approximation. For every source configuration $h(x)$ we can find a five-dimensional field configuration

$$h(x) \to \hat{h}(x, x_5) \tag{2.6}$$

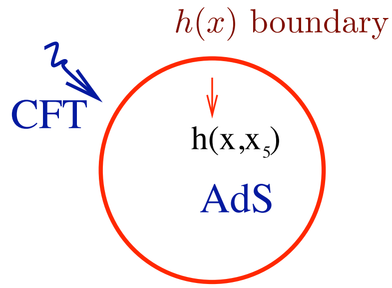

Figure 7: The prescription for extending sources to the bulk. We use the Euclidean version for clarity.

obtained by demanding that $h(x, x_5)$ solves the five-dimensional equations of motion derived by S_{AdS}. The extension from boundary to bulk is unique if we impose suitable boundary conditions at the horizon. The gravitational path integral in the saddle point approximation reduces to the exponential of the on-shell action

$$Z_{\mathrm{QG}}[h(x, x_5)\Big|_{\partial} = h(x)] = e^{iS_{AdS_{d+1}}(\hat{h})} + \cdots , \tag{2.7}$$

and we can identify:

$$e^{W(h)} = \left\langle e^{\int hO} \right\rangle_{QFT} = e^{iS_{AdS_5}(\hat{h})} \tag{2.8}$$

where on the left hand side we have a functional depending on an arbitrary four dimensional (off-shell) configuration $h(x)$ and on the right hand side we have the (on-shell) value of the five dimensional Lagrangian, evaluated on the solution of the equations of motion that reduces to $h(x)$ at the boundary. All this is similar to a Dirichlet problem in electrostatic.

The precise meaning of the previous formulae will be explored in the rest of these notes. However, a few immediate comments are in order.

- The equation of motions in AdS are second order and we need to specify two boundary conditions in order to find a unique solution. Both boundary conditions require some care. First of all, we cannot simply set $\hat{h}(x, \text{boundary}) = h(x)$ since solutions of the equations of motion diverge or vanish at the boundary. The typical example is the metric; as we saw, the metric blows up at the boundary. The right condition at the AdS boundary is of the form $\hat{h}(x, x_5) \sim f(x_5)h(x)$ and it will be extensively discussed in the next Section. The second boundary condition is to be imposed in the interior of AdS. We shall mainly work with the Euclidean version of the theory where things simplify. In the Euclidean, AdS_5 is a ball and the center of the space is just a point. We shall require regularity of the solution at the center of the ball. Working instead in Minkowski, and using Poincaré coordinates, we have to impose a suitable condition at the horizon in order to have an unique extension.

- We should stress that, in the semi-classical limit, we used the AdS equations of motion. An off-shell theory in four dimensions corresponds to an on-shell theory in five dimensions. This is a general feature of all the AdS/CFT inspired correspondences.

- The prescription assumes a choice of *radial* coordinate x_5. In different coordinates the boundary of AdS looks different. In global coordinates, using $x_5 = \rho$, the boundary is $S^3 \times \mathbb{R}$ and we will be considering the CFT defined on $S^3 \times \mathbb{R}$. On the other hand, in Poincarè coordinates the boundary is $\mathbb{R}^4$ (or $\mathbb{R}^{3,1}$ in the Lorenzian signature) and we will be considering the CFT in flat space. A change of coordinates in the bulk corresponds in general to a conformal transformation on the boundary. And indeed flat space and $S^3 \times \mathbb{R}$ are conformally equivalent

$$dr^2 + r^2 d\Omega_3 = r^2 \left(\frac{dr^2}{r^2} + d\Omega_3 \right) = e^{2t} \left(dt^2 + d\Omega_3 \right) . \tag{2.9}$$

  Of course, the CFT observables in the different cases will be related by the action of the conformal group.

- We haven't said how to map CFT operators to fields in the bulk. This will be specified by the details of the two theories and will be available when we have a constructive way of determining dual pairs. This is provided by string theory and will be discussed in Sections 3 and 4, where we shall also see how to map observables in specific examples. For the moment, let us notice that the field that couples to an operator can be often found using symmetries. h and O have the same $O(2,4)$ quantum numbers. In particular, there are obvious couplings in the case of conserved currents: we introduce a background gauge field by covariantizing the boundary action. The natural linearized couplings

$$L_{CFT} + \int d^4x \sqrt{g} (g_{\mu\nu} T_{\mu\nu} + A_\mu J_\mu + \phi F_{\mu\nu}^2 + \cdots) \tag{2.10}$$

  suggest that the operator associated with the graviton is the stress-energy tensor and the operator associated with a gauge fields in AdS is a current. Note that conservation of stress-energy tensor or currents are associated with the gauge invariance at the level of the sources (meaning that the field theory functional $W(A_\mu, g_{\mu\nu})$ is a gauge invariant functional of A_μ and $g_{\mu\nu}$). This obviously extends by consistency to the bulk theory. We see from this a general fact: global symmetries in the CFT correspond to gauge symmetries in AdS. We couldn't refrain from adding a particular scalar operator which is present in all gauge theories: $O = \mathrm{Tr}F^2_{\mu\nu}$. We coupled it to a source ϕ; the corresponding bulk field, in many explicit examples, is the string theory dilaton.

2.2 Physics in the bulk

One surprising thing about the correspondence is the existence of a fifth radial coordinate in the gravitational picture which is needed for the holographic interpretation. Let us see its role in more details.

A crucial ingredient in all the models obtained by the AdS/CFT correspondence is the identification of the radial coordinate in the supergravity solution with an energy scale in the dual field theory. The identification between radius and energy follows from the form of the AdS metric in Poincaré coordinates (we put $R = 1$ when no confusion is possible)

$$ds^2 = \frac{dx_\mu^2 + dz^2}{z^2}. \tag{2.11}$$

A dilatation $x_\mu \to \lambda x_\mu$ in the boundary CFT corresponds in AdS to the $SO(4,2)$ isometry

$$x_\mu \to \lambda x_\mu, \qquad z \to \lambda\, z. \tag{2.12}$$

We see that we can roughly identify $u = 1/z$ with an energy scale μ. The boundary region of AdS ($z \ll 1$) is associated with the UV regime in the CFT, while the horizon region ($z \gg 1$) is associated with the IR. This is more than a formal identification: as we shall see, holographic calculations of Green functions or Wilson loops associated with a specific reference scale μ are dominated by bulk contributions from the region $u = \mu$.

Let us consider now the dynamics of bulk fields in AdS_5. For the purposes of the correspondence, the boundary values of fields are arbitrary functions of four-spacetime coordinates x_μ, while the profile of fields in the fifth directions is set on-shell by the equations of motion. For simplicity, let us consider the case of a massive field ϕ in AdS_5 dual to some operator O in the CFT. We consider the Euclidean continuation of the metric (2.11). Given a scalar field $\phi(z, x_\mu)$ with action

$$S \sim \int dx^5 \sqrt{g}(g^{mn}\partial_m\phi\partial_n\phi + m^2\phi^2) = \int dz dx \frac{1}{z^5}\left(z^2(\partial_z\phi)^2 + z^2(\partial_\mu\phi)^2 + m^2\phi^2\right)$$

the equation of motion reads

$$\partial_z\left(\frac{1}{z^3}\partial_z\phi\right) + \partial_\mu\left(\frac{1}{z^3}\partial^\mu\phi\right) = \frac{1}{z^5}m^2\phi \tag{2.13}$$

Look first at the z behaviour. Consider first a mode independent of x_μ. The equation reduces to

$$z^5\partial_z(z^{-3}\partial_z\phi) = m^2\phi \tag{2.14}$$

which has two independent power-like solutions $\phi \sim z^\Delta$ with

$$m^2 = \Delta(\Delta - 4) \tag{2.15}$$

Since under a dilation $z \to \lambda z, x_\mu \to \lambda x_\mu$, Δ corresponds to a scaling dimension for the field and, as we shall see, will be identified with the conformal dimension of the dual operator O.

Denoting simply by Δ the largest solution of the quadratic equation (2.15), we find the near boundary behaviour of an on-shell field

$$\phi \sim \phi_0 z^{4-\Delta} + \phi_1 z^\Delta \tag{2.16}$$

The coefficient ϕ_0 and ϕ_1 correspond to the two linearly independent solutions of the second order equation of motion. They can be distinguished by the fact that the solution corresponding to ϕ_0 is not normalizable at the boundary,

$$\int dx^5 \sqrt{g}|\phi|^2 = \infty \tag{2.17}$$

while the one corresponding to ϕ_1 is. If we now include the x_μ dependence the previous behaviour is modified to

$$\phi(z, x_\mu) \sim (\phi_0(x)z^{4-\Delta} + O(z)) + (\phi_1(x)z^\Delta + O(z)) \tag{2.18}$$

where we can still identify the coefficients $\phi_{0,1}(x)$ of the two linearly independent solutions, which will still grow as $z^{4-\Delta}$ and z^Δ with corrections depending on both z and x_μ.

Now we are ready to discuss the boundary conditions to be imposed on a field near the boundary. We see that the leading term of a solution of the equation of motion can be singular if $\Delta > 4$ or vanishes if $\Delta < 4$. It approaches a constant only in the case $\Delta = 4$. In order to have a consistent prescription we need to impose at the boundary $z = 0$

$$\phi(z, x_\mu) \to z^{4-\Delta}\phi_0(x_\mu)\,. \tag{2.19}$$

$\phi_0(x)$ is the boundary value of our field to be identified with the source of the dual operator O. Once the value of $\phi_0(x)$ is specified, we have a unique regular solution that extends to all of AdS_5. In particular $\phi_1(x)$ will be determined as a functional of $\phi_0(x)$ by imposing the equations of motion and regularity at the center.

Let us see explicitly how it works in the simple case of a massless scalar field $m^2 = 0$. In this case $\Delta = 4$. It is convenient to perform a Fourier decomposition of modes on $\mathbb{R}^4$. The Fourier mode $\phi_p(z)e^{ipx}$ satisfies

$$z^5\partial_z(z^{-3}\partial_z\phi_p(z)) - p^2z^2\phi_p(z) = 0 \tag{2.20}$$

which, with $\phi_p = (pz)^2 y(pz)$, reduces to a Bessel equation

$$(pz)^2\frac{d^2y}{d(pz)^2} + (pz)\frac{dy}{d(pz)} - (4 + (pz)^2)y = 0 \tag{2.21}$$

whose general solution is $A_pI_2(pz) + B_pK_2(pz)$ [3]. Correspondingly $\phi_p \sim B_p(1+\cdots) + A_p(z^4+\cdots)$ as expected from (2.18) for a field with $\Delta = 4$. K_2 is the non-normalizable solution and I_2 the normalizable one. Since $I_2(pz)$ is exponentially growing for large z, regularity of the solution in AdS$_5$ requires $A_p = 0$ and we are left with $K_2(pz)$, which is exponentially vanishing for large z.

We can now go further and actually compute the effective action $W(\phi_0)$ depending on the boundary conditions ϕ_0, identified with the field theory source. For $\Delta = 4$, (2.19) set the asymptotic value of the solution equal to ϕ_0. In general computations in AdS_5, various quantities in the game diverge for $z \to 0$ and it is convenient to introduce a cut off and impose boundary conditions at $z = \epsilon$. At the end of the computation one sends ϵ to zero. This allows to keep track of local divergent pieces of the effective action and it is a general prescription for computing correlation functions in AdS_5. So we impose

$$\phi_p(z = \epsilon) \equiv \phi_p^0 \tag{2.22}$$

so that the solution is

$$\phi_p(z) = \frac{(pz)^2K_2(pz)}{(p\epsilon)^2K_2(p\epsilon)}\phi_p^0 \tag{2.23}$$

[3] $x^2y'' + xy' - (x^2 + a^2) = 0$ with $a \in \mathbb{N}$ has solutions $I_a(x) \sim J_a(ix)$ and $K_a(x)$, with asymptotic behaviour, $I_a(x) \sim x^a + \cdots$ and $K_a(x) \sim \frac{1}{x^a}(1 + \cdots + c_ax^{2a}\log x)$ for $x \to 0$ and $I_a(x) \sim \frac{e^x}{\sqrt{x}}$ and $K_a(x) \sim \frac{e^{-x}}{\sqrt{x}}$ for $x \to \infty$.

$W(\phi_0)$ is now obtained by evaluating the five-dimensional Lagrangian on the solution of the equations of motion. The computation can be simplified using a standard trick: by integrating by parts

$$S_{AdS} \sim \int_{\text{boundary}} \sqrt{g}\phi\partial^n\phi + \int \sqrt{g}\phi(-\Box + m^2)\phi \tag{2.24}$$

the second term is zero on the equations of motion and the action reduces to a boundary contribution. In our case,

$$S_{AdS} \sim \frac{\phi\partial_z\phi}{z^3}\Big|_{z=\epsilon} \tag{2.25}$$

and inserting the solution of the equations of motion

$$\phi(z,x) = \int dp^4 e^{ipx}\phi_p(z) \tag{2.26}$$

we obtain

$$W(\phi_0) = S_{AdS} \sim \int dpdp'\delta(p+p')\phi_p^0\phi_{p'}^0\frac{1}{z^3}\partial_z \log\phi_p(z)\Big|_{z=\epsilon} \tag{2.27}$$

from which we see that

$$\begin{aligned}\langle O(p)O(p')\rangle = \frac{\delta W}{\delta\phi_p^0\delta\phi_{p'}^0} &\sim \delta(p+p')p^4\frac{1}{(pz)^3}\frac{d}{d(pz)}\log\phi_p(z)\Big|_{z=\epsilon} \\ &\sim p^4\log(p\epsilon) + \sum_k \frac{1}{\epsilon^k}(\text{polynomial in } p) + O(\epsilon)\end{aligned}$$

since $\phi_p(z) = a_0 + a_2(pz)^2 + a_4(pz)^4 + c_4(pz)^4\log pz + \cdots$. We see that there are divergent terms in ϵ but they are local polynomials in p. These analytic terms are irrelevant in a quantum field theory computation since they can be reabsorbed by local counter-terms. In the $\epsilon \to 0$ limit the relevant contribution is

$$\langle O(p)O(p')\rangle \sim p^4\log(p\epsilon) \tag{2.28}$$

which after Fourier transform to coordinate space becomes

$$\langle O(x)O(x')\rangle = \frac{1}{(x-x')^8} \tag{2.29}$$

in agreement with CFT expectations for an operator of dimension four.

An analogous computation can be performed for $m^2 \neq 0$ and $\Delta \neq 4$. In this case the boundary condition (2.19) requires

$$\phi_p(z=\epsilon) \equiv \phi_p^0\epsilon^{4-\Delta} \tag{2.30}$$

so that $\phi(z) \sim z^{4-\Delta}$ for $z \to 0$ and the solution is

$$\phi_p(z) = \frac{(pz)^2 K_{\Delta-2}(pz)}{(p\epsilon)^2 K_{\Delta-2}(p\epsilon)} \phi_p^0 \epsilon^{4-\Delta} \tag{2.31}$$

As before

$$\langle O(p) O(p') \rangle \sim p^{2\Delta-4} + \text{analytic terms} \tag{2.32}$$

which after Fourier transform to coordinate space becomes

$$\langle O(x) O(x') \rangle = \frac{1}{(x-x')^{2\Delta}} \tag{2.33}$$

Let us note that the introduction of a cut-off ϵ is not only a matter of convenience, since the $\epsilon \to 0$ limit does not commute with other expansions performed to obtain the result. In particular, the cut-off prescription is the right one for obtaining the right normalization of the two point functions consistent with Ward identities when vector fields are included [18].

This computation confirms the interpretation of Δ as the conformal dimension. There are various observations to be made about this identification and its relation with the mass in AdS_5

$$R^2 m^2 = \Delta(\Delta - 4) \tag{2.34}$$

where we have restored the AdS radius,

- $m^2 \geq 0$ only for $\Delta \geq 4$. There are certainly theories where $\Delta < 4$. In fact the unitary bound is $\Delta \geq 1$. Operators with $\Delta < 4$ correspond to fields with negative mass in AdS_5. However they are not tachyons. Energy is positive as long as the Breitenlohner-Freedman bound $m^2 R^2 \geq -4$ is satisfied [19]; the curvature gives indeed a positive contribution to the energy of a scalar field propagating in AdS. The minimal value $m^2 R^2 = -4$ corresponds to $\Delta = 2$.

- Still a puzzle. Unitary bound requires $\Delta \geq 1$. Using masses greater than $-4/R^2$ we can obtain all operators with $\Delta \geq 2$. What happens to the operators with $1 \leq \Delta < 2$? Recall that we chosen Δ as the largest solution of equation (2.34). This because typically only the largest solution is greater than the unitary bound. However, precisely for $-4 \leq m^2 R^2 \leq -3$, equation (2.34) has two different solutions satisfying the unitary bound, one with $1 \leq \Delta \leq 2$ and one with $2 \leq \Delta \leq 3$. One then has two different choices for imposing boundary conditions: they amount to choice ϕ_0 or ϕ_1 as boundary value of the bulk field. The two different choices lead to correlation functions for two different operators, one with $1 \leq \Delta \leq 2$ and one with $2 \leq \Delta \leq 3$ [20].

- The relation between mass and conformal dimension for fields of arbitrary spin is

$$\begin{array}{rll} \text{scalar}\,\phi & (j_1,j_2)=(0,0) & m^2=R^2\Delta(\Delta-4) \\ \text{vector}\,A_\mu & (j_1,j_2)=(\frac{1}{2},\frac{1}{2}) & m^2=R^2(\Delta-1)(\Delta-3) \\ \text{symm}\,g_{\mu\nu} & (j_1,j_2)=(1,1) & m^2=R^2\Delta(\Delta-4) \\ \text{antisymm}\,B_{\mu\nu} & (j_1,j_2)=(1,0)+(0,1) & m^2=R^2(\Delta-2)^2 \\ \text{spin}\,\dfrac{1}{2}\,\psi & (j_1,j_2)=(\frac{1}{2},0)+(0,\frac{1}{2}) & m=R(\Delta-2) \\ \text{spin}\dfrac{3}{2}\,\psi_\mu & (j_1,j_2)=(\frac{1}{2},1)+(1,\frac{1}{2}) & m=R(\Delta-2) \end{array}$$

  The mass is a function of the three quantum numbers (Δ, j_1, j_2), or more geometrically, of the Casimirs of the conformal group $O(2,4)$. For a scalar field, the mass just corresponds to the quadratic Casimir. In general, mass is an ambiguous concept in AdS because of the coupling to the curvature (for example, for a scalar field we have the term ϕR^2). We chose a definition that is consistent with the dual interpretation in terms of conformal fields. For example a massless graviton and a massless gauge field correspond to operators with dimension four and three, respectively. This is consistent with the fact that conformal invariance requires conserved currents to have canonical dimension.

2.3 Construction of correlation functions

We now sketch the general construction of n-point correlation functions for a CFT with a gravitational dual given by an effective action in AdS_5. For all reasonable S_{AdS}, we can inductively construct a set of Green functions that satisfy all requirements of a local quantum field theory. We shall perform the computation in a theory with AdS_5 vacuum, but it will be clear from the construction how to extend the prescription to other spaces that have the same topological structure as AdS_5. Our aim is to provide a general picture and not a working technology. For this the reader is referred to the very good existing reviews [7, 8].

We shall focus for simplicity on a set of scalar fields ϕ_i with mass m_i interacting through a local Lagrangian $L_{AdS}(\phi)$. We denote with ϕ_i^0 the boundary value of the fields obtained by imposing (2.19) where Δ_i satisfies (2.15); ϕ_i^0 is identified on the quantum field theory side with the source for the dual operators O_i of dimension Δ_i. The CFT generating function is given by evaluating the bulk action on the solution of the equations of motion with the prescribed boundary conditions. A n-point function is obtained by differentiating the on-shell bulk action with respect to the sources ϕ_i^0 and setting $\phi_i^0 = 0$ afterwards

$$\langle O_1 \cdots O_n \rangle = \frac{\delta^n S}{\delta\phi_1^0 \cdots \delta\phi_n^0}\Big|_{\phi_i^0=0} . \tag{2.35}$$

Since on-shell fields vanish when the sources ϕ_i^0 are turned off, it is obvious that an interaction in the bulk action with more than n fields will not contribute to this derivative. Therefore, in order to compute n-point functions we can keep in the action only the terms with at most n-fields.

We shall now consider in turn 1-, 2- and 3-point functions. Up to order $n = 2$ we just need to keep the quadratic terms in the Lagrangian. The equations of motion are then a set of Klein-Gordon equations in the bulk, assuming that the kinetic terms are canonically normalized. To extend a field ϕ with mass m and conformal dimension Δ from the boundary to the bulk we need a Green function or boundary-to-bulk propagator as shown in Figure 8:

$$\phi(z, x_\mu) = \int dx'_\mu K(z, x_\mu - x'_\mu)\phi^0(x'_\mu) \tag{2.36}$$

which satisfies

$$\begin{aligned} &\left(-\Box + m^2\right) K = 0 \\ &K \to z^{4-\Delta}\,\delta(x_\mu - x'_\mu) \qquad z \to 0 \end{aligned} \tag{2.37}$$

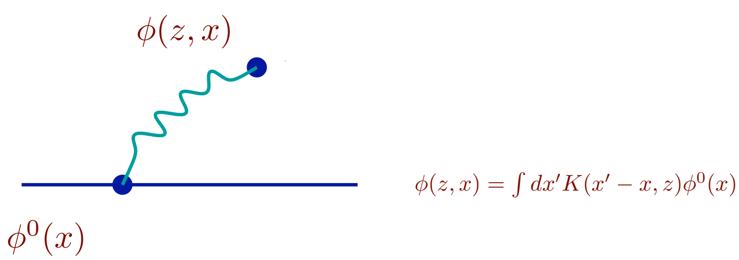

Figure 8: The boundary-to-bulk propagator.

It is easy to find a solution working in Euclidean space and treating AdS_5 as a ball in $\mathbb{R}^5$. More precisely, we compactify the boundary to S^4 by adding the point $z = \infty$. We saw in the previous section that the Klein-Gordon equation has a particular x_μ- independent solution z^Δ (see (2.13) and discussion thereafter). This solution is zero on most of the boundary ($z = 0$) and infinity at a particular point

($z = \infty$); it looks like a delta function corresponding to the insertion of a source at $z = \infty$. Since the equations of motion are covariant under the Euclidean conformal group $O(1,5)$ we can find the generic Green function (2.37) by mapping $z = \infty$ to a generic point on the boundary by a conformal transformation. In particular we can send it to ($z = 0, x_\mu = 0$) by

$$\begin{aligned} z &\rightarrow \frac{z}{z^2 + x_\mu^2} \\ x_\mu &\rightarrow \frac{x_\mu}{z^2 + x_\mu^2} \end{aligned}$$

obtaining the required Green function

$$K(z, x_\mu) = c\frac{z^\Delta}{(z^2 + x_\mu^2)^\Delta} \tag{2.38}$$

where c is some normalization constant. The value of c is not particularly important for us but it can be found by computing

$$\begin{aligned} &\int dx K(z,x)\phi^0(x) = cz^{4-\Delta}\int dx \frac{z^{2\Delta-4}}{(z^2+x^2)^\Delta}\phi^0(x) \\ &= cz^{4-\Delta}\int dy \frac{\phi^0(zy)}{(1+y^2)^\Delta} \rightarrow c\int \frac{dy}{(1+y^2)^\Delta} z^{4-\Delta}\,\phi^0(0)\,, \qquad z \rightarrow 0 \end{aligned}$$

so that $c^{-1} = \int \frac{dy}{(1+y^2)^\Delta}$.

The solution of the equations of motion is then given by equation (2.36). We can expand the solution in powers of z,

$$\phi(z, x_\mu) \sim \phi_0(x)(z^{4-\Delta} + O(z)) + \phi_1(x)(z^\Delta + O(z)) \tag{2.39}$$

where

$$\phi_1(x) = c\int dx' \frac{\phi_0(x')}{|x - x'|^{2\Delta}}\,. \tag{2.40}$$

As in the previous section, the value of the on-shell action reduces to a boundary term

$$S_{AdS} \sim \int_{\text{boundary}} \sqrt{g}\phi\partial^n\phi + \int \sqrt{g}\phi(-\Box + m^2)\phi \tag{2.41}$$

since the second term is zero on the equations of motion and we obtain

$$S_{AdS} \sim \frac{1}{z^3}\phi\partial_z\phi\Big|_{\text{boundary}} \tag{2.42}$$

By inserting (2.39) we see that we have some divergent terms proportional to $\phi^0(x)^2$ or other local functions of ϕ_0 [4]. These are contact terms which can be reabsorbed by local counter-terms and we disregard them. The finite contribution is non local and given by

$$S_{AdS} \sim \int dx\phi^0(x)\phi^1(x) \sim \int dxdx' \frac{\phi^0(x)\phi^0(x')}{|x-x'|^{2\Delta}} \tag{2.43}$$

The first thing we note is that in general the 1-point function is given by

$$\langle O(x)\rangle = \frac{\delta S_{AdS}}{\delta\phi^0(x)}\Big|_{\phi^0=0} = \phi^1(x)|_{\phi^0=0} \tag{2.44}$$

the normalizable term in the solution of the equations of motion. Since ϕ^1 is proportional to ϕ^0, the 1-point function vanishes identically. This is in agreement with conformal invariance since an operator with non zero dimension cannot have non zero vacuum expectation value without breaking the symmetry under dilations. However the identification of the normalizable term in the solution with a VEV [20, 21] is important in application of the AdS/CFT correspondence to non conformal theories.

The 2-point function is instead given by

$$\langle O(x)O(x')\rangle = \frac{\delta^2 S_{AdS}}{\delta\phi^0(x)\delta\phi^0(x')}\Big|_{\phi^0=0} = \frac{1}{(x-x')^{2\Delta}} \tag{2.45}$$

consistently with the computation in the previous section.

For computing n-point function we need to keep up to the n-adic terms in the action,

$$S_{AdS} = \int dx^5 \left(\frac{1}{2}\sum_i (\partial\phi_i)^2 + \frac{m_i^2}{2}\phi_i^2 + \sum_{k=3}^{n} \lambda_{i_1 \ldots i_k}\phi_{i_1}\ldots\phi_{i_k}\right) \tag{2.46}$$

We are considering for simplicity fields with canonic kinetic terms, no higher derivatives interactions and we are neglecting couplings to other gauge and gravity fields. All these other ingredients can be incorporated in the AdS/CFT correspondence without any conceptual effort (but with some technical effort). The equations of motion now have higher order terms and cannot be solved exactly. We can however set a perturbation expansion. The typical equation $(-\Box + m^2)\phi = \lambda\phi^n$ can be solved in power series of λ if we know the Green functions for the Klein-Gordon equation in the bulk. At order λ^0 we know that the solution is

$$\phi(z,x)^{\mathrm{zero}} = \int K(z, x-x')\phi^0(x')$$

[4] All the subleading terms in 2.39 with powers of z in between z^{Δ} and $z^{4-\Delta}$ contribute divergent terms. However one can check that all these terms are local functions of ϕ^0. ϕ^1 and its subleading terms are instead non local functions of the source $\phi_0(x)$.

where K is the boundary-to-bulk propagator (2.37). At $O(\lambda)$ we have

$$\phi^{\text{one}}(z,x) = \lambda \int G(z-z',x-x')(\phi^{\text{zero}}(z',x'))^n$$

where now G is a bulk-to-bulk Green function, that is the solution of

$$(-\Box + m^2)G(z-z',x-x') = \frac{1}{\sqrt{g}}\delta(z-z')\delta(x-x') \tag{2.47}$$

with (z,x) and (z',x') arbitrary points in the bulk. Explicit expressions for G can be found in [7]. We can then reinsert ϕ^{one} in the right-hand side of the equation of motion and determine ϕ^{two}. This sets a perturbative expansion for the solution. Since all ϕ are functions of the source ϕ^0 we can stop the expansion after a finite number of steps: all contributions containing more than n powers of ϕ^0 do not give contribution to equation (2.35). One can even set a graphical Feynman description of this perturbative series: the n points on the boundary are connected by boundary-to-bulk propagators to points in the bulk where we insert vertices of the Lagrangian; vertices in the bulk are connected to each other by bulk-to-bulk propagators and we integrate on their position as in Feynman rules.

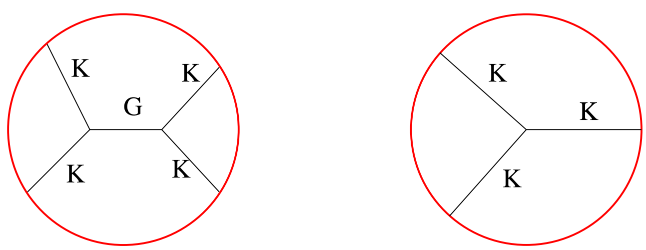

Figure 9: Contributions to 4-point and 3-point functions in a theory with cubic interaction. K are boundary-to-bulk and G bulk-to-bulk propagators. 3-point functions are exhausted by the contribution shown.

For the simple case of a 3-point function, the three external points can only connect a cubic vertex in the bulk. All other graphs will give higher powers of ϕ^0. There is no need for bulk-to-bulk propagators. If the cubic interaction is $\lambda_{ijk}\phi_i\phi_j\phi_k$ one obtains

$$\langle O_i(x_i)O_j(x_j)O_k(x_k)\rangle \sim \frac{\lambda_{ijk}}{|x_i-x_j|^{\Delta_i+\Delta_j-\Delta_k}|x_j-x_k|^{\Delta_j+\Delta_k-\Delta_i}|x_k-x_i|^{\Delta_k+\Delta_i-\Delta_j}}$$

in agreement with the requirements of conformal invariance that fix the coordinate dependence of the 3-point function.

Starting with 4-point functions, we would need bulk-to-bulk propagators. The functional dependence a 4-point function is not fixed by conformal invariance but must be consistent with channel decomposition and the OPE expansion in the CFT. Actual computations indeed confirm this: the correlation functions obtained by the AdS/CFT prescription satisfy all requirements of a consistent local quantum field theory.

Let us finish this section with some comments,

- As we saw, divergent contributions appear in the correlation function computation. Local terms in $W(\phi^0)$ can be always eliminated by local counter-terms. However, this requires at least regularizing the effective action with a cut-off as we did in section 2.2. This can be done in general and in accord with all Ward identities in the presence of gauge fields. And this should be done in a real calculation. In fact, the two point functions computed as we did, without introducing a cut-off, lead to a wrong normalization. For the purposes of these notes, this is an irrelevant detail. As already said, the reader interested in actual calculation technology should refer to the existing literature.

- The same computation can be done for vector, spinor and tensor fields. Each requires its own propagators. We again refer to the literature for explicit expressions. For example, the two-point function for a massless vector field would give
$$\langle J_\mu(x)J_\nu(x')\rangle = \frac{\delta_{\mu\nu}}{|x-x'|^6} - 2\frac{(x-x')_\mu(x-x')_\nu}{|x-x'|^8} \tag{2.48}$$
consistent with a field of dimension 3, the canonical dimension of a current (recall $m^2R^2 = (\Delta-1)(\Delta-3)$ for vector fields). The tensor structure shows that the current is conserved. We should stress again a basic fact about vector fields: gauge invariance in the bulk corresponds to a global symmetry in the boundary with conserved currents. Gauge invariance requires zero mass in the bulk and therefore $\Delta = 3$; conformal invariance now implies that a vector operator J_μ with $\Delta = 3$, which saturates the unitary bound, is conserved $\partial^\mu J_\mu = 0$.

2.4 Anomalies

As we have seen the conserved currents J^μ associated with global symmetries of the CFT and the stress-energy tensor $T^{\mu\nu}$ couple to gauge fields in the bulk and the

AdS/CFT correspondence allows to compute the QFT generating functional

$$e^{W(A_\mu, g_{\mu\nu})} = \langle e^{\int d^4x\sqrt{g}(g_{\mu\nu}T^{\mu\nu}+A_\mu J^\mu)}\rangle_{CFT}\,. \tag{2.49}$$

Here we assume that the currents J_μ and $T_{\mu\nu}$ are conserved in the CFT when all the sources are turned off, in particular there are no ABJ anomalies and no violation of the equations of conservation due to dynamical gauge fields. However, the symmetry is fully preserved at quantum level only if the functional W is invariant under gauge transformation of A_μ and diffeomorphisms. The t'Hooft anomalies for global symmetries and the Weyl anomaly for scale invariance are detected by the failure of W to be gauge invariant under A_μ and $g_{\mu\nu}$, respectively. This kind of anomalies for global symmetries are neither a sickness of the theory nor the indication that the symmetry is absent, it just indicates that conservation of the current is violated in the presence of background fields. t'Hooft anomalies are indeed very useful to classify different phases of quantum field theories.

Let us assume, for simplicity, that the CFT has a set of abelian symmetries with currents J^i_μ and corresponding background fields A^i_μ. The t'Hooft anomalies of the symmetries are captured in the bulk by a topological Chern-Simons term in the effective theory

$$S_{CS} = -f_{ijk}\int d^5x\epsilon^{\mu\nu\rho\sigma\tau}A^i_\mu F^j_{\nu\rho}F^k_{\sigma\tau}\,. \tag{2.50}$$

Indeed, under a gauge transformation $\delta A^i_\mu = \partial_\mu\alpha^i$, the action S_{CS} is not invariant and picks up a boundary contribution

$$\delta S_{CS} = -f_{ijk}\int d^5x\epsilon^{\mu\nu\rho\sigma\tau}\partial_\mu\left(\alpha^i F^j_{\nu\rho}F^k_{\sigma\tau}\right) = -f_{ijk}\int d^4x\epsilon^{\nu\rho\sigma\tau}\left(\alpha^i F^j_{\nu\rho}F^k_{\sigma\tau}\right)\Big|_{boundary}\,, \tag{2.51}$$

so that the variation of W under a gauge transformation is

$$\delta W = -f_{ijk}\int d^4x\epsilon^{\nu\rho\sigma\tau}\left(\alpha^i F^j_{\nu\rho}F^k_{\sigma\tau}\right)\,, \tag{2.52}$$

which is the standard form of the four-dimensional anomaly variation of W. More explicitly from (2.49), at linear order in α, in flat space

$$\delta W = \langle\int d^4x\partial^\mu\alpha^i J^i_\mu\rangle = -\langle\int d^4x\alpha^i\partial^\mu J^i_\mu\rangle \tag{2.53}$$

so, up to coefficients we have been quite sloppy about, we obtain the standard violation of the current conservation due to the chiral anomaly

$$\partial^\mu J^i_\mu = f_{ijk}\epsilon_{\nu\rho\sigma\tau}F^j_{\nu\rho}F^k_{\sigma\tau}\,. \tag{2.54}$$

The anomaly arises perturbatively from triangle fermionic loop diagrams with insertions at the vertices of the currents J^i_μ. The coefficients f_{ijk} are then proportional to $\mathrm{Tr} Q_i Q_j Q_k$ where the Q_i are the charge operators and the trace is taken over all the fermions of the theory.[5] If the symmetries J^j_μ and J^k_μ were gauged in the CFT, the i-th symmetry would be broken. The ABJ anomaly is an example of this. For us the symmetries are global, and A^j_μ are just background fields. The currents are still conserved when all background fields are turned off.

Another very interesting object to compute with the AdS/CFT correspondence is the Weyl anomaly. Conformal invariance breaks when the CFT is coupled to an external metric, or equivalently when the theory is defined on a curved background. In fact $< T^\mu_\mu > \neq 0$ when $g_{\mu\nu} \neq 0$. By general covariance, one can prove that

$$T^\mu_\mu = -a\, E_4 - c\, I_4 \tag{2.55}$$

where E_4, I_4 are the two invariants we can make with the Riemann tensor

$$\begin{aligned} E_4 &= \frac{1}{16\pi^2}\left(R^2_{\mu\nu\tau\rho} - 4R^2_{\mu\nu} + R^2\right) \\ I_4 &= -\frac{1}{16\pi^2}\left(R^2_{\mu\nu\tau\rho} - 2R^2_{\mu\nu} + \frac{1}{3}R^2\right) \end{aligned} \tag{2.56}$$

(and cannot be reabsorbed with local counter-terms). c and a are called central charges. They generalize the familiar central charge c of two dimensional conformal field theories. It is known that c cannot be zero in a unitary two-dimensional CFT, and the same is true of a and c in unitary four dimensional CFTs. For example, in free theories, we have a formula in terms of the number N_i of fields of spin i:

$$c = \frac{12N_1 + 2N_{1/2} + N_0}{120} \qquad a = \frac{124N_1 + 11N_{1/2} + 1N_0}{720} \tag{2.57}$$

The non-vanishing of the trace of the stress-energy tensor is equivalent to the fact that the functional

$$e^{W(g)} = \left\langle e^{\int dx \sqrt{|g|} g_{\mu\nu} T^{\mu\nu}} \right\rangle \tag{2.58}$$

is not invariant under Weyl rescaling $\delta_\lambda g_{\mu\nu} = \lambda g_{\mu\nu}$. In fact $< T^\mu_\mu >= \delta_\lambda W$. This looks like something that is amenable to an holographic computation: the AdS/CFT correspondence in fact just computes the functional $W(g)$ for external sources. In fact, starting with the Einstein action with cosmological constant

$$S = -\frac{1}{16\pi G_N}\int dx^5 \sqrt{g}\,(\mathcal{R} - \Lambda) \tag{2.59}$$

[5]We can always redefine the fields such all the fermions have the same chirality. This is commonly done in supersymmetry. The cubic trace formula is valid under this assumption.

we can compute $W(g)$ using holography. After adding boundary terms, regularizing and removing divergences, we can compute $< T^\mu_\mu >$ and we reproduce the functional form (2.55) predicted by conformal invariance [22]. This is technical calculation that it is too long to report here but it is very instructive and it is strongly suggested as a complementary reading.

As a surprising result of this computation, we have the prediction that for all CFT described by AdS [22]:

$$c = a \tag{2.60}$$

and the common value is determined in terms of the cosmological constant

$$a = c = \frac{\pi R^3}{8G_N} \sim (\Lambda)^{-3/2} \tag{2.61}$$

Equation (2.60) is the first result that restrict the class of theories that have a weakly coupled holographic description. Only theories with $c = a$ can have a dual description based on an effective Lagrangian for Einstein gravity coupled to other fields.

An analogous computation in AdS_3 would give

$$c = \frac{3R}{2G_N} \tag{2.62}$$

where c is the central charge of the CFT_2, which can be defined through the Weyl anomaly by

$$T^\mu_\mu = -\frac{c}{24\pi} R\,. \tag{2.63}$$

2.5 Wilson loops

In gauge theories, another natural observable is the Wilson loop, defined, for every closed contour C and representation R of the gauge group, by the path ordered integral of the holonomy of the gauge field along the path

$$W_R(C) = Tr_R P e^{i\int_C A^a_\mu T^a dx^\mu} \tag{2.64}$$

where T^a are the generators in the representation R. It has the following intuitive interpretation. Given a pure gauge theory, we introduce external massive sources (quarks) transforming in a representation R of the gauge group. The loop C corresponds to the propagation of a quark-antiquark pair along C, from its creation to its

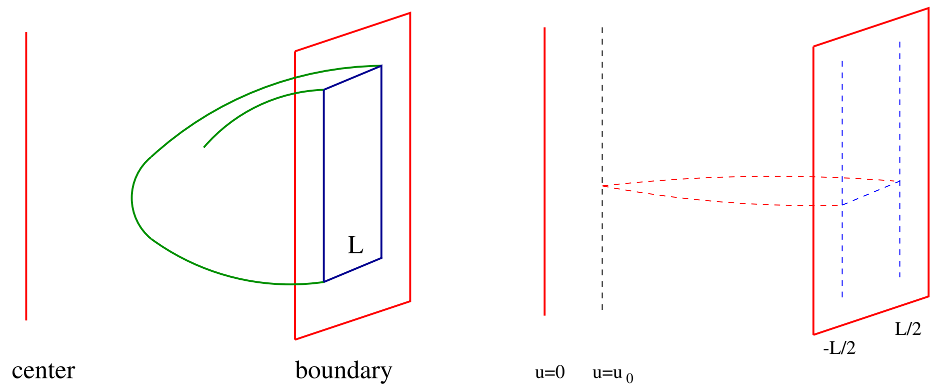

Figure 10: The string world-volume that minimizes the area in AdS. It enters deeply into the space.

disappearing and measures the free energy of this configuration. For a rectangular Wilson loop in Euclidean space with length L in space and height T in time,

$$W_R(C) = e^{-TE_I(L)} \tag{2.65}$$

where $E_I(L)$ is the energy of a pair of quarks at distance L.

The Wilson loop is a signal for confinement if it grows as (the exponential of) the area of the loop C. In a confining theory indeed external quarks have an energy which grows linearly with distance $E = m_q + m_{\bar{q}} + E_I$ with $E_I = \tau L$ since they are connected by a color flux tube, or string, with tension τ. It follows that, for a rectangular loop $W = e^{-TE_I} = e^{-\tau TL}$, and more generally

$$W(C) \sim e^{-\tau A(C)} \tag{2.66}$$

where $A(C)$ is the area of the loop, or equivalently the area of the world-sheet for the propagation of the string. In this picture, the quarks are considered as external non dynamical sources (for example quarks with a very large mass) and $W(C)$ just captures the dynamics of the gauge fields in the theory. Confinement in QCD is indeed a property of the glue vacuum.

We can define an analogous quantity in AdS. An external source is inserted at the boundary and we may attach to it a string. This is very natural in the explicit realizations of the AdS/CFT correspondence where the gravitational background is embedded in a string vacuum. We are lead to consider a string whose endpoint lies on a contour C on the boundary. The natural action for the string is the Nambu-Goto action $\int dx^2 \sqrt{g}$, which is proportional to its area. We can now define a very natural observable in AdS

$$-\log W(C) = (\text{minimal area surface with boundary C}) \tag{2.67}$$

This is identified with the value of some Wilson loop in the dual CFT. We shall see in section 3 and 4 that this identification is very natural in explicit realizations of AdS/CFT correspondence.

In a flat space-time, the surface of minimal area with rectangular C would lie entirely on the boundary, giving an obvious confining behaviour $S \sim LT$, good for a confining theory, not certainly for a CFT. However, things are different in AdS. The point is that the AdS metric diverges on the boundary $u = \infty$:

$$ds^2 = u^2(dx_\mu dx_\mu) + \frac{(du)^2}{u^2} \tag{2.68}$$

(as usual we put $R = 1$ when possible) and it is energetically favorable for the string to enter inside AdS. As in figure 10, the string will penetrate deeply in the interior of the space where the gravitational interaction is weaker.

Choose a parameterization of the world-sheet by coordinates σ and τ. The string world-sheet in AdS will be given by an embedding $X^M(\sigma,\tau)$ and the action is

$$S = \int_C d\sigma d\tau \sqrt{\det_{ab}\left(g_{MN}\partial_a X^M \partial_b X^N\right)}\,. \tag{2.69}$$

We can perform a simple calculation for a time invariant configuration of two external sources separated by a distance L. In this case, as in Figure 10, we can choose $t = \tau\,, x = \sigma\,, U = U(\sigma) \equiv U(x)$ and we obtain the action

$$S \sim \int_{-L/2}^{L/2}\int_0^T dt\sqrt{(\partial_x u)^2 + u^4} \sim T\int_{-L/2}^{L/2}\sqrt{(\partial_x u)^2 + u^4} \tag{2.70}$$

We are taking T very large in order to have a very large strip and not to bore about the bottom and top of the rectangle. To find the minimal area is just a classical exercise with the Euler-Lagrange equations. In particular, since the problem is time invariant, the quantity (the Hamiltonian)

$$H = \frac{\delta L}{\delta(\partial_x u)}\partial_x u - L = \frac{u^4}{\sqrt{(\partial_x u)^2 + u^4}} \tag{2.71}$$

is conserved. Its value can be computed at the turning point of the string, which by symmetry is at $x = 0$. Since at the turning point $u'(0) = 0$ we have $H = u(0)^2$. We obtain a differential equation for u

$$u' = u^2\sqrt{\frac{u^4}{u(0)^4} - 1} \tag{2.72}$$

which we can solve by

$$x = \int_0^x dx = \frac{1}{u(0)} \int_1^{u/u(0)} \frac{dy}{y^2\sqrt{y^4-1}} \tag{2.73}$$

First of all, note that, at the boundary $u = \infty$, $x = L/2$ and we obtain a relation between L and the turning point

$$L/2 = \frac{1}{u(0)} \int_1^{\infty} \frac{dy}{y^2\sqrt{y^4-1}} \sim \frac{1}{u(0)} \tag{2.74}$$

The action evaluated on the solution reads

$$S = T \int_{-L/2}^{L/2} \sqrt{(\partial_x u)^2 + u^4} = 2Tu(0) \int_1^{\infty} \frac{y^2 dy}{\sqrt{y^4-1}} \tag{2.75}$$

This integral is linearly divergent. The interpretation of this divergence is that we are really computing the energy of a pair of quarks including their large renormalized self-energy $m_q + m_{\bar{q}} + E_I$. The energy of a single quark can be estimated by a long linear string from $u = \infty$ to $u = 0$. We are only interested in the potential energy of the sources, therefore we subtract two linearly divergent contributions and we obtain a finite result

$$S = 2Tu(0) \int_1^{\infty} \left(\frac{y^2}{\sqrt{y^4-1}} - 1 \right) dy \sim Tu(0) \sim T\,\frac{1}{L} \tag{2.76}$$

With more effort, Wilson loops can be computed for more general contours. Let us make some observations.

- We see that the result for a Wilson loop is consistent with conformal invariance: by dimensional reasons, in absence of dimensionful parameters, the potential energy should go like $1/L$. We shall see later what happens in backgrounds which holographically realize confinement. If we restore factors of R and the tension τ of the string, the final result is $E_I \sim (\tau R^2)/L$. We shall discuss again this behaviour in section 3.3.4.

- We see from equation (2.74) that for large separation between the sources the turning point goes to the center of AdS. As we already anticipated, this has a very natural holographic interpretation: probing large distances in quantum field theory means probing the horizon. More generally, from $L \sim 1/u(0)$, we see that field theory UV computations ($L \ll 1$) take contributions from region with large u, while IR computations ($L \gg 1$) from region with small u, according to the interpretation of u as an energy scale.

- The regularization of the action is similar in spirit to the elimination of local terms in the holographic computation of correlation functions. We see that, even if the theory in AdS is classical, we need to consistently implement a regularization and a sort of renormalization of physical quantities. In the case of a Wilson loops $W(C)$ we need to subtract from the area of the world-sheet bounded by C an infinite multiple of the circumference of C.

2.6 The entanglement entropy

Another interesting observable in QFT that raised a lot of interest in the last decades is the entanglement entropy. Suppose that space is divided into two complementary regions A and B and the Hilbert space is also factorized into $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.[6] Suppose that the system is in the pure state $|\Omega\rangle$. We define the reduce density matrix for the region A

$$\rho_A = \mathrm{Tr}_B |\Omega\rangle\langle\Omega| = \sum_{\text{basis}\,|m\rangle\, in\, \mathcal{H}_B} \langle m|\Omega\rangle\langle\Omega|m\rangle \,, \tag{2.77}$$

and the entanglement entropy

$$S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \,. \tag{2.78}$$

This quantity suffers generically from UV divergences from the boundary between A and B. The form of the entanglement entropy for the ground state in a d-dimensional CFT behaves as

$$S_A = a_1 \left(\frac{l_A}{\epsilon}\right)^{d-2} + a_2 \left(\frac{l_A}{\epsilon}\right)^{d-4} + \cdots + C \log \frac{l_A}{\epsilon} + \text{const} \,, \qquad d\,\text{even} \tag{2.79}$$

$$S_A = b_1 \left(\frac{l_A}{\epsilon}\right)^{d-2} + b_2 \left(\frac{l_A}{\epsilon}\right)^{d-4} + \cdots + F \,, \qquad d\,\text{odd} \tag{2.80}$$

where ϵ is a UV cut-off and l_A the size-scale of region A. The coefficients C and F are interesting. They are proportional to central charges in even dimensions and to sphere partition functions in odd dimensions. For example in 2d the entanglement entropy of a segment A is given by

$$S_A = \frac{c}{3} \log \frac{l_A}{\epsilon} \tag{2.81}$$

where c is the central charge of the CFT$_2$. In four dimensions C is proportional to the a central charge defined in the previous subsections and in three dimensions F

[6] This statement is usually subtle but we will be sloppy here.

is proportional to the S^3 free energy. All these quantities are known or conjectured to decreases along a renormalization group flow. Casini and Huerta have used entanglement entropy techniques to prove it in general, deriving and generalizing previous results and conjectures.

Riu and Takayanagi has provided an intriguing holographic formula for the entanglement entropy [23]

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N} \tag{2.82}$$

where γ_A is the minimal surface in AdS at fixed time whose boundary is the space region A and that is homotopic to A. The similarity with the Hawking-Bekenstein formula for the entropy of a black hole is manifest. The formula has been generalized by Rangamani-Hubeny-Takanayagi. As a quick exercise we can apply the holographic formula to AdS$_3$

$$ds^2 = R^2 \frac{-dt^2 + dx^2 + dz^2}{z^2}\,, \tag{2.83}$$

and the space region on the boundary given by the interval $A = \{x \in (-l_A/2, l_A/2)\}$. The minimal surface at fixed time with boundary A is the geodesic between the points $(-l_A/2, 0)$ and $(l_A/2, 0)$ in the plane (x, z). We need to regularize it by introducing a cut off ϵ. We then take the points $(-l_A/2, \epsilon)$ and $(l_A/2, \epsilon)$. The "area" of γ_A is

$$\int ds = R \int \frac{dz}{z} \sqrt{1 + (x'(z))^2} = \int dz \mathcal{L}[x'(z), x(z), z]\,, \tag{2.84}$$

and a simple extremization using the Euler-Lagrange equation for $x(z)$

$$\partial_z \frac{\delta \mathcal{L}}{\delta x'(z)} = 0 \tag{2.85}$$

gives the semi-circle extremal solution

$$x = \frac{l_A}{2} \cos\eta\,, \qquad z = \frac{l_A}{2} \sin\eta\,, \qquad \eta \in \left(\frac{2\epsilon}{l_A}, \pi - \frac{2\epsilon}{l_A}\right)\,. \tag{2.86}$$

The area is then $\sim 2R \log l_a/\epsilon$ and the entanglement entropy is

$$S_A = \frac{R}{2G_N} \log \frac{l_A}{\epsilon} = \frac{c}{3} \log \frac{l_A}{\epsilon} \tag{2.87}$$

predicting $c = \frac{3R}{2G_N}$ in agreement with the holographic computation (2.62) based on the Weyl anomaly.

For more details on entanglement entropy and its holographic versions see for example the nice lectures by T. Hatman: http://www.hartmanhep.net/topics2015/.

2.7 Which quantum field theory?

Given an effective Lagrangian for gravity coupled to other fields and an AdS_5 vacuum we have constructed a set of correlation functions satisfying the axioms of a local conformal field theory. What kind of theory is it? Moreover, is it true that all four dimensional CFTs have a holographic dual? We already saw that there are restrictions. If we assume an effective weakly coupled Lagrangian that reduces to Einstein gravity the central charges of the CFT are equal: $c = a$. This is not a general feature of CFTs, as the free field theory case clearly shows. Moreover, we considered only theories with maximal spin equal to two. Therefore all CFT operators should have maximal spin two. Again, it is very easy to find theories where conformal operators have arbitrarily high dimensions, as the free field theory case shows. We see that, in this way, we can describe completely only theories with very few operators with low spin. We shall see in the next section that in the specific realizations of the AdS/CFT correspondence these restrictions are naturally implemented and without any contradiction. But this restricts the class of CFTs with a weakly coupled gravitational description.

We could include higher derivative interactions and higher spin fields in our effective Lagrangian and hope to obtain a complete description of any CFT. However, we typically face problems with ghosts and consistency of higher spin theories. For example, the free field theory case, where we have an infinite number of conserved currents of arbitrary spin, shows clearly that we should deal with a theory with infinite massless higher spin fields in the bulk. All these theories typically make sense only if embedded in a consistent string background. We can say that every CFT has a holographic dual, but this is a string theory dual with all complications about string theory. In particular there could be no regime where the gravitational dual is weakly coupled; in this case, we will not be able to compute CFT correlators using a classical theory.

2.8 The non-conformal realm

Holography, that works so well for conformal gauge theories, certainly has to play a role in the description of non conformal, realistic theories. In the description of non conformal theories we typically give up the AdS form of the metric for some more general metric which is only asymptotically AdS near the boundary

$$ds^2 = \frac{dz^2 + dx_\mu dx^\mu}{z^2} + O(z)\,. \tag{2.88}$$

Most of what we discussed before about calculation of correlation functions and other observables depended only on the existence of a boundary where the metric has a double pole and can be also applied to asymptotically AdS spaces. The natural interpretation is that we are discussing a theory that becomes conformal in the UV. The metric can still have four-dimensional Poincaré invariance, as in the description of local four-dimensional theories with a conformal UV completion, or it can break it as in the case of theories at finite temperature, which are the two cases we will focus on. In explicit realizations, one considers higher dimensional metrics, but, for simplicity, in this section we restrict to the five-dimensional case; all the results can be extended to the higher dimensional case. There are also examples of holographic pairs where even the asymptotic AdS condition is modified, but we will not treat them in these notes.

2.8.1 Finite temperature and black holes

The easiest way to break conformal invariance is to introduce temperature and consider a thermal state. Temperature introduces a scale of energy and breaks conformal invariance.

The basic statement in quantum field theory is that a thermal state can be described by Wick rotating the theory and compactifying the Euclidean time. More precisely, the thermal partition function of a theory defined on a spatial manifold M with Hilbert space $\mathcal{H}$ and Hamiltonian H can be computed as an Euclidean path integral on $M \times S^1$

$$\mathrm{Tr}_{\mathcal{H}} e^{-\beta H} = \int_{M\times S^1} d[\Phi] e^{-S_E(\Phi)} \,, \tag{2.89}$$

where Φ is the set of fields of the theory and the radius of the circle is related to the temperature by $2\pi R_{S^1} = \beta = \frac{1}{T}$.

This can be easily seen by looking at a quantum mechanical example. The partition function is given by

$$\mathcal{Z} = \sum_m e^{-E_n/T} = \mathrm{Tr}\left(e^{-H/T}\right) \,, \tag{2.90}$$

where $|n\rangle$ are the energy eigenstates of the system with energies E_n. By Wick-rotating the standard path integral formula expressing the propagator as a sum over paths connecting two points

$$\langle x|e^{-iHt}|x'\rangle = \int_{x(0)=x}^{x(t)=x'} [dx(t)] e^{iS[x(t)]} \tag{2.91}$$

we obtain

$$\mathrm{Tr}\left(e^{-\beta H}\right) = \int dx \langle x|e^{-\beta H}|x\rangle = \int dx \int_{x(0)=x}^{x(-i\beta)=x} [dx(t)] e^{iS[x(t)]} \tag{2.92}$$

$$= \int dx \int_{x(0)=x}^{x(\beta)=x} [dx(t_E)] e^{-S_E[x(t_E)]} = \int_{x(t_E)=x(t_E+\beta)} [dx(t_E)] e^{-S_E[x(t_E)]}\,, \tag{2.93}$$

where we defined $t = -it_E$ and the final path integral in taken over *all* paths that are periodic in time. This formula is easily generalized to a quantum field theory, expressing the thermal partition function as a Euclidean path integral with fields that are periodic of period $\beta = 1/T$ in the Euclidean time. More precisely, a more complete comparison reveals that we need to compactify the Euclidean time and take *periodic* boundary conditions for bosons and *antiperiodic* boundary conditions for fermions.

Let us focus as usual on four dimensions for simplicity. The thermal partition function in flat space is the Euclidean path integral on $\mathbb{R}^3 \times S^1$. What is the holographic description of a CFT at finite temperature? We should be able to create this state for any CFT. The corresponding physics should be universal and therefore it should be captured by the basic low-energy Lagrangian

$$S = \int dx^5 \sqrt{g}\,(\mathcal{R} - \Lambda)\ . \tag{2.94}$$

We need a solution that, in Euclidean signature, has a boundary $\mathbb{R}^3 \times S^1$. Luckly enough, black holes (or better black branes) precisely do that.

The euclidean black three-brane

$$ds^2 = \frac{R^2}{z^2}\left(\sum_{i=1}^{3} dx_i^2 + \left(1 - \frac{z^4}{z_0^4}\right) d\tau^2 + \frac{dz^2}{1 - \frac{z^4}{z_0^4}}\right) \tag{2.95}$$

is a solution of the Einstein theory (2.94) which asymptotes AdS_5 for small values of z. Moreover, the solution is smooth if and only if the euclidean time τ is periodic and therefore the boundary is $\mathbb{R}^3 \times S^1$. Indeed, the geometry has an apparent singularity at $z = z_0$. If we expand the metric around $z \sim z_0$, we obtain

$$ds^2 \sim R^2\left(4\frac{z_0 - z}{z_0^3} d\tau^2 + \frac{d(z_0 - z)^2}{4z_0(z_0 - z)} + \ldots\right) = R^2\left(\frac{4}{z_0^2}\rho^2 d\tau^2 + d\rho^2 + \ldots.\right) \tag{2.96}$$

with $\rho = \sqrt{(z_0 - z)/z_0}$. This becomes the two-dimensional flat metric in polar coordinates only if τ is an angular variable with radius $R_0 = \frac{z_0}{2}$. For all other choices of periodicity, the metric would have a conical singularity at $z = z_0$. Moreover the metric admits only one spin structure where fermions change sign along S^1 [7]. For all these

[7] One way of see it is to consider that τ is not a good coordinate everywhere; near z_0 it corresponds

reasons, the black three-brane solution is the natural candidate for the holographic description of the thermal CFT at temperature

$$T = \frac{1}{2\pi R_0} = \frac{1}{\pi z_0}\,. \tag{2.97}$$

We can strengthen this interpretation by computing the partition function using the holographic dictionary in euclidean signature

$$\mathcal{Z} = e^{-F/T} = e^{-S_E}\Big|_{on-shell}\,, \tag{2.98}$$

using the action

$$-\frac{1}{16\pi G_N}\left(\int dx^5\sqrt{g}\,(\mathcal{R}-\Lambda) + 2\int d^4x\sqrt{\gamma}K + c\int d^4x\sqrt{\gamma}\right)\,. \tag{2.99}$$

In this expression the second term is the Gibbons-Hawking term, it contains the trace of the extrinsic curvature at the boundary and it is necessary to have a well-defined variational problem for deriving the equations of motion in the presence of the boundary. The third term is a local counterterm needed to cancel the divergences of the action. For the details and the explicit computation one can see for example [24]. The action evaluated on the black three-branes gives the free energy

$$F = -T\log\mathcal{Z} = -\frac{\pi^3R^3}{16G_N}VT^4\,, \tag{2.100}$$

where V is the regularized volume of $\mathbb{R}^3$. Notice that the result is the one expected for a CFT. The free energy, being an extensive quantity, is correctly proportional to V and, in the absence of any other dimensionful parameter, it must scale as T^4 for dimensional reasons. The corresponding entropy is

$$S = -\frac{\partial F}{\partial T} = \frac{\pi^3R^3}{4G_N}VT^3\,. \tag{2.101}$$

Of course, that black holes and black objects have an associated thermodynamics is an old prediction, going back to the work of Hawking and Bekenstein in the late sixties. The AdS/CFT correspondence is just consistent with the old results for the temperature and the entropy. According to the laws of black hole thermodynamics

to the polar angle in the plane $(z_0 - z, \tau)$ whose value is not defined at the origin. In local cartesian coordinates, the wave functions for a fermion is single valued; however, it is easy to see that if we coordinate transform it to polar coordinates the same wave-function will change sign in a loop around the origin.

- The temperature of a black hole is related to the surface gravity κ by

$$T = \frac{\kappa}{2\pi} \,. \tag{2.102}$$

The surface gravity is defined through the covariant derivative of the null Killing vector at the horizon, but the best way to compute it in black hole physics is to go to Euclidean signature and expand near the horizon

$$ds^2 = \kappa^2 \rho^2 d\tau^2 + d\rho^2 + \cdots \,. \tag{2.103}$$

The inverse temperature $\beta = 1/T$ is then just the periodicity of the time direction τ that allows for a smooth metric. This is precisely what we did in holography.

- The entropy of a black hole is given by Bekenstein-Hawking formula

$$S = \frac{A}{4G_N} \,, \tag{2.104}$$

in terms of the area of the horizon. In our case, the horizon is $\mathbb{R}^3$. Regularizing the volume, we find

$$S = \frac{1}{4G_N} \int_{z=z_0, fixed\, \tau} \sqrt{g} d^3x = \frac{1}{4G_N} \frac{R^3}{z_0^3} V = \frac{\pi^3 R^3}{4G_N} V T^3 \,, \tag{2.105}$$

which is exactly the result found with holography.

The AdS/CFT correspondence is thus confirming old results in quantum gravity, and offering a novel view-point on the statistical degrees of freedom of black objects in terms of a dual CFT. One should also mention that the thermodynamics of asymptotically flat black holes is quite unstable, while the thermodynamics of asymptotically AdS black holes is perfectly nice and fine, being associated to the quite regular statistical mechanics of a dual quantum field theory.

2.8.2 Confining theories

Pure glue theories (with no supersymmetry or $\mathcal{N} = 1$) confine, have a mass gap and a discrete spectrum of massive glueballs. We will now describe how these features can be realized through a gravitational dual.

In the description of non conformal theories in flat space we give up the AdS form of the metric but we keep four-dimensional Poincaré invariance [8],

$$ds^2 = e^{2A(z)} \left(dz^2 + dx_\mu dx^\mu\right) . \tag{2.106}$$

We shall study only the case where the metric (2.106) is asymptotic to AdS_5 for small z. We also assume that it is everywhere regular. Since the space has a boundary isomorphic to $\mathbb{R}^{1,3}$, we can apply most of the technology of the AdS/CFT correspondence.

With respect to the conformal case, we also give up the association of bulk fluctuations with conformally invariant operators. Fluctuations of the background are now associated with bound states of the dual gauge theory. In QCD-like theories these would be glueballs and mesons. Baryons, as usual in large N Lagrangian, will appear as solitonic objects, typically wrapped branes. Other non-perturbative objects that characterize the dynamics of strongly coupled gauge theories, like monopoles, flux tubes or domain walls, typically appear as bound states made of extended objects, strings or branes.

The computation of correlation functions and Wilson loop proceeds exactly as in the conformal case. The results however are different.

- **Confinement:** In general, a criterion for confinement is the following: the warp factor $e^{2A(z)}$ multiplying the four-dimensional part of the metric must be bounded above zero. We can see this using a Wilson loop. As in the conformal case, we introduce heavy external sources at the boundary of AdS and study their energy by analyzing a string connecting them. Since the metric is still blowing up at the boundary and decreasing in the interior, the string will find it energetically favorable to reach the IR part of the background. In particular, the string will minimize its energy by reaching the point z_0 where the warp factor has a minimum. For large L, the minimal energy configuration consists of three straight segments: two long strings at fixed x_μ connecting the boundary to the point z_0, and a string at fixed z_0 stretching for a distance L along the four-dimensional spacetime directions as in Figure 11. The infinite energy of the long string from $z = 0$ to $z = z_0$ is interpreted as the bare mass of the external source. All the relevant contribution to the potential energy between

[8] In explicit realizations, one typically considers higher dimensional metrics, but, for simplicity, in this section we restrict to the five-dimensional case; all the results can be extended to the higher dimensional case.

two external sources is then due to a string localized at $z = z_0$ and stretched in the x direction. The total energy

$$E(L) = m_q + m_{\bar{q}} + e^{2A(z_0)}\tau L \tag{2.107}$$

then gives the linear increasing potential characteristic of confinement. We denoted with τ the tension of the string; in any explicit realization this will be fixed by the string scale M_s. The theory has stable finite tension strings, which

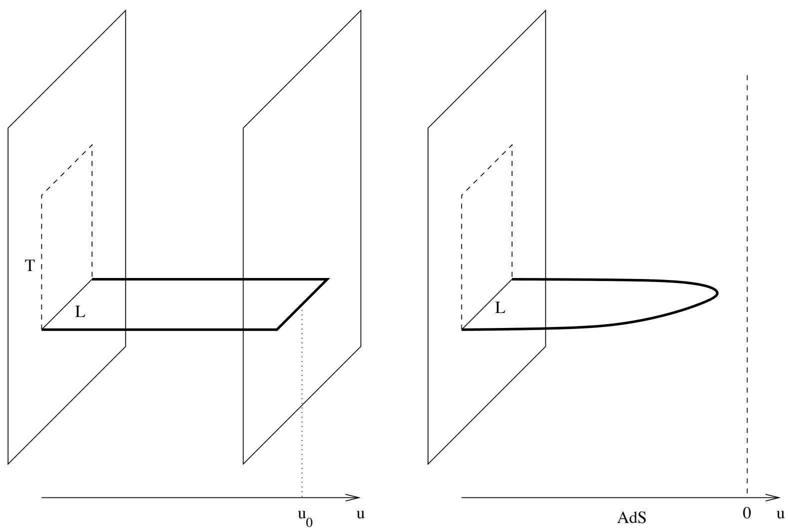

Figure 11: The quark-antiquark potential in the confining and in the conformal case.

we identify with the color flux tube of the dual gauge theory. They will live in the region of the solution where the warp factor has its minimum value e^{2A_0} and their tension will be given by $e^{2A_0}\tau$. This situation should be contrasted with the AdS case, where the metric vanishes in the IR at $z = \infty$: the vanishing of the metric is responsible for the $1/L$ behaviour of the Wilson loop.

- **The Glueball Spectrum**: In a theory with mass gap and discrete spectrum, we expect poles in the two point functions corresponding to the physical states (mesons, glueballs,...)

$$\langle \phi(k)\phi(0)\rangle = \sum_i \frac{A_i}{k^2 + M_i^2} \tag{2.108}$$

where M_i^2 are the masses of the glueballs. We use for convenience the Euclidean version of the theory. The poles appear for unphysical values of $k^2 = -M_i^2$.

The correlation functions can be computed as in the conformal case by evaluating the action on the equation of motion for a field ϕ dual to the operator O.

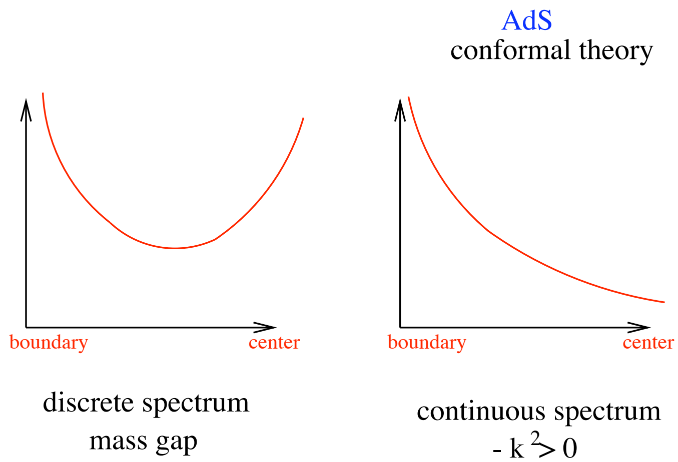

Figure 12: The potential in the confining and conformal case.

In the case of a minimally coupled scalar field, the equation of motion is

$$\partial_z \left(e^{3A(z)}\partial_z\phi\right) + \partial_\mu \left(e^{3A(z)}\partial^\mu\phi\right) = e^{5A(z)}m^2\phi \tag{2.109}$$

At small z the theory is asymptotically AdS and the solution still behaves as $\phi_k = z^{4-\Delta}(A(k) + O(z)) + z^\Delta(B(k) + O(z))$, where $R^2m^2 = \Delta(\Delta - 4)$. The contribution proportional to A is non-normalizable while the one proportional to B is normalizable. Regularity of the solution for large z determines $B(k)$ and the subleading terms as functions of $A(k)$ exactly as in the conformal case. By normalizing the field

$$\hat{\phi}_k(z) = \phi_k^0 \left[z^{4-\Delta}(1 + O(z)) + \frac{B(k)}{A(k)}z^\Delta(1 + O(z))\right] \tag{2.110}$$

with the value of the boundary source, we see that 2-point Green functions have poles if and only if $A(k) = 0$; or in other words if there exist normalizable regular solutions of the equations of motion. In the physical region $k^2 \geq 0$ there are no such solutions [9]. However nothing prevents the existence of solutions in the unphysical region $k^2 < 0$. The corresponding values of k^2 determines the masses of bound states through $M^2 = -k^2$.

We have thus reduced the problem of finding the spectrum to that of finding the regular normalizable solutions of the equations of motion. We can further reduce

[9] In fact the action reduces to a boundary term as in (2.43) that vanishes for $A(k) = 0$; on the other hand the Euclidean action is definite positive for $k^2 \geq 0$ and can vanish only for a solution which is identically zero. Recall that, in the case $-4 \leq R^2m^2 \leq 0$, the action is still positive definite due to the effect of curvature.

this problem to a Schroedinger-like one. It is easy to see that the equation of motion of a massive scalar field with mode expansion

$$\phi(x_\mu, z) = \phi(z)e^{ikx}, \qquad k^2 = -M^2 \tag{2.111}$$

can be written as

$$-\psi'' + (\frac{9}{4}(A')^2 + \frac{3}{2}A'' + m^2 e^{2A})\psi \equiv E\psi\,, \qquad E = -k^2 \tag{2.112}$$

where $\phi = e^{-3A/2}\psi$. We see that the spectrum is given by the positive eigenvalues $E = M^2 = -k^2$ of a Schroedinger equation. Boundary conditions for this equation should be worked out from the original problem but are straightforward. Normalizability of ϕ

$$\int \sqrt{g}|\phi|^2 < \infty \tag{2.113}$$

requires vanishing of ψ at the boundary $z = 0$. Conditions at $z = \infty$ are imposed by the regularity of the solution.

In the case of AdS_5, the potential in the Schroedinger equation is $\sim 1/z^2$, and we have a continuum spectrum starting from zero (see Figure 12), appropriate for a confining theory. On the other hand, the typical potential for a confining theory coincides with that of AdS_5 only for small z, and it goes to a constant value or blows up for large z; it has therefore a discrete spectrum M_i^2 bounded from below which gives the glueball masses. This happens in particular in the case where $A(z)$ is finite and bounded above zero.

We shall discuss explicit examples of regular backgrounds based on string theory in section 4.2.

2.8.3 The RG flow

In general, the background (2.106) should solve the equations of motion of a local five-dimensional Lagrangian, which will be used for all computations based on holography. In this section we shall discuss general properties of Poincaré-four invariant solutions of an arbitrary five-dimensional theory. Simple exercises in classical General Relativity lead to interesting conclusions.

Start with a local five dimensional gravitational theory

$$\mathcal{L} = \sqrt{-g}\left[-\frac{\mathcal{R}}{4} - \frac{1}{2}G^{ij}\partial\phi_i\partial\phi_j + V(\phi)\right] \tag{2.114}$$

where, for simplicity, we have only considered scalar fields. Consider only configurations that preserve four-dimensional Poincaré invariance: a metric of the form (2.106) and scalar fields depending only on the coordinate y. In order to simplify the equations of motion it is useful to write the metric in the coordinates

$$ds^2 = dy^2 + e^{2Y(y)} dx_\mu dx^\mu \,. \tag{2.115}$$

AdS_5 is recovered for $Y(y) \sim y/R$. An easy computation, that the reader is strongly advised to perform, shows that all the Einstein and scalar equations of motion following from eq. (2.114) can be deduced from the effective Lagrangian

$$\mathcal{L} = e^{4Y}\left[3\left(\frac{dY}{dy}\right)^2 - \frac{1}{2}G_{ij}\frac{d\varphi^i}{dy}\frac{d\varphi^j}{dy} - V(\varphi)\right], \tag{2.116}$$

supported by the zero energy constraint

$$3(Y')^2 - \frac{1}{2}G_{ij}(\varphi^i)'(\varphi^j)' + V(\varphi) = 0\,. \tag{2.117}$$

In particular, there are only two independent equations that comes from Einstein and scalar equations of motion

$$\begin{aligned} \left(G_{ij}\varphi^{j\prime}\right)' + 4G_{ij}Y'\varphi^{j\prime} &= \frac{\partial V}{\partial \varphi_i}, \\ 6(Y')^2 &= G_{ij}\varphi^{i\prime}\varphi^{j\prime} - 2V. \end{aligned} \tag{2.118}$$

An obvious solution of these equations with a boundary is obtained at the critical points of the potential $\frac{\partial V}{\partial \varphi_i} = 0$; if the critical value of $V$is negative, we obtain an AdS_5 solution of the equations of motion with constant scalar fields. In fact $Y(y) = y/R$ where $1/R^2 = -V_{\rm crit}/3$. The value of V at the critical point determines the cosmological constant and the radius of AdS. By the general discussion given in this section, we expect that the gravitational theory in this vacuum describes a dual CFT.

Other solutions with a boundary can be obtained as follows. Start with a five-dimensional theory with a critical point of the potential which, without loss of generality, we take at $\varphi_i = 0$. This AdS vacuum corresponds to some dual CFT. We can expand the action to quadratic order around the AdS vacuum and read the masses m_i^2 of the scalar fields; call Δ_i the dimension of the dual operator, obtained using $R^2m^2 = \Delta(\Delta - 4)$. We now look for more general solutions with asymptotics: $Y(y) \to y/R$ and $\varphi_i(y) \to 0$ for $y \to \infty$. Since asymptotically the background is AdS_5, the scalar fields behave as

$$\varphi_i(y) = A_i e^{-(4-\Delta_i)y} + B_i e^{-\Delta_i y}, \tag{2.119}$$

for large y. According to the general rules of the AdS/CFT correspondence, A_i is associated with a source for the dual operator O_i, while as seen in equation (2.44), B_i can be associated with the 1-point function for O_i. We are then naturally led to the following interpretation of the new solutions. We associate solutions behaving as $e^{-(4-\Delta)y}$ with deformations of the original CFT with the operator O_i

$$L_{CFT} \to L_{CFT} + \int dx^4 A_i O_i \tag{2.120}$$

On the other hand, solutions asymptotic to $e^{-\Delta y}$ (the subset with $A_i = 0$) are associated with a different vacuum of the theory, where the operator O_i has a non-zero VEV proportional to B_i [20, 21].

Solutions of this type can be interpreted as the gravitational dual of a Renormalization Group flow (RG) in quantum field theory. The deformation of the original CFT breaks conformal invariance and induces a running of deformation parameters and coupling constants with the scale. In the gravitational description this RG flow is described by the non-zero profile of the scalar fields. The energy scale can be roughly identified with the fifth dimensional coordinate, with the region of large y corresponding to the UV region. The functions $\varphi_i(y)$ then encode the running with the scale of the deformation parameters.

Interesting is the case where a CFT perturbed by a relevant operator O_i flows in the IR to another fixed point. The supergravity description corresponds to the case where the potential V has another critical point for non-zero value of the scalar field φ_i. The $5d$ description of the RG flow between the two CFTs is a kink solution, which interpolates between the two critical points [25]. Explicitly this is given by a solution with asymptotics: $Y(y) \to y/R_{UV,IR}$ for $y \to \pm\infty$; $\varphi_i(y) \to 0$ for $y \to \infty$, while $\varphi_i(y) \to \varphi_{i\,IR}$ for $y \to -\infty$. We associate larger energies with increasing y. Most of the solutions however leave a critical point and go to infinity in the space of φ_i. They correspond to theories which are non-conformal in the IR. Unfortunately most of these solutions have naked singularities.

Now some technical observations.

- Some of the squared masses around a critical point can be negative. As we have already said, due to the non-zero curvature, these modes are not *tachyonic*: recall that a mode is stable iff $m^2R^2 \geq -4$ (see section 2.2). We have
  - Negative mass modes $\to$ Relevant operators ($\Delta < 4$)
  - Massless modes $\to$ Marginal operators ($\Delta = 4$)

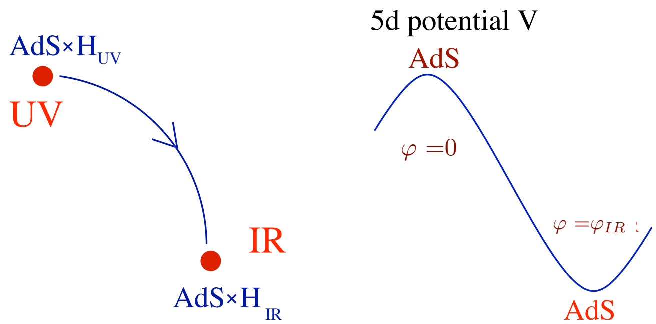

Figure 13: Schematic picture of the RG flow.

 – Positive mass modes → Irrelevant operators ($\Delta > 4$)

In particular, the critical point of V does not need to be a minimum. In the interesting cases of CFT with relevant operators, it is a maximum, with some unstable directions associated with fields with $0 \geq R^2m^2 \geq -4$. In particular the flow between two CFTs induced by a relevant operator interpolates between a maximum and a minimum as in Figure 13; the deformation must be relevant in the UV to leave the critical point and irrelevant in the IR to reach a new one.

- The problem of finding solutions simplifies for potentials of simple form, occurring typically if some supersymmetry is present [26]. If the potential V can be written in terms of a superpotential W as

$$V = \frac{1}{8}G^{ij}\frac{\partial W}{\partial \varphi^i}\frac{\partial W}{\partial \varphi^j} - \frac{1}{3}\left|W\right|^2. \tag{2.121}$$

we can find particular solutions by reducing the second order equations to first order ones. It is easy to check that a solution of

$$\begin{aligned} \frac{d\varphi^i}{dy} &= \frac{1}{2}G^{ij}\frac{\partial W}{\partial \varphi^j}, \\ \frac{dY}{dy} &= -\frac{1}{3}W. \end{aligned} \tag{2.122}$$

also solve the second order equations of motions. In supersymmetric theories the reduction from second to first order is obtained by looking at the vanishing of the supersymmetry variations instead than the equations of motion. It is possible to show that for various gauged supergravity supersymmetric solutions follow from a superpotential, even if this is not a general statement.

The existence of a holographic RG flow has striking consequences for conformal field theories at strong coupling. The most remarkable one is the possibility of proving a c-theorem for all theories with an AdS dual. As discussed in section 2.4, there are two central charges a, c in the superconformal algebra in four dimensions. It has been conjectured in [27] that, for general four dimensional conformal field theories, a is decreasing along the RG flow and thus it is the candidate central charge for a c-theorem in four dimensions. There are examples that, on the contrary, show that c, which enters in the OPE of two stress energy tensors, is not in general decreasing. For theories with an AdS dual $a = c$ and we can recover the value from the cosmological constant $a = c \sim R^3 \sim (\Lambda)^{-3/2}$. It is possible to extrapolate the value of the central charge all along an RG flow. We can define a c-function that is monotonically decreasing [25,26] and reduces to the previous result at the fixed points. The obvious choice

$$c(y) \sim (Y')^{-3} \tag{2.123}$$

makes the job. The monotonicity of c can be easily checked from the equations of motion (2.118) and the boundary conditions of the flow [25]. The equations of motion indeed give

$$Y'' = -\frac{4}{3} G_{ab} \varphi_a' \varphi_b' \tag{2.124}$$

In a consistent theory, the kinetic terms for scalar fields need to be positive definite to avoid ghosts. The previous equation then shows the decreasing of the c-function for all sensible theories. More generally, without resorting on a specific Lagrangian, we can reduce the c-theorem to a positivity condition for energy [26]. The equations of motion for five dimensional gravity coupled to matter with a stress energy tensor $T_{\alpha\beta}$ give

$$-2Y'' \sim (T_0^0 - T_r^r) \tag{2.125}$$

The c theorem is then equivalent to the weak positive energy condition,

$$\xi^\alpha \xi^\beta T_{\alpha\beta} \geq 0 \qquad\qquad \xi \,\text{null vector} \tag{2.126}$$

that is expected to hold in all physically relevant supergravity solutions. Let us stress that the value of c is well defined only at a fixed point, where it represents a central charge. In QFT, the value of c along the flow is scheme dependent. Similarly, in supergravity there are several possible definitions of monotonic functions interpolating between the central charges at the fixed points.

2.8.4 Some remarks

Let us finish the discussion of non-conformal models with some general comments.

- It is very easy to find solutions with Poincaré invariance and a boundary, as we discussed above. However, it is very difficult to find regular solutions without naked singularities. Regularity is a basic requirement in the AdS/CFT correspondence. We can have a five dimensional effective theory with singularities but these should be resolved when the model is embedded in a consistent string theory. Singular solutions are nevertheless often used, in particular in phenomenological applications like AdS/QCD, since in most cases correlation functions computed through holography give finite results despite the singularity.

- The most successful regular solutions describing confining theories in string theory arise directly from ten dimensional constructions. Most of what we say, from the criteria for confinement to the picture of an RG flow, straightforwardly extend to higher dimensional solutions.

- We only consider asymptotically AdS solutions. In this framework we can only describe field theories that become conformal in the UV. This condition can be relaxed in specific constructions. In particular, the best understood example of a dual for a $\mathcal{N}=1$ gauge theory, the Klebanov-Strassler solution [28], is based on a background which is asymptotic to the AdS_5 metric only up to logarithmic corrections depending on the radial coordinate. Other regular solutions, for example the Maldacena-Nunez one [29], are based on even more exotic form of holography for NS branes.

3 Explicit examples: the conformal case

In this section we analyze in details the founding example of the AdS/CFT correspondence, the duality between $\mathcal{N}=4$ SYM and the type IIB string background $AdS_5\times S^5$. We shall be able to give a precise map between observables in the two theories and apply and extend the general results of section 2.

The basic tool for providing dual pairs is the ability of realizing gauge theories on the world-volume of D-branes in string theory. D-branes are extended objects with non zero tension and they modify the surrounding space-time. The gravity dual is obtained by taking a near-brane (or near-horizon) limit of the D-brane geometry.

We shall work at the level of effective actions without requiring any specific knowledge of string theory. All the necessary ingredients will be reviewed in the following.

3.1 String theory, supergravity and D-branes

In this section, we give a quick presentation of the string theory ingredients of the game. The following is not intended as a tutorial in string theory, but just a basic description of the context and players, and an attempt to describe them in terms of effective field theories. We refer to [30–33] for more details on string theory, supergravity, p- and D-branes.

A consistent quantization of gravity is obtained by replacing elementary particles with extended objects. The fundamental ingredient of the theory is a string with tension $M_s^2 = \frac{1}{2\pi\alpha'}$. The vibrations of this extended object give rise, upon quantization, to some massless modes and a tower of particles of mass $m^2 \sim \frac{1}{\alpha'}$. The connection with gravity comes from the fact that the spectrum of a closed string contains a massless tensor of rank two, which can be decomposed in a symmetric tensor, an antisymmetric tensor and a scalar field,

$$g_{\mu\nu}\,, \qquad\qquad B_{\mu\nu}\,, \qquad\qquad \phi \tag{3.1}$$

In particular, the massless spin 2 can be identified with the graviton. The scalar ϕ has also an important role and it is called the dilaton.

It will be important in the following to consider also open strings. The quantization of an open string contains at the massless level a tensor of rank one, that is a vector field A_μ. This will be extremely important for the constructions involving D-branes.

There is a variety of consistent string and superstring theories. At low energy, the effective action of the massless modes of a closed string reduces to General Relativity coupled to matter fields or to supergravity when supersymmetry is present. We shall often ignore the fermionic part of the effective action, thus avoiding all technical details about supergravity theories.

One peculiar point of supergravities in higher dimensions is the presence of antisymmetric tensor fields with a gauge invariance, generalizations of the electromagnetic field. These are given by multi-indices potentials $C_{\mu_1\cdots\mu_k}$ invariant under

$$C_{\mu_1\cdots\mu_k} \to C_{\mu_1\cdots\mu_k} + \partial_{\{\mu_1}\Lambda_{\mu_2\cdots\ldots\mu_k\}} \tag{3.2}$$

which generalizes gauge transformations. The gauge invariant object is the curvature

$$F_{\mu_1\cdots\mu_{k+1}} = \partial_{\{\mu_1}C_{\mu_2\cdots\ldots\mu_{k+1}\}} \tag{3.3}$$

and the corresponding kinetic term is

$$\int dx^{10} F_{\mu_1\cdots\mu_{k+1}} F^{\mu_1\cdots\mu_{k+1}} \tag{3.4}$$

We shall see that these antisymmetric fields play an important role in the correspondence.

3.1.1 Type IIB supergravity

There are two consistent maximally supersymmetric string theories and both live in 10 dimensions. They are called type II string theories. Maximal supersymmetry means $\mathcal{N} = 2$ in ten dimensions, or better the existence of 32 real supercharges [10]. At low energy, we can consider a supersymmetric effective action for the massless fields of string theory. Supersymmetry restricts both the field content and the effective action up to two derivatives. In fact, there are only two supermultiplets with 32 supercharges and the corresponding supergravity is uniquely fixed by supersymmetry. They are obviously called type II supergravities, the non-chiral A and the chiral B [11]. We are mostly interested in type IIB supergravity.

[10] Counting in terms of real supercharges allows to compare theories in different dimensions and avoid complications related to the properties of spinors that wildly depend on the dimension of space-time. For example a $\mathcal{N} = 1$ theory in four dimensions, where the supersymmetric parameter is a complex Weyl fermion with four real components, has 4 supercharges. The maximally supersymmetric $\mathcal{N} = 4$ gauge theory in four dimensions has 16 supercharges.

[11] Focusing on bosons, type II supergravity has a common set of NS-NS fields $(g_{\mu\nu}, \phi, B_{\mu\nu})$, and a set of R-R fields, consisting of antisymmetric tensors C_k, for all odd k in type IIA and for all even k in type IIB. The names NS and R refers to Neveu-Schwartz and Ramond, respectively, and reflect the explicit world-sheet construction. In generic dimensions the electric-magnetic duality $F_{\mu\nu} \to \epsilon_{\mu\nu\tau\rho}F^{\tau\rho}$ is generalized by $F_{\mu_1\cdots\mu_{k+1}} = \epsilon_{\mu_1\cdots\mu_{10}}F^{\mu_{k+2}\cdots\mu_{10}}$ $(F_{k+1} = *F_{9-k})$ so if we have a potential C_k with k-indices we also have its magnetic dual C_{8-k}. Obviously, they are mutually non local and they cannot appear simultaneously in a local Lagrangian. When two potentials C_k are electric-magnetically dual only the one with the lower number of indices is included in the type II Lagrangian.

The supersymmetry type IIB multiplet contains the following bosonic fields: a metric $g_{\mu\nu}$, a scalar ϕ, called the dilaton, a 2-indices antisymmetric tensor $B_{\mu\nu}$, and three 0-,2-,4-, potentials (C_0, C_2, C_4). Supersymmetry requires that C_4 is self-dual ($F_5 = *F_5$). The fermionic content is given by a spin 3/2 and a chiral spin 1/2 fermion. The action is completely fixed by supersymmetry and reads [12]

$$\begin{aligned} S_{IIB} = \quad & \frac{1}{(2\pi)^7(\alpha')^4}\int dx^{10}\sqrt{g}e^{-2\phi}\left(R + 4(\partial\phi)^2 - \frac{1}{12}H^2_{\mu\nu\tau}\right) \\ & -\frac{\sqrt{g}}{2}\left((F_\mu)^2 + \frac{\tilde{F}^2_{\mu\nu\tau}}{3!} + \frac{\tilde{F}^2_{\mu\nu\tau\rho\sigma}}{5!}\right) \\ & -\frac{1}{2}\epsilon_{\mu_1\cdots\mu_{10}}C_{\mu_1\cdots\mu_4}H_{\mu_5\cdots\mu_7}F_{\mu_8\cdots\mu_{10}} + \text{fermions} \end{aligned} \tag{3.5}$$

where we denoted with $H_{\mu\nu\tau}$ the curvature for $B_{\mu\nu}$ and by $\tilde{F}_{\mu_1\cdots\mu_{p+1}}$ the modified curvature $F_{\mu_1\cdots\mu_{p+1}} - B_{\{\mu_1\mu_2}F_{\mu_3\cdots\mu_{p+1}\}}$ [13]. We haven't written the fermionic part of the Lagrangian since we shall not need it and it is completely fixed by supersymmetry. We shall not use much the bosonic Lagrangian itself. The full action is invariant under 32 supersymmetries given by two Majorana-Weyl parameters in 10 dimensions, $\epsilon = \epsilon_1 + i\epsilon_2$.

Equation (3.5) is the effective action of type IIB string theory at low energy, restricted to massless modes and two-derivatives. We can identify two parameters. One is $2\pi\alpha' = 1/M_s^2$ which determines the string mass M_s and controls the masses of the string modes, $m^2 \sim 1/\alpha'$. A second one is implicitly given by the vacuum expectation value of the dilaton $g_s =< e^\phi >$ which can be arbitrary since there is no potential for ϕ. g_s determines the string coupling constant and controls string interactions and quantum corrections. Both parameters can correct the effective action. If we integrate out the massive fields we get higher derivative corrections weighted by powers of α'. If we add the quantum corrections we get corrections (with

[12]Actually, type IIB supergravity has equations of motion and no Lagrangian, due to the presence of a self-dual potential. (3.5) is nevertheless a good approximation to a Lagrangian: its equation of motion (with unconstrained C_4) are the equations of motion of type IIB supergravity to which we need to add the self-duality constraint $F_5 = *F_5$.

[13]These couplings are required by supersymmetry and imply that the gauge invariance $\delta B_{\mu\nu} = \partial_{\{\mu}\Lambda_{\nu\}}$ should be combined with the transformation $\delta A_{\mu_1\cdots\mu_p} = \Lambda_{\{\mu_1}F_{\mu_2\cdots\mu_p\}}$. None of these details will be used in the following.

arbitrary derivatives) weighted by powers of g_s. Schematically [14]

$$\int dx^{10}\sqrt{g}e^{-2\phi}\mathcal{R}+\cdots \quad + \quad \sqrt{g}e^{-2\phi}\left(\alpha'\mathcal{R}^2+\cdots\right)$$
$$+ \quad \sum_{g=1}^{\infty} e^{(2g-2)\phi}\sqrt{g}\,(\cdots)$$

Any physical quantity has an expansion of form

$$\sum_{g=0}^{\infty} g_s^{2g-2} f_g(\frac{\alpha'}{R^2}) \tag{3.6}$$

Note that, in order to have a well defined dimensionless expansion parameter, we weighted α' with the typical radius R of the space-time in the vacuum where the computation is performed.

String theory gives an explicit way of computing all these corrections. The loop expansion for a string is given by fattened Feynman graphs, represented by Riemann surfaces where the genus is the number of loops (see figure 6). At each order in perturbation theory we have precisely one graph, which can be computed using the world-sheet formulation of string theory and the technology of two-dimensional conformal field theories. The result is a non-trivial function of α' encoding the effect of all massless and massive string modes.

3.1.2 D-branes

Type II supergravities (and string theories) have solitonic extended charged objects, called p-branes, which generalize strings and membranes. They are massive objects that extend in p-spacelike directions and interact with the gravitational and gauge fields through the coupling

$$S_{\text{p-brane}} = \tau\int dx^{p+1}\sqrt{g} + q\int dx^{p+1}A_{p+1} \tag{3.7}$$

where the integral is taken on the world-volume of the brane. The first term is the generalization of the Nambu-Goto action for strings and it defines the tension τ of the brane. A massive object in General Relativity moves by minimizing the volume swept. The second term is the coupling to the A_{p+1} antisymmetric tensor field; it

[14]The RR fields kinetic term enters the action without a factor of $e^{-2\phi}$, in order to simplify gauge couplings. It should be consider a part of the tree level action at all effects.

generalizes the coupling of a charged particle to a vector, which is given by the integral of the one-form potential on the world-line of the particle,

$$q \int dt \vec{A}\,\vec{v} = q \int dx^\mu A^\mu \equiv q \int A_1 \,. \tag{3.8}$$

The parameter q is the p-brane charge. As in electro-magnetism, it is useful to measure the charge by the flux on a sphere surrounding the source. For an extended object with $p+1$ space-time directions, the transverse space is $\mathbb{R}^{9-p}$ and the correct definition is [15]

$$\int_{S^{8-p}} *F_{p+2} = N \tag{3.9}$$

The flux N is proportional to the charge q by α' and numerical factors. N is more useful than q because N is an integer by the Dirac quantization condition, similarly to what happens in electro-magnetism.

We can see p-branes as solitons. In fact, there exist classical solutions of type II supergravity with sources corresponding to massive charged p-branes, or, in other words, solutions of the coupled action

$$S_{II}(g, B, \phi, F) + S_{\text{p-brane}} \tag{3.10}$$

These solutions are called black p-brane and generalize rotating black holes. They have $SO(1,p) \times SO(9-p)$ isometry (corresponding to an object extending in the first $p+1$ coordinates of space-time), N units of flux for the A_{p+1} potential as in equation (3.9), and a finite energy per unit volume in the p space-like directions, which we can identify with the tension of the brane

$$\mathcal{E} = \frac{E}{\mathrm{Vol}_d} = \tau \,. \tag{3.11}$$

The black p-brane solutions are very similar to Kerr black holes, with a singularity surrounded by an inner and outer horizon. As in the case of rotating black holes, we avoid naked singularities only if the energy density is greater than the charge,

$$\mathcal{E} = \tau \geq \frac{N}{(2\pi)^p g_s (\alpha')^{(p+1)/2}} \,. \tag{3.12}$$

When the bound is saturated, we speak of extremal p-branes. They are particularly important because the saturation of the bound is equivalent to the preservation of part of the supersymmetries of the theory. An extremal p-brane preserves exactly half of the 32 supersymmetries of type II supergravity. For this reason it is called

[15] As we see from integrating the equation of motion $d * F_{p+2} \sim q\delta$ on the transverse space.

a BPS object [16]. The extremal p-brane lying in the first $p+1$ directions (including time) corresponds to the solution

$$\begin{aligned} ds^2 &= H^{-1/2}(r)dx_\mu^2 + H^{1/2}(r)dy^2 \\ A_{0\cdots p} &= H(r) \\ e^\phi &= g_s H(r)^{(3-p)/4} \\ H(r) &= 1 + \frac{cg_s N(\alpha')^{(7-p)/2}}{r^{7-p}} \end{aligned} \tag{3.13}$$

with c a numerical factor. y are directions transverse to the brane and $r^2 = \sum_i y_i^2$ is the radial distance. Note that, perhaps not unexpectedly, $H(y_i)$ is the solution of the Laplace equation in the transverse space $\mathbb{R}^{9-p}$ with a delta function source at the origin. The solution has an $SO(1,p) \times SO(9-p)$ isometry group and preserves 16 supersymmetries. One can check all these statements by explicit computations (which we shall not report here).

The horizon of an extremal p-brane collapses on the singularity. Something very special happens for 3-branes: it is good exercise in General Relativity to check that the metric for an extremal 3-brane is actually completely regular. Note also that the dilaton becomes constant.

There is a generalization of the extremal solution where H is replaced by a more general harmonic function

$$H(y_i) = 1 + cg_s(\alpha')^{(7-p)/2} \sum_{a=1}^{N} \frac{1}{|y - y_a|^{7-p}} \tag{3.14}$$

This solution has still charge N, an energy density that saturates the unitary bound (3.12) and it preserves 16 supercharges. While (3.13) corresponds to a brane of charge N sitting at the point $\vec{y} = 0$, (3.14) corresponds to N branes of unit charge sitting at the arbitrary points $\vec{y}_a$. This is called a multi-center solution. The extremality of the multi-center solution has an important consequence: p-branes do not exert force on each other. They can be separated and moved around in space-time without any cost in energy. Typically, if we put two massive objects with the same charge at a certain distance, they will attract or repulse by a combination of gravitational and gauge interaction. However, for extremal p-branes the energy density of a configuration is proportional to its charge. Since charge is additive, the energy density of a system of

[16] In the language of supersymmetry the inequality follows from the supersymmetry algebra in presence of central charges (given by the charge of the p-brane) and its saturation is the BPS condition.

identical extremal branes is equal to the sum of the energy density of the single branes: the potential energy vanishes. In fact, precisely by equality of tension and charge, the gravitational attraction is compensated by the gauge repulsion. This phenomenon is typical of BPS objects and also happens in the physics of four-dimensional monopoles in supersymmetric gauge theories.

In string theory extremal p-branes have an explicit characterization as planes where open strings can end. These objects are called D-branes and we shall mostly use this terminology in the following.

3.1.3 Collective coordinates

Solitonic objects usually carry collective coordinates. The same happens for p-branes: there are fields living on their world-volume.

Some of these collective coordinates are related to fluctuations of our extended object in the transverse directions. We can easily see where they come from if we look closely to the Nambu-Goto action. The position of the brane in space-time is specified by a set of functions $X^M(x^a)$ where x^a are coordinates on the world-volume while X^M are the coordinates of the ten-dimensional space-time. The functions $X^M(x^a)$ describe the embedding and the shape of the D-brane in space-time. The metric on the brane is given by

$$g_{MN}dX^M dX^N = g_{MN}\partial_a X^M \partial_b X^N dx^a dx^b \tag{3.15}$$

and the Nambu-Goto action reads

$$\int dx^{p+1}\sqrt{g} = \int dx^{p+1}\sqrt{\det_{a\,b}\left(\partial_a X^M \partial_b X^N g_{MN}\right)} \tag{3.16}$$

For a p-brane embedded in the first $p+1$ coordinates and sitting at a point $(\phi^{p+2}\cdots\phi^{10})$ we can set $X^a = x^a$, $a = 0,\cdots,p+1$ and $X^i = \phi^i$, $i = p+2,\cdots 10$. To study fluctuations, we allow the positions ϕ^i to vary slowly as functions of the x^a. Computing the induced metric and expanding it in powers of the fluctuations we see that

$$\int \sqrt{\det_{a\,b}\left(g_{ab} + g_{ij}\partial_a\phi^i\partial_b\phi^j\right)} = \int \sqrt{g} + \frac{1}{2}\sqrt{g}g_{ij}\partial_a\phi^i\partial^a\phi^j + ... \tag{3.17}$$

At the order of two derivatives we obtain the standard kinetic term for $9-p$ scalar fields living on the world-volume of the brane. Obviously, this result was expected: it is the only term with two derivatives that is covariant under space-time symmetries. The scalar fields parametrize the position of the brane in space-time.

In addition to scalar fields parametrizing the $9-p$ transverse positions, there are other fields on the brane. The nature of these modes and their effective Lagrangian are difficult to find for generic p-branes. However in the case of extremal p-branes supersymmetry comes to a rescue. Since an extremal brane preserves half of the supersymmetry of the background, the world-volume theory should have 16 supercharges. Fortunately, there is exactly one supermultiplet with 16 supercharges in any dimensions and this is the vector supermultiplet. It is not a coincidence that a vector multiplet contains exactly $9-p$ scalar fields. This uniquely fixes the world-volume fields living on an extremal p-brane. The Lagrangian is also fixed by supersymmetry at the level of two derivatives. It can be obtained by the dimensional reduction of ten dimensional YM theory, as we shall see explicitly.

The previous argument is confirmed by an explicit calculation in string theory, where the world-volume modes can be studied and incorporated by allowing open strings to end on a D-brane. By quantizing open strings ending on extended planes we indeed find massless excitations corresponding to a vector multiplet with 16 supercharges. We also find a tower of massive string modes with squared masses of order $1/\alpha'$. The effective action for world-volume fields and the interaction with the background can be determined by the open plus closed string perturbative expansion. In particular one finds a nice generalization of (3.7) (for a brane of unit charge),

$$\frac{1}{(\alpha')^{(p+1)/2}}\int dx^{p+1}\left(e^{-\phi}\sqrt{\det\left(g+(2\pi\alpha' F+B_2)\right)}+e^{2\pi\alpha' F+B_2}\wedge\sum_k C_k\Big|_{\mathrm{p+1-form}}\right) \tag{3.18}$$

The first term is called the Dirac-Born-Infeld action and generalizes the Nambu-Goto action by including gauge fields and their coupling to the bulk field $B_{\mu\nu}$; the second one is called the Wess-Zumino term and generalizes the coupling to A_{p+1} to the other RR forms [17]. The dilaton factor comes from the tension of a D-brane (see equation (3.12)). By expanding up to two-derivatives in the gauge fields the previous expression, similarly to what we did for the scalar fields, we would get the effective action for gauge fields. Doing it explicitly for the case of a D3-brane in type IIB we obtain

$$\int dx^4 e^{-\phi}\, F_{\mu\nu}^2+C_0\,\epsilon^{\mu\nu\tau\rho}F_{\mu\nu}F_{\rho\sigma} \tag{3.19}$$

which is indeed the action for a gauge vector. The background value of the dilaton fixes the coupling constant while the background value of the scalar C_0 fixes the theta angle. Note that the expansion in derivatives is equivalent to an expansion in α'.

[17]In the Wess-Zumino term only the term in the expansion of $e^{F+B}\wedge\sum C$ with $p+1$ indices, which can be integrated on the world-volume, should be kept.

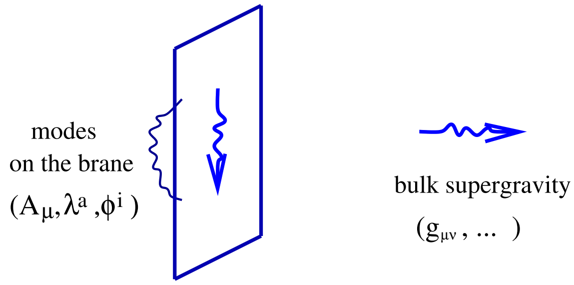

Figure 14: The coupled brane-bulk system.

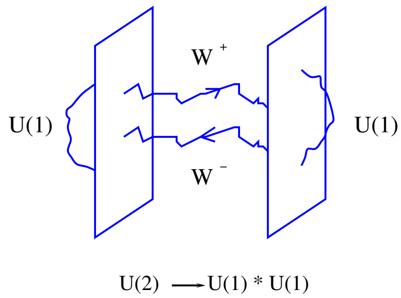

Figure 15: Non-abelian gauge symmetries are naturally realized on the world-volume of coinciding D-branes. When D-branes coincide, open strings connecting different branes become massless giving an enhanced non-abelian symmetry.

3.1.4 Non-abelian gauge fields from D-branes

It will be of utmost importance for us that a set of N Dp-brane carries on its world-volume a YM theory with gauge group $U(N)$. Most of the recent interest in D-branes comes from this fact.

The solution (3.13) with charge N can be interpreted as the superposition of N elementary Dp-branes with unit charge. It is in fact the limit of the more general solution (3.14) when all the D-brane positions coincide. We already know that D-branes carry vector multiplets. In string theory these multiplets come from the quantization of open strings ending on the D-branes as in Figure 15. Open strings ending on the same D-brane give rise to a massless vector multiplet. On the other hand, we also have open strings connecting different D-brane which give rise to massive vector multiplets, since the string, which has a tension $1/\alpha'$, has a non zero length. The mass of these modes is then proportional to the distance between branes: $m \sim |\vec{\phi}_1 - \vec{\phi}_2|/\alpha'$. When D-branes coincide, however, the masses of these vector multiplets vanish. By counting the number of open strings that become massless, we obtain N^2 vector fields, consistent with an enhanced $U(N)$ symmetry (see Figure 15 for the $N=2$ case).

We shall see now that this spectrum of states is well reproduced by a world-volume analysis. The bosonic part of the maximally supersymmetric YM theory in $p+1$ dimensions is obtained by dimensional reduction from super YM in ten. The bosonic part of the 10-dimensional theory is remarkably simple

$$-\int dx^{10}\,\mathrm{Tr}F_{\mu\nu}^2\,, \qquad\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+i[A_\mu,A_\nu] \tag{3.20}$$

The dimensional reduction is obtained by considering configurations that depend only on the first $p+1$ coordinates. We also split the ten dimensional vector in a $p+1$ dimensional vector A_μ , $\mu=0,\cdots p+1$ and $9-p$ scalars $\phi^i=A_i$, $i=p+2,\cdots 10$. Since all derivatives in the $9-p$ transverse directions vanish, we obtain a very simple action for the scalar fields ϕ^i

$$\sum_{i=p+2}^{10}\mathrm{Tr}\left(\partial_\mu\phi^i+i[A_\mu,\phi^i]\right)^2+\sum_{i\,,j=p+2}^{10}\mathrm{Tr}[\phi^i,\phi^j]^2 \tag{3.21}$$

The scalar potential is a sum of squares, as required by supersymmetry, and its vacua are given by the configurations

$$[\phi^i,\phi^j]=0 \tag{3.22}$$

In a $U(N)$ theory the ϕ^i are hermitian matrices. Since they commute, they can be simultaneously diagonalized by a gauge transformation. In the vacuum corresponding to diagonal matrices [18] $\phi^i=\mathrm{diag}\{\phi^i_1,\cdots,\phi^i_N\}$ the gauge group $U(N)$ is broken to $U(1)^N$. In fact, thinking of A^{pq}_μ as an N by N hermitian matrix, it is easy to see that the diagonal components remain massless (giving the $U(1)^N$) while the off-diagonal ones acquire a mass

$$|A^{pq}_\mu|^2\sum_{i=p+2}^{10}|\phi^i_p-\phi^i_q|^2 \tag{3.23}$$

The diagonal VEV for the ϕ^i can be interpreted as the position of the N D-branes in space-time. There is a one-to-one correspondence between the set of vacua of the world-volume theory and the minimal energy configuration of a set of D-branes in space-time. Moreover, the mass of the gauge fields in any given vacuum nicely fits with the space-time description: the diagonal gauge fields correspond to open strings connecting the same D-brane and the off-diagonal ones to open strings connecting different D-branes with a mass proportional to the distance. Finally, we see what

[18] Due to a residual discrete gauge symmetry corresponding to the Weyl group of $U(N)$, a permutation of the eigenvalues give a gauge equivalent configuration.

happens when all the D-branes coincide. In field theory, we have an inverse Higgs mechanism. Some $W^{\pm}$ bosons becomes massless enhancing the symmetry to $U(N)$. In the space-time picture the open string connecting different branes become massless.

We see that the set of vacua of a $U(N)$ world-volume field theory matches the space-time expectation for a set of N BPS objects.

3.1.5 A closer look to D3-branes

We shall be interested in particular in D3-branes. As already mentioned the supergravity solution is regular and the dilaton is constant. On the field theory side, the world-volume theory on N D3-branes is $\mathcal{N}=4$ SYM with gauge group $U(N)$ (16 real supercharges). The theory contains a gauge field, six scalar fields ϕ^i (parametrizing the six transverse directions) and four Weyl fermions ψ_a. A D3-brane has a $SO(1,3)\times SO(6)$ global symmetry. The first factor is the Lorentz group while the second is the $SU(4)_R \sim SO(6)$ R-symmetry of the theory, which rotates the four supercharges Q^a_α. Scalar and fermions transform in the $\underline{6}$ and $\underline{4}$ representation of the R-symmetry group, respectively. Although we shall not use it much, we report the Lagrangian for completeness

$$S=\frac{1}{g^2_{YM}}\int dx^4 \mathrm{Tr}(-\frac{F^2_{\mu\nu}}{2}-i\bar{\psi}^a \not{D}\psi_a-(D_\mu\phi^i)^2+C^{ab}_i\psi_a[\phi^i,\psi_b]+\left[\phi^i,\phi^j\right]^2)$$

with C^{ab}_i Clebsh-Gordan coefficients for the decomposition $\underline{4}\times\underline{4}\to\underline{6}$. The supersymmetry transformations are generated by Q^a_α acting on fields as $\delta^a\chi=[\epsilon^\alpha Q^a_\alpha+\bar{\epsilon}^{\dot{\alpha}}\bar{Q}^a_{\dot{\alpha}},\chi]$ and sending

$$\begin{aligned}
\phi^i &\to C^{iab}\psi_{\alpha\, b}\\
\psi_{\beta\, b} &\to F^+_{\mu\nu}\sigma^{\mu\nu}_{\alpha\beta}\delta^a_b+[\phi^i,\phi^j]\epsilon_{\alpha\beta}C^a_{ijb}\\
\bar{\psi}^b_{\dot{\beta}} &\to C^{ab}_i\bar{\sigma}^\mu_{\alpha\dot{\beta}}D_\mu\phi^i\\
A_\mu &\to \sigma^{\dot{\beta}}_{\mu\alpha}\bar{\psi}^a_{\dot{\beta}}
\end{aligned} \tag{3.24}$$

We see that the scalar potential agrees with the general discussion in the previous section. The form of the Lagrangian makes manifest the $SU(4)_R$ symmetry. There is a simpler form of this Lagrangian using the language of $\mathcal{N}=1$ supersymmetry. The $\mathcal{N}=4$ multiplet can be decomposed in a $\mathcal{N}=1$ gauge multiplet W_α, containing A_μ and ψ^4, and three complex chiral fields Φ_i, containing $\phi^i+i\phi^{i+3}$ and

ψ^i for $i = 1, 2, 3$. The Lagrangian is then

$$\frac{1}{g_{YM}^2}\int dx^4 \left(\int d\theta^2 W_\alpha^2 + \int d\theta^4 \sum_{i=1}^{3} \bar{\Phi}_1 \Phi_i + \int d\theta^2 \epsilon_{ijk} \Phi_i \Phi_j \Phi_k \right) \tag{3.25}$$

A disadvantage of this formulation is that only a $U(1)_R \times SU(3)$ subgroup of the R-symmetry $SU(4)_R$ is manifest.

From the explicit expression (3.19), we know that the kinetic term of a D3-brane should be multiplied by $1/g_s$. We are thus led to the identification of the Yang-Mills coupling on a D3-brane with the constant value of the dilaton $g_{YM}^2 \sim g_s$. More generally, from the same equation, we identify

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} = \frac{C_0}{2\pi} + \frac{i}{g_s} \tag{3.26}$$

where, for reference, we reintroduced the correct numerical factors. As discussed above there is a moduli space of vacua where the scalar fields ϕ^i are diagonal with arbitrary entries. These are flat directions of the gauge theory.

More importantly, the theory is conformal. As already discussed in section 1.1.2, the beta function vanishes up to three-loops and it is believed to vanish at all orders. All correlation functions of elementary fields are finite at all order, when a suitable regularization is used. With supersymmetry the conformal group is enhanced to $SU(2,2|4)$ obtained by $O(2,4)$ by adding the global $SU(4)_R$ generators, the four Weyl supercharges Q_α^a , $a = 1, ...4$ and the four so-called conformal supercharges $S_{\dot{\alpha}}^a$. These are required to close the superconformal algebra $[K, Q] \sim S$. In particular, if we include the S generators, we collect 32 real supersymmetry generators, the 16 supercharges and the 16 conformal supersymetries. We shall not use explicitly the superconformal group; the reader can find more details about it in the Appendix.

3.2 The near-horizon geometry

The original Maldacena conjecture states that $\mathcal{N} = 4$ SYM is dual to the type IIB string background $AdS_5 \times S^5$. The conjecture arose from the observation that the two theories can be obtained by the same decoupling limit $\alpha' \to 0$, performed on the world-volume theory and on the back-reacted metric in space-time.

$\mathcal{N} = 4$ SYM theory can be realized on the world-volume of N parallel D3-branes in Type IIB, as depicted in Figure 15 and 16. The D3 branes interact with the bulk fields which live in 10 dimensions. If we expand in powers of α' the leading terms in

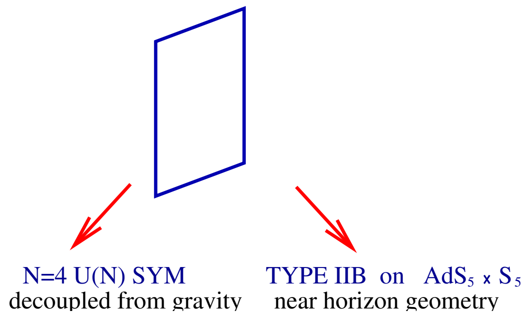

Figure 16: The two limits.

the action for the coupled brane/bulk system are

$$\frac{1}{g_s}\int d^4x F_{\mu\nu}^2 + \frac{1}{\alpha'^4}\int d^{10}x\sqrt{g}Re^{-2\phi} + \cdots \tag{3.27}$$

where $g_{YM}^2 \sim g_s$. We see that, in the limit $\alpha' \to 0$, we can turn off the interaction with the bulk: the brane theory decouples from the bulk and reduces to $\mathcal{N} = 4$ SYM. The limit should be performed in a careful way in order to keep field theory quantities finite. The mass of a gauge boson in the Higgs phase of $\mathcal{N} = 4$ YM is given, in the D3 brane description, by the mass of a stretched open string $m = \Delta r/\alpha'$, where Δr is the distance between branes. If we want to keep these masses finite we take the limit

$$\begin{array}{cc} \alpha' \to 0 & \\ g_s & \text{fixed} \\ N & \text{fixed} \\ \phi^i = \frac{r^i}{\alpha'} & \text{fixed} \end{array} \tag{3.28}$$

In other words, we zoom on the region containing the branes.

On the other hand, a D3-brane is a solution of the equation of motion of type IIB supergravity and deforms the background as in equation (3.13),

$$\begin{aligned} ds^2 &= H^{-1/2}dx_\mu dx^\mu + H^{1/2}(dr^2 + r^2\Omega_5) \\ F^{(5)} &= \text{flux of a charged object} \\ e^\phi &= g_s \equiv \frac{g_{YM}^2}{4\pi} \\ H &= 1 + \frac{g_{YM}^2 N\alpha'^2}{r^4} = 1 + \frac{g_{YM}^2 N}{\alpha'^2\phi^4} \end{aligned}$$

This is the background induced by a tensionful 3-brane charged under the field C_4; far from the brane the deformation vanishes and the metric becomes flat. If we take the limit (3.28) on the solution we obtain

$$ds^2 \sim \alpha' \left\{ \frac{R^2}{\alpha'} \frac{(d\phi)^2}{\phi^2} + \frac{\alpha' \phi^2}{R^2} (dx)^2 + \frac{R^2}{\alpha'} \Omega_5 \right\} , \qquad R^2 = \alpha' \sqrt{g_{YM}^2 N}$$

which we recognize as the product of two Einstein spaces of constant curvature: $AdS_5 \times S^5$. Starting from an asymptotically flat space-time and performing a limit we have obtained a curved space. This is possible because the limit amounts to discard the constant term in $H(r)$ thus decoupling the asymptotically flat part of the metric. Equivalently, we are taking $r \to 0$ and zooming on the region where the branes sit. For this reason this procedure is called a near-horizon limit.

This observation led Maldacena to formulate his conjecture and propose the equivalence between $\mathcal{N} = 4$ SYM and the type IIB string background $AdS_5 \times S^5$.

A first obvious check concern symmetries on the two sides. These are reported in the following self-explaining table:

$\mathcal{N} = 4\,\text{SYM}$	Type IIB on $AdS_5 \times S^5$
Conformal group $O(4, 2)$	Isometry group $O(4, 2)$ of AdS_5
Supersymmetries =16 Q + 16 S	32 supersymmetries
R-symmetry $SU(4)_R$	Isometry of S^5: $SO(6) = SU(4)$

In a word, the symmetry on both sides is the superconformal group $SU(2, 2|4)$. The only subtle point in these identifications regards supersymmetry. The D3 brane solution has 16 supercharges. However its near horizon region has an enhanced supersymmetry: one can check that $AdS_5 \times S^5$ is a maximally supersymmetric background of type IIB. On the field theory side, the 16 supercharges of the YM theory are enhanced to 32 by adding the conformal supersymmetries.

Since S^5 is compact, we can perform a dimensional reduction and think about the gravity theory as an effective five-dimensional theory with an infinite numbers of fields and an AdS_5 vacuum.

It is interesting to compare parameters in the two theories. The radius of both AdS_5 and S^5 is given by $R^2 = \alpha'\sqrt{x}$, where $x = g_{YM}^2 N$ is the t'Hooft coupling. This suggests some relation to the large N limit. In fact, the CFT has two dimensionless parameters, x and N and these can be matched with the two string parameters, g_s and α' as follows

$$\begin{aligned} 4\pi g_s &= \frac{x}{N} \\ \frac{R^2}{\alpha'} &= \sqrt{x} \end{aligned} \tag{3.29}$$

The dual string theory is useful when it is weakly coupled, reducing to an effective supergravity. This happens if $g_s \to 0$ (string loops suppressed) and $\alpha'/R^2 \to 0$ (higher derivatives terms and massive string modes suppressed). This regime corresponds to the large N limit and the strong coupling ($x \to \infty$) of the CFT.

- We see that the t'Hooft limit $N \to \infty, x = g_{YM}^2 N$ fixed, is naturally implemented in the correspondence.
- We also see that we have a *duality*: the weak coupling regime of the gravity theory corresponds to the strong coupling regime of the CFT. This is what makes the correspondence useful. We can learn everything about $\mathcal{N} = 4$ SYM at weak coupling using Feynman graphs and perturbative expansion.

Let us finish this section with some comments.

- We can apply the methods of section 2 and compute Green functions of the CFT using a classical gravitational theory. This method will work in the large N limit and strong coupling. The correspondence is valid for all N and x, but for computations at finite N and x we need the full string theory. We can organize a t'Hooft expansion in $1/N$ and $1/x$, as in Figure 17. The CFT expansion in $1/N$ is the loop expansion of string theory; both are organized as a Riemann surface expansion. The expansion in $1/x$ is the expansion in higher derivatives. In field theory, at each order in $1/N$, we need to re-sum the infinite graphs of given topology obtaining an highly non trivial function $f_g(x)$ of the t'Hooft coupling. In the limit $x \to \infty$, $f_0(x)$ is computable in supergravity, the corrections in $1/x$ corresponding to the higher derivatives corrections of string theory. The correspondence is useful at the moment only at large N and strong coupling. To go beyond the planar limit is probably hopeless, since it requires computing string loops. On the other hand, to go beyond the strong

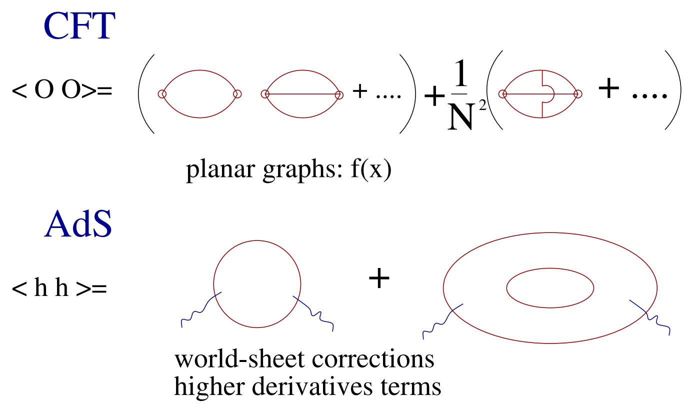

Figure 17: The string expansion versus the the large N expansion

coupling involves computing world-sheet corrections, which are, in principle, more tractable than loop corrections. In flat space, for example, all the α' corrections are computable. In the AdS case, the analogous computation is made difficult, but perhaps not impossible, by the presence of RR-fields.

- Taking a near horizon limit is a general method for obtaining the gravity dual of the gauge theory living on a set of D-branes. This works every time the limit is able to decouple consistently brane and bulk physics. For example, we can find many other examples with less supersymmetry by placing D-branes in curved geometries or considering complicated set of intersecting branes. Some examples will be briefly discussed in the following.

3.3 Matching the spectrum

We shall now give a map between observables in the dual theories. In both cases we shall classify them in terms of the quantum numbers of the superconformal group: the dimension Δ, the spin (j_1, j_2) and the R-symmetry representation.

3.3.1 The field theory side

The theory is finite and (using a suitable regularization) elementary fields are not renormalized. However, gauge invariance requires observables to be composite operators. As well-known, composite operators need additional renormalization: Green functions have short-distance singularities when two or more elementary fields coincide. The conformal dimension of a composite operator can be renormalized

$$\Delta_O = \text{canonical dim} + \gamma_O(g_{YM}) \tag{3.30}$$

where the anomalous dimension $\gamma_O = \Lambda \frac{\partial}{\partial \Lambda} \log Z_O$ is obtained from the wave-function renormalization of the operator: $O_{\text{ren}} = Z_O^{-1} O_{\text{bare}}$.

Some particular operators are protected by the superconformal algebra and their dimension is not renormalized; this certainly happens to the conserved currents (and their partners under supersymmetry) that have canonical dimension. The multiplets of conserved currents are special cases of the so-called short multiplets, which contain less states than a generic representation, as discussed in the Appendix. They saturate an unitary bound and, as a consequence, the dimension of the operators in such multiplets are uniquely determined by the Lorentz and R-symmetry quantum numbers.

- In $\mathcal{N} = 4$ SYM we have various conserved currents, the stress-energy tensor $T_{\mu\nu}$, the supercharges Q^a_α and the $SU(4)_R$ R-currents $(R^a_b)_\mu$. They all belong to the same supermultiplet

$$(Tr\phi_i\phi_j - \text{trace}, \cdots, Q^a_\alpha, \cdots (R^a_b)_\mu, \cdots, T_{\mu\nu}) \tag{3.31}$$

  whose lowest component is a scalar field. Since the currents are conserved and have canonical dimensions, the same is true for their supersymmetric partners.

- The relevant short multiplets of $\mathcal{N} = 4$ SYM are the generalization of the chiral multiplets of $\mathcal{N} = 1$ supersymmetry. Recall that in $\mathcal{N} = 1$ a multiplet is chiral when is annihilated by half of the supersymmetries (let's say the $\bar{Q}$); the corresponding superfield depends only on the θ coordinates (and not $\bar{\theta}$) and it is shorter than a generic multiplet. A chiral multiplet satisfies the unitary bound

$$\Delta = \frac{3}{2}R \tag{3.32}$$

  where R is the R-charge. This bound implies a non-renormalization theorem: Δ and R can be re-normalized but their ratio is not. The analogous case in $\mathcal{N} = 4$ is a multiplet annihilated by half of the supercharges. It has maximum spin two with a scalar lowest state transforming in the symmetric traceless representation of rank k of $SO(6) \sim SU(4)_R$. The dimension of the lowest state is fixed by the shortening condition to be $\Delta = k$ [19]. Since k is an integer, it follows that

[19] A more general short multiplet with lowest scalar state transforming in the $[p, k, p]$ representation of $SU(4)$ with dimension $\Delta = k + 2p$ plays a role in analyzing multi-trace operators.

the dimension of the lowest state (and of all its partners under supersymmetry) cannot be renormalized. In the following, the words *short or chiral* in $\mathcal{N}=4$ will refer to this particular multiplet.

An example of chiral multiplet A_k of $\mathcal{N}=4$ SYM is obtained by applying supersymmetries to the operator

$$Tr\phi_{\{i_1}\cdots\phi_{i_k\}} - \text{traces} \tag{3.33}$$

of canonical dimension k. A_2 corresponds to the supercurrent multiplet. Quite remarkably, one can prove that the A_k are short multiplets with protected dimensions and that they are the only single-trace short multiplets of $\mathcal{N}=4$ SYM.

All single-trace operators not lying in one of the A_k are not protected by the superconformal symmetry and are in general renormalized, unless some miracle or some dynamical reason intervenes.

- **Example:** Consider the quadratic operators made with the six scalar fields of $\mathcal{N}=4$ SYM: $\mathrm{Tr}\phi_i\phi_j$. These operators are symmetric in i and j due to the cyclic property of the trace. We have 21 independent operators. In terms of $SU(4)_R$ representations, the trace $\mathrm{Tr}\sum_i\phi_i\phi_i$ is a singlet and the symmetric traceless part spans a $\underline{20}$. An explicit computation shows that the singlet is renormalized while the $\underline{20}$ is not,

$$\begin{array}{lccc} \underline{20}: & \mathrm{Tr}\left(\phi_i\phi_j - \frac{\delta_{ij}}{6}\sum_i\phi_i\phi_i\right) & \Delta = 2. & \\ \underline{1}: & \mathrm{Tr}\left(\sum_i\phi_i\phi_i\right) & \Delta = 2 + O(g_{YM}) & (3.34) \end{array}$$

  The scalar operator in the $\underline{20}$ is not renormalized since it belongs to the short multiplet A_2; it is the lowest component of the supermultiplets of currents. On the other hand, the trace part does not belong to any multiplet of currents and there is no reason why it should not receive corrections. $\mathrm{Tr}\phi_i\phi_i$ is the lowest component of a long unprotected multiplet, called the Konishi multiplet, and its dimension is renormalized.

- One can check the non-renormalization of A_k just by using the $N=1$ subalgebra and its bounds. The $\mathcal{N}=4$ Lagrangian is written in $\mathcal{N}=1$ notations in (3.25). It has a $U(1)_R$ symmetry which give charge 1 to θ and charge 2/3 to Φ_i. As we said, in $\mathcal{N}=1$ supersymmetry, the chiral multiplets are short and their lowest component saturates the unitary bound $\Delta=3R/2$. The $\underline{20}$

multiplet, when decomposed in $\mathcal{N}=1$ supersymmetry, contains among other things the chiral multiplet $\mathrm{Tr}\Phi_i\Phi_j$ which is protected and has dimension $\Delta=2$ since its R-charge is $R=4/3$. By supersymmetry the entire $\mathcal{N}=4$ multiplet has protected dimension. On the other hand, the singlet $\mathrm{Tr}\phi_i\phi_i$ corresponds in $\mathcal{N}=1$ notations to $\bar{\Phi}_i\Phi_i$ which is not chiral and not protected. Analogously the multiplet A_k contains, after decomposition, the chiral multiplet $\Phi_{\{i_1}\cdots\Phi_{k\}}$ with protected dimension k. Note however that the products $\phi_{i_1}\cdots\phi_k$, where the indices are not symmetrized, are not protected. In fact, although $\Phi_{i_1}\cdots\Phi_k$ looks as chiral multiplets, they are not. In fact, by the equations of motion of $\mathcal{N}=4$ SYM we see that

$$[\Phi_i,\Phi_j]\sim\epsilon_{ijk}D^2\bar{\Phi}_k \tag{3.35}$$

so that the anticommutator of two scalars is not the lowest component of a chiral field, but rather a descendant under supersymmetry of something else. The non-renormalization condition only applies to the lowest component of a chiral field.

3.3.2 The gravity side

Since S^5 is compact we can perform a KK reduction of the ten-dimensional fields to five-dimensions. This is similar to the KK reduction of fields on a circle given by expansion in Fourier modes. If we have a space-time $\mathbb{R}^{1,3}\times S^1$, with a circle of radius R, the equations of motion of a five-dimensional massless field can be diagonalized by considering Fourier modes $\phi(x,y)=\sum_k\phi_k(x)e^{iky/R}$ with $k\in\mathbb{Z}$,

$$-\Box_5\phi(x_\mu,y)=-\Box_4\phi(x_\mu,y)-\partial_y^2\phi(x_\mu,y)=\sum_k e^{iky/R}\left(-\Box\phi_k+\frac{k^2}{R^2}\phi_k\right) \tag{3.36}$$

We thus obtain an infinite tower of KK modes with mass k^2/R^2.

The situation is similar on S^5. We need to diagonalize the Laplacian on S^5 and this is done using spherical harmonics. The bosonic massless modes in ten-dimensions are

$$(g_{\mu\nu},B_{\mu\nu},C_{\mu\nu},\phi,C_0,A^+_{\mu\nu\rho\sigma}) \tag{3.37}$$

They give rise to five-dimensional modes by expansion on spherical harmonics on the sphere.

Example: for a scalar for example

$$\phi(x,y)=\sum\phi^I(x)Y_I(y) \tag{3.38}$$

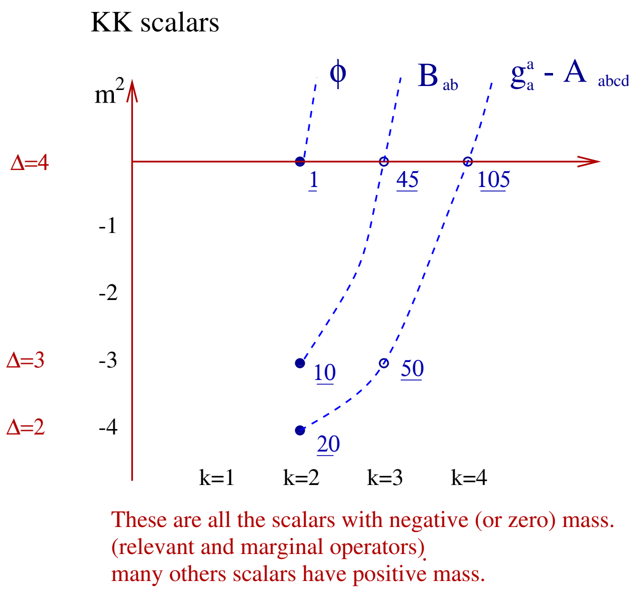

Figure 18: The lines connect states belonging to the same KK tower. Multiplets of supersymmetry are arranged in vertical lines. There is one multiplet for all integers $k \geq 2$.

where Y_I are the eigenfunctions of the scalar Laplacian on S^5. The Y_I can be constructed as $Y_I(y_i) = \hat{y}_{\{i_1} \cdots \hat{y}_{i_I\}}$ − traces using a unit vector on S^5 $\sum_{i=1}^{6} \hat{y}_i^2 = 1$ and satisfy $-\Box Y_I = k(k+4)Y_I$. This construction generalizes the case of familiar spherical harmonics $Y_l(\theta, \phi)$ which are eigenfunctions of the Laplacian on S^2 with eigenvalue $l(l+1)$. This construction can be generalized to fields with Lorentz indices, as $B_{\mu\nu}$ etc.

The KK spectrum was computed and organized in supersymmetry multiplets in the eighties [34, 35]. In figure 18 the reader can find the lowest scalar excitations. For each state we indicated the $SO(6)$ representation. The mass can be read on the vertical axis. Modes on the same vertical line belong to the same supermultiplet. One can show that the entire spectrum consists of a series of short multiplets A'_k of the superconformal algebra, labelled by an integer $k \geq 2$. The lowest state in A'_k is a scalar in the k-fold symmetric representation of $SO(6)$ with mass $m^2 = k(k-4)/R^2$ and the maximum spin in the multiplet is two. For example, $k = 2$ is the graviton multiplet,

$$(g_{\mu\nu},\ \psi^i_\mu :\ \text{in the}\, \underline{4}, A_\mu :\ \underline{15},\ B_{\mu\nu} :\ \underline{6} + \underline{\bar{6}},\ \lambda :\ \underline{4} + \underline{10}_c,\ \text{scalars} :\ \underline{1}_c + \underline{10}_c + \underline{20}) \tag{3.39}$$

Note that even if this is called the *massless* multiplet, only the gauge fields are strictly massless: the mass is not a Casimir of the superconformal group and varies inside a multiplet.

Let us make some observations.

- In the classical KK expansion on a circle, the zero-modes are massless and separated from massive modes by a quantity of order $1/R^2$. We can decouple massive modes by taking large R. Due to the non-zero curvature, in AdS there is no separation: all the KK modes have a mass of the same order of the zero-modes. In fact strictly massless modes are those with gauge invariance (graviton,etc...). But as we already noticed even some of their supersymmetric partners are massive.

- Even if we cannot decouple the massive multiplets A'_k with $k \geq 3$ by taking the internal manifold large, it makes sense to write an effective action for the massless multiplet A'_2. This is the $\mathcal{N} = 8$ gauge supergravity in five-dimensions. It is believed to be a consistent truncation of the theory in the sense that every solution of its equations of motion can be uplifted to a solution of the ten-dimensional equations of motion.

3.3.3 The comparison

There is a complete correspondence between the KK spectrum and the single-trace short multiplets of $\mathcal{N} = 4$ SYM: in each case we have precisely one short multiplet of the superconformal algebra for every $k \geq 2$. The massless multiplet A'_2 on the gravity side, which contains the gravitons, the gravitino and the $SO(6)$ gauge fields, corresponds to the supercurrent multiplet on the field theory side. This is the usual manifestation of the fact that gauge symmetries in the bulk correspond to global symmetries in the boundary. The multiplets A'_k corresponds to the field theory multiplet A_k with lowest component

$$Tr\phi_{\{i_1} \cdots \phi_{i_k\}} - \text{traces}\,. \tag{3.40}$$

It is an useful exercise to check explicitly that the two multiplets contain fields with the same quantum numbers and compare their masses and dimensions using the standard AdS rules discussed in section 2.2.

Exercise: identify all scalars in Figure 18, by using susy, mass/dimension relations and $SU(4)$ quantum numbers. The result is (schematically)

Up to now, we have discussed the KK modes. What happens to the string modes? In the supergravity limit, all stringy states are very massive and decouple.

SU(4) rep.	operator	multiplet/dim.
$\underline{20}$	$\mathrm{Tr}\phi_{\{i}\phi_{j\}} - \mathrm{traces}$	$A_2\ \ \Delta = 2$
$\underline{50}$	$\mathrm{Tr}\phi_{\{i}\phi_j\phi_{k\}} - \mathrm{traces}$	$A_3\ \ \Delta = 3$
$\underline{10}_c$	$\mathrm{Tr}\lambda_a\lambda_b + \phi^3$	$A_2\ \ \Delta = 3$
$\underline{105}$	$\mathrm{Tr}\phi_{\{i}\phi_j\phi_k\phi_{p\}} - \mathrm{traces}$	$A_4\ \ \Delta = 4$
$\underline{45}_c$	$\mathrm{Tr}\lambda_a\lambda_b\phi_i + \phi^4$	$A_3\ \ \Delta = 4$
$\underline{1}_c$	$\mathrm{on-shell\,Lagrangian}$	$A_2\ \ \Delta = 4$

Table 1: Marginal or irrelevant scalar operator in the KK spectrum.

In the CFT these states correspond to operators with anomalous dimension $x^{1/4}$

$$m^2 = \frac{\Delta(\Delta-4)}{R^2} \sim \frac{1}{\alpha'} = \frac{\sqrt{x}}{R^2} \qquad \rightarrow \qquad \Delta \sim x^{1/4}\,. \tag{3.41}$$

Since they have infinite dimension, they should decouple from all the OPE and Green functions. We then have a very strong prediction: in $\mathcal{N}=4$ SYM at strong coupling only the set of protected operators A_k have finite dimensions; all the other non protected operators have an anomalous dimension that diverges at strong coupling as $x^{1/4}$. In particular all multiplets containing spin greater than two and many other including the Konishi multiplet should have infinite dimension. We thus have a very large separation between a set of operators with maximum spin two and all the other operators in the theory. This separation, as discussed in section 2.7, is necessary for every CFT with a weakly coupled dual. All quantum field theory indications are consistent with the fact that this separation actually occurs for $\mathcal{N}=4$ SYM at strong coupling.

Let us finish with some important comments.

- Some of the KK modes have negative mass. The corresponding mode is stable if $m^2R^2 \geq -4$ (Breitenlhoner-Freedman bound). In N=4, all operators have $\Delta \geq 2$ and therefore $m^2 \geq -4$. Stability is a consequence of supersymmetry. Non-supersymmetric solutions might be unstable and the corresponding CFT non-unitary.

- The separation of scales is a property of strong coupling. In $\mathcal{N}=4$ SYM at weak coupling all operators, including the non protected ones, have small dimensions, computable in power series of $x = g_{YM}^2 N$. The dual description of this weakly coupled regime requires a stringy limit where the radius of AdS is small; in this situation many stringy states have low masses and correctly reproduce the CFT spectrum at weak coupling, including operators with arbitrary spin.

- The exact correspondence between KK modes and protected operators is a peculiarity of $\mathcal{N} = 4$ SYM and it does not extend to theories with less supersymmetry. The reason why the KK spectrum of $\mathcal{N} = 4$ SYM contains only chiral multiplets of the form discussed is that all other short, semishort and long multiplets of the $\mathcal{N} = 4$ superconformal algebra contain fields with spin greater than two which cannot appear in supergravity. The $\mathcal{N} = 1$ superalgebra has even long multiplets with maximum spin two. And in fact, in examples of $\mathcal{N} = 1$ dual pairs there are KK modes corresponding to non-protected operators with finite dimension.

- We discussed only single trace operators. In the AdS/CFT correspondence, multi-trace operators correspond to multi-particles states in the bulk.

There is a final important remark. It is believed that the correspondence applies to an $\mathcal{N} = 4$ SYM theory with $SU(N)$ gauge group. The extra $U(1)$ factor living on the D3 branes is IR free and its dynamics is not captured by the duality. A first indication that the gauge group is $SU(N)$ comes from the KK spectrum. In a $U(N)$ theory $\mathrm{Tr}\phi_i$ is a gauge invariant operator with protected dimension. However, the corresponding KK mode can be gauged away and it is not a physical state [34]: the full A'_1 multiplet is absent from the KK spectrum. Further strong indications that the correspondence describes $SU(N)$ gauge theories come from dual pairs with $\mathcal{N} = 1$ supersymmetry where one can identify states dual to baryonic operators, which would be not gauge invariant in a $U(N)$ theory.

3.3.4 Correlation functions, Wilson loops and all that.

The methods developed in section 2 can be applied to $\mathcal{N} = 4$ SYM.

- Green functions for operators in A_k can be computed at strong coupling using supergravity as explained in section 2. As a curious result, 3-point functions for the lowest component of A_k, which in principle depends multiplicatively by an arbitrary function of the t'Hooft coupling x, agree with the free-field theory result [36]. This is a hint for some non-renormalization theorem.

- In $\mathcal{N} = 4$ SYM the computation of Wilson loops can be seen in a better light: external sources can be naturally obtained by separating one brane from the

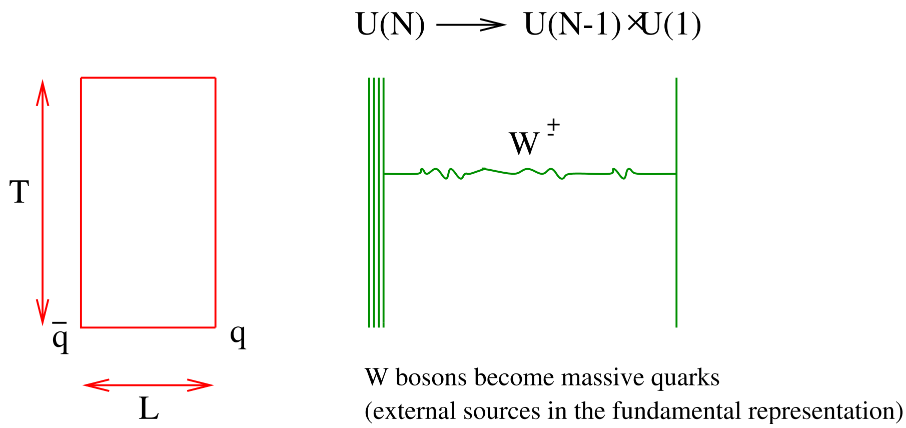

Figure 19: In a systems of branes, external sources in the fundamental representation of the gauge group can be obtained by moving one brane far away from the others

other and breaking the group $U(N) \to U(N-1)\times U(1)$ as in Figure 19. At large N, $U(N-1) \sim U(N)$. Since the near-horizon focuses on the stack of branes, moving away one of them corresponds to pull it to the boundary. The $W^\pm$ bosons then play the role of external sources: they are described by long and very massive strings connecting the boundary with the bulk and they transform in the fundamental representation of the unbroken $U(N-1) \sim U(N)$ group. This interpretation makes clear why, in the prescription described in section 2.5, we renormalize the Wilson loop by subtracting the mass of two very long strings: this is the mass of the external quarks. In section 2.5 we obtained the result $E \sim \frac{R^2}{L} \sim \frac{\sqrt{g_{YM}^2 N}}{L}$. The coefficient in this formula is a genuine strong coupling result, as the presence of a square root indicates. At weak coupling we would find $E = \frac{g_{YM}^2 N}{L}$. There is no non-renormalization theorem for the basic Wilson loop.

3.4 Other $\mathcal{N} = 1$ CFTs from D-branes

In this section we present a brief detour on the other $\mathcal{N} = 1$ CFTs that admit an AdS dual. The content requires some more advanced knowledge of $\mathcal{N} = 1$ supersymmetry. The results of this section will not be used in the following.

3.4.1 Deformation of $\mathcal{N} = 4$

We can obtain other CFT by deforming $\mathcal{N} = 4$ SYM. It is difficult to say when a deformed Lagrangian is conformal or, more generally, flow to a conformal theory in the IR. However we have some tools for dealing with supersymmetric theories.

- The Leigh and Strassler argument (LS) [37]: the understanding of this argument requires some advance knowledge of supersymmetry but the result is very easy to manage. Consider a $\mathcal{N} = 1$ gauge theory with chiral matter fields transforming in some representations R_a and a superpotential W. Anomalies and supersymmetry relate the N=1 gauge beta function to the dimension of matter fields via,
$$\beta(g) \sim 3\left[T(\mathrm{adj}) - \sum T(R_a)(3 - 2\Delta_a)\right] \tag{3.42}$$
where a runs over chiral matter fields and T denotes the second Casimir. The requirements for conformal invariance are exhausted by the vanishing of the previous expression combined with the fact that the sum of dimensions of the fields appearing in each term of the superpotential is three.

- We can also obtain information from the dual supergravity side, when the operator used for deforming the Lagrangian is dual to a KK mode. In this case we can use the results of section 2.8.3. In particular we can study the deformations of $\mathcal{N} = 4$ SYM by all the modes contained in Table 1 using supergravity.

Let us discuss first the marginal case. We see from Table 1 that we have marginal modes transforming in the $\underline{1}, \underline{45}$ and $\underline{105}$. When added to the original Lagrangian as deformation, these operators preserve conformal invariance at first order in h_i. We are interested in exactly marginal operators that preserve conformal invariance for all values of the parameters. They provide lines of fixed points continuously connected to the original CFT. It is impossible to study this problem in full generality, however:

- The scalar $\underline{1}_c$ is an example of exactly marginal deformation: it preserves N=4 supersymmetry and the corresponding perturbation of the N=4 theory is simply a change in the complexified coupling constant τ. This corresponds to an exactly marginal deformation, because the N=4 Yang-Mills theory is conformal for each value of the coupling. Its supergravity description is extremely simple and it corresponds to vary the constant values of the dilaton and axion C_0, which are arbitrary since there is no potential for ϕ and C_0.

- There is an exactly marginal deformation of $\mathcal{N} = 4$ SYM preserving $\mathcal{N} = 1$ supersymmetry given by the superpotential

$$\mathcal{W} = h\epsilon_{ijk}\mathrm{Tr}\Phi_i\Phi_j\Phi_k + Y_{ijk}\mathrm{Tr}\Phi_i\Phi_j\Phi_k, \tag{3.43}$$

In this equation, Φ_i are the three adjoint chiral fields of $\mathcal{N} = 4$ SYM in $\mathcal{N} = 1$ notations and Y_{ijk} is a generic symmetric tensor transforming in the $\underline{10}$ of $SU(3)$. By expanding in components we see that the deformations in the Lagrangian are cubic and quartic terms contained in the $\underline{45}$ and $\underline{105}$. The N=4 theory is recovered for $Y_{ijk} = 0$ and $h = g_{YM}$. The 11 complex parameters in the superpotential can be reduced to 4 independent ones by using the (complexified) $SU(3)$ global symmetry. A convenient parameterization of the superpotential is

$$h\,\mathrm{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2) + h'\mathrm{Tr}(\Phi_1^3 + \Phi_2^3 + \Phi_3^3) \tag{3.44}$$

There is a particular relation between the four complex parameters g_{YM}, h, h', β for which the theory is superconformal. In the specific case, the gauge beta function (3.42) vanishes if

$$\Delta_{\Phi_1} + \Delta_{\Phi_1} + \Delta_{\Phi_1} = 3 \tag{3.45}$$

On the other hand, the requirement that the superpotential terms have formal dimension three, gives the condition $\Delta_{\Phi_i} = 1$ for all $i = 1, 2, 3$. All together, the equations for the vanishing of all beta functions are satisfied if

$$\Delta_{\Phi_i}(g_{YM}, h, h', \beta) = 1 \tag{3.46}$$

Since there is an obvious permutation symmetry, the Φ_i must have the same dimension. We thus conclude that we have a single equation for four unknowns. This yields a three-dimensional complex manifold of fixed points. Although will be never able to solve equation (3.46) but in perturbation theory, the LS argument is strong enough to conclude the existence of the manifold of fixed points. The supergravity dual of this manifold is obtained by deforming $AdS_5 \times S^5$ with the corresponding KK modes. From Table 1 and figure 18 we see that we need a deformation of the metric but also the inclusion of a non-zero background for the antisymmetric complex two-form of type IIB. A perturbative analysis of the supergravity equations of motion, performed up to second order, reveal that it is possible to find an AdS_5 solution with these new modes turned on, in agreement with the fact that the corresponding deformations are exactly marginal. The solution with arbitrary parameters is only known at few orders in perturbation theory [25,38]. For the case $h' = 0$, called the β deformation of $\mathcal{N} = 4$ SYM however a full supergravity description exists [39].

Now we turn to the relevant deformations of $\mathcal{N} = 4$ SYM. We see from Table 1 that we have relevant modes in the $\underline{20}, \underline{50}$ and $\underline{20}_c$. In particular, we can describe almost all massive deformations of $\mathcal{N} = 4$. The only exception corresponds to the mass term $\sum \phi_i \phi_i$ which is dual to a string mode. We can use supergravity to study the existence of possible IR fixed points. They should correspond, as discussed in section 2.8.3, to kink solutions of type IIB supergravity that connect two different AdS_5 vacua. The existence of these kink solutions is better investigated in the contest of a five-dimensional reduction of the theory. We do not have a full consistent 5d Lagrangian containing all relevant states in Table 1. However we do have a consistent Lagrangian for the massless multiplet A_2', $\mathcal{N} = 8$ gauged supergravity. The scalar sector of the gauged supergravity contains the complexified dilaton $\underline{1}_c$, the $\underline{20}$ and $\underline{10}_c$, for a total of 42 scalars. In particular it describes all KK modes dual to mass terms. As discussed in section 2.8.3, IR fixed points correspond to the critical points of the gauged supergravity potential. The full potential is too hard to study, however:

- All critical points with at least $SU(2)$ symmetry have been classified [40]. A central critical point, with $SO(6)$ symmetry and with all the scalars λ_a vanishing, corresponds to the unperturbed $\mathcal{N} = 4$ YM theory. There are three N=0 theories with residual symmetry $SU(3)\times U(1)$, $SO(5)$ and $SU(2)\times U(1)^2$. They correspond to non-zero VEV for some of the scalars in the $\underline{10}$, $\underline{20}$, and $\underline{10} + \underline{20}$, respectively. Then there is an $\mathcal{N} = 2$ point (this corresponds to a $\mathcal{N} = 1$ field theory) with symmetry $SU(2) \times U(1)$, obtained giving VEV to scalars in the $\underline{10} + \underline{20}$. The three N=0 theories are unstable and correspond to non-unitary CFTs. The $\mathcal{N} = 2$ theory is stable by supersymmetry. The central charges of these theories can be computed from the value of the cosmological constant at the critical points.

- The supersymmetric IR point can be explicitly identified in field theory [26,41]. It is obtained by adding a mass term for one of the chiral fields of $\mathcal{N} = 4$ when written in $\mathcal{N} = 1$ notations,

$$h\mathrm{Tr}\Phi_3[\Phi_1, \Phi_2] + m^2\Phi_3^2 \tag{3.47}$$

  Integrating out the massive field Φ_3, we obtain an $SU(N)$ theory with two adjoint fields $\Phi_i, i = 1, 2$ and a superpotential

$$\lambda\mathrm{Tr}[\Phi_1, \Phi_2]^2, \qquad \lambda \sim \frac{h^2}{m^2} \tag{3.48}$$

It is easy to show that the theory flows to an IR fixed points. The vanishing of the gauge beta function and the requirement of scaling invariance of the superpotential give the same constraint

$$2(\Delta_{\Phi_1} + \Delta_{\Phi_2}) = 3 \tag{3.49}$$

Since the theory has an $SU(2)$ symmetry rotating Φ_1 and Φ_2, the fields Φ_i have the same dimensions and the previous equation reduces to a single constraint

$$\Delta_\Phi(g_{YM}, h) = 3/4 \tag{3.50}$$

With two couplings, g_{YM} and λ, we have a line of fixed points. Using some more sophisticated methods, one can even compute the central charge of of the IR fixed point: $\frac{a_{IR}}{a_{N=4}} = \frac{27}{32}$ [26]. It is probably superfluous to say that it corresponds exactly with the supergravity prediction based on gauged supergravity: $\left(\frac{\Lambda_{UV}}{\Lambda_{IR}}\right)^{3/2} = \frac{27}{32}$. The supergravity dual of the Renormalization Group flow from $\mathcal{N} = 4$ to the IR fixed point was found in [26] using five dimensional gauged supergravity and successively uplifted to ten dimensions [42]; as expected from Table 1 and Figure 18, the solution is a warped AdS_5 compactification on a squashed sphere with non zero B field.

3.4.2 Other $\mathcal{N} = 1$ theories.

We can obtain an infinite class of $\mathcal{N} = 1$ theories by taking a near horizon limit of systems of D3-branes in curved geometries. Since the AdS/CFT correspondence focuses on the near brane region and every smooth manifold is locally flat, we will find new models only when the branes are placed at a singular point of the transverse space [43–45]. An interesting class of theories makes use of conifold singularities. We place branes at the singularity of Ricci-flat manifolds C_6 whose metric has the conical form

$$ds^2_{C_6} = dr^2 + r^2 ds^2_{M_5}. \tag{3.51}$$

One can prove that C_6 is Ricci-Flat iff M_5 is a five-dimensional Einstein manifold [44, 46]. The AdS/CFT correspondence is then formulated with the background $AdS_5 \times M_5$, which is the near horizon geometry of the previous metric. The gauge theory is maximally supersymmetric $\mathcal{N} = 4$ SYM with gauge group $U(N)$ in the case $M_5 = S^5$ and a less supersymmetric gauge theory depending on M_5 otherwise.

- Famous examples of this construction are orbifolds on $\mathcal{N} = 4$ SYM or the Klebanov-Witten theory, pictured in Figure 20. They give examples of the so-

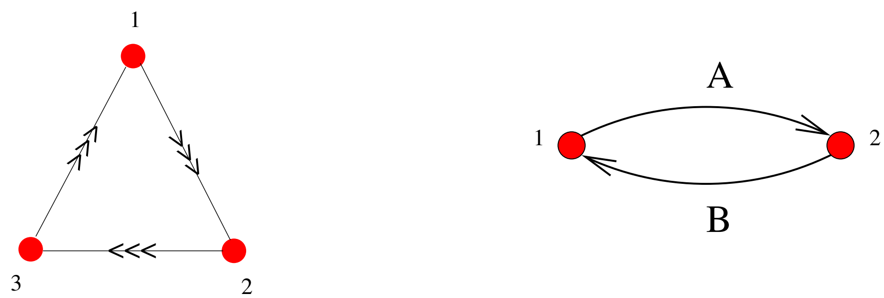

Figure 20: The quiver corresponding to $M_5 = S^5/\mathbb{Z}^3$ (the orbifold $C_6 = \mathbb{C}^3/\mathbb{Z}_3$) and $M_5 = T^{1,1}$ (the conifold $C_6 : xy = zw$ in $\mathbb{C}^4$).

called quiver gauge theories, with product of $U(N)$ gauge groups (where N is the number of branes) and bi-fundamental or adjoint matter fields.

- The correspondence between M_5 and CFTs can be worked out explicitly when the cone $C_6 = C(M_5)$ is a toric manifold using Tiling techniques [47, 48] [20].

- The five-manifold M_5 has typically non trivial topology, with two and three cycles. Three cycles lead to the existence of baryons in these theories [52] and indicate that the gauge group is $SU(N)$. Their existence is also the basis for the most successful attempts to describe confining theories using the AdS/CFT correspondence [28].

We shall not discuss further this general class of theories since they have a very rich structure that will take us too far.

Other examples of $\mathcal{N} = 1$ CFTs can be obtained by considering flux compactifications of type IIB. Much less is known about this class of backgrounds, in principle very large; the typical example in this class is the warped compactification corresponding to the IR point of the $\mathcal{N} = 1$ adjoint deformation of $\mathcal{N} = 4$ SYM [42]. Other classes of (quite exotic) $\mathcal{N} = 2$ CFTs are obtained from M theory compactifications on Riemann surfaces [53].

[20]Toric roughly means that there are three $U(1)$ isometries. The corresponding quivers can be drawn as a tiling of a two-dimensional torus. One can show that the these quivers are conformal (using a LS argument), that the field theory mesonic moduli space is $(\mathrm{Sym}C_6)^N$ [49] and that the spectrum computed via a-maximization [50] coincide with the prediction of supergravity [51].

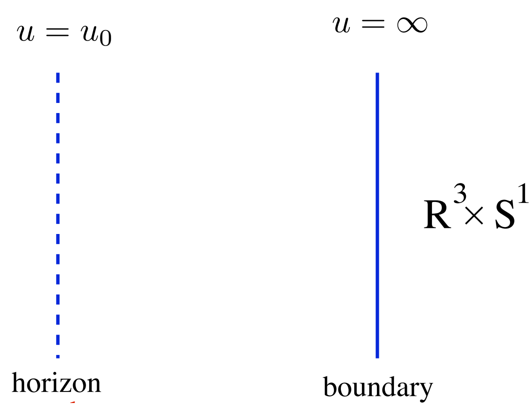

Figure 21: The black brane describing $\mathcal{N} = 4$ SYM at finite temperature.

4 Explicit examples: the non-conformal case

We shall now study euclidean $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ on $\mathbb{R}^3 \times S^1$ with anti-periodic boundary conditions for fermions. Interpreting the compactified direction as the euclidean time we obtain a description of $\mathcal{N} = 4$ SYM at finite temperature T. On the other hand, by interpreting S^1 as a spatial direction, we can study the compactification of $\mathcal{N} = 4$ SYM to three dimensions. Since the boundary conditions breaks supersymmetry we are left at low energy with a non-supersymmetric YM theory that confines. Using black D3-branes we can then simultaneously fulfil two purposes: study $\mathcal{N} = 4$ SYM at finite temperature T and describe a simple toy model for confining theories.

4.1 The black three-brane and finite temperature SYM

We consider a set of N D3-branes defined on a space-time with topology $\mathbb{R}^3 \times S^1$. The solution for an Euclidean black three-brane with this boundary topology is known and its near horizon geometry is given by:

$$ds^2 = R^2 \left[\left(u^2 \sum_{i=1}^{3} dx_i^2 + u^2 \left(1 - \frac{u_0^4}{u^4}\right) d\tau^2 \right) + \frac{du^2}{u^2 (1 - \frac{u_0^4}{u^4})} + d\Omega_5 \right] \tag{4.1}$$

where, as usual, $R = \sqrt{4\pi g_s N \alpha'}$. This metric is the product of the metrics for a round S^5 and the black brane solution discussed in section 2.8.1 (with the usual change of coordinate $u = 1/z$). The geometry has an horizon at $u = u_0$. As discussed in section 2.8.1 the metric is smooth if τ is an angular variable with radius $R_0 = \frac{1}{2u_0}$.

Moreover the metric admits only one spin structure where fermions change sign along S^1. For all these reasons, the black three-brane solution is the natural candidate for the gravitational dual of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ at finite temperature

$$T = \frac{1}{2\pi R_0} = \frac{u_0}{\pi} \,. \tag{4.2}$$

We already study the thermodynamics of this system in section 2.8.1. Here we can be more precise about the dependence on N and g_s of free energy and entropy. Quantities in section 2.8.1 are expressed in terms of the five-dimensional Newton constant. We can determine it by dimensionally reduce the Einstein term in the ten-dimensional type IIB theory[21]

$$\frac{1}{(2\pi)^7(\alpha')^4 g_s^2}\int d^{10}x\sqrt{g}R \;\longrightarrow\; \frac{1}{(2\pi)^7(\alpha')^4 g_s^2}R^3\mathrm{Vol}(S^5)\int d^5x\sqrt{g}R \,, \tag{4.3}$$

so that

$$\frac{1}{16\pi G_N} = \frac{1}{(2\pi)^7(\alpha')^4 g_s^2}R^5\pi^3 \,. \tag{4.4}$$

Now, from the holographic dictionary we can express

$$\sqrt{x} = \frac{R^2}{\alpha'} \,, \qquad \frac{x}{N} = 4\pi g_s \qquad \longrightarrow \qquad g_s^2 = \frac{R^8}{16\pi^2 N^2(\alpha')^4} \,, \tag{4.5}$$

and we obtain the relation

$$\frac{R^3}{4G_N} = \frac{N^2}{2\pi} \,. \tag{4.6}$$

We then find from (2.100) the $\mathcal{N} = 4$ SYM free energy in the t'Hooft limit

$$F = -\frac{\pi^2}{8}N^2T^4V \,, \tag{4.7}$$

which correctly scales like N^2, which is proportional to the number of degrees of freedom. In a free theory, we could compute the free energy of a gas of free gluons, fermions and scalars in thermal equilibrium at temperature T. This is just a text book exercise about black body radiation. At large N there are $N^2 - 1 \sim N^2$ different species of particles, and it is not difficult to compute

$$F = -\frac{\pi^2}{6}N^2T^4V \,, \tag{4.8}$$

for a gas of non interacting gluons. We see that the free energy is not protected against quantum corrections but stay finite at strong coupling. In particular

$$F(x = \infty) = \frac{3}{4}F(x = 0) \,. \tag{4.9}$$

[21] Let us pretend that there is an action although there is none.

We can also consider $\mathcal{N} = 4$ SYM on $S^3 \times S^1$. If we denote by l the radius of S^3 now the physics depends on the dimensionless quantity lT. There are due five-dimensional euclidean solutions with boundary topology $S^3 \times S^1$:

- The AdS Schwarzschild black hole

$$ds^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2\Omega_3\,, \qquad f(r) = 1 - \frac{\mu}{r^2} + \frac{r^2}{R^2}\,, \tag{4.10}$$

  where μ is related to the black hole mass. The metric is smooth if τ has period $\beta = \frac{1}{T} = \frac{2\pi R^2 r_+}{2r_+^2 + R^2}$ where r_+ is the horizon radius, the larger root of $f(r)$. As before, there is just one spin structure where fermions change sign along S^1.

- Thermal Euclidean AdS

$$ds^2 = \left(1 + \frac{r^2}{R^2}\right) d\tau^2 + \frac{dr^2}{1 + \frac{r^2}{R^2}} + r^2\Omega_3\,, \tag{4.11}$$

  which is just AdS_5 in global coordinates with the time direction compactified. Since the circle is not contractible, the periodicity $\beta = 1/T$ is arbitrary and there are two different spin structure, one where fermions are periodic and another where they are anti-periodic.

Since there are two possible solutions with the same boundary and anti-periodic spin structure, there are two saddle point contributions to the gravitational path integral and we need to figure out which one dominates. It turns out that there is a phase transition, discovered by Hawking and Page well before the AdS/CFT correspondence, with critical temperature

$$T_c = \frac{3}{2\pi R}\,. \tag{4.12}$$

For $T > T_c$ the black hole solution dominates, and the free energy scale as $O(N^2)$ as for a gas of gluons. For $T < T_c$ the thermal AdS dominates. Since there is no horizon in thermal AdS the entropy and the free energy are zero in the semiclassical limit, and it will be of order $O(1)$ when correction are added. This is interpreted as a confinement/deconfinement transition. This is different from the usual quark-gluon plasma deconfinement transition for QCD at infinite volume. Indeed such transition would not apply to $\mathcal{N} = 4$ SYM or a generic CFT, since they do not confine on $\mathbb{R}^3$. The phenomenon we are observing is peculiar of the physics of a gauge theory in finite volume. In principle, in finite volume we cannot have a phase transition. Here it is possible only because we are sending N to infinity. For fixed radius of S^3, at low temperature the physical states of $\mathcal{N} = 4$ SYM on a sphere are color singlets, and there are $O(1)$ of them. This is a sort of confinement (for kinematical reasons, not dynamical as in QCD!). At high temperature instead we see states made with deconfined gluons and there are $O(N^2)$ of them.

Notice that, since the physics depends only on lT, the large l limit where S^3 is replaced by $\mathbb{R}^3$ drives the system to high temperature where only the black hole phase exists. On $\mathbb{R}^3 \times S^1$ the thermal physics is captured by the black hole solution, as we already discussed.

The reader may wonder what is the meaning of the thermal solution with periodic spin structure. It turns out that it describes a supersymmetric partition function on $S^3 \times S^1$ and it computes a Witten index

$$\mathrm{Tr}(-1)^F e^{-\beta H} \,. \tag{4.13}$$

There is a huge literature, both in field theory and holography, about supersymmetric indices, but we have no time to discuss them here.

4.2 The black three-brane and three-dimensional confinement

One aim of the correspondence is the description of non-conformal theories that reduce at low energy to pure YM theories. We will now describe a simple toy model for confining theories based on a the very same black D3-brane solution [2]. We shall explicitly confirm the picture of confinement discussed in section 2.8.2.

We study again euclidean $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ on $\mathbb{R}^3 \times S^1$ for a circle of radius R_0 but we know interpret S^1 as a spatial direction. As in the case of finite temperature, we require that the fermions have anti-periodic boundary conditions along S^1

$$\begin{aligned} \psi(y) &= -\psi(y + 2\pi R_0) \to \psi = \sum_{\mathbb{Z}+1/2} \psi_k e^{iky/R} \\ m^2_{\psi_k} &= \frac{k^2}{R_0^2} > 0 \quad (|k| > 0) \,. \end{aligned} \tag{4.14}$$

Conformal invariance is broken by the compactification and supersymmetry by the boundary conditions. The fermions get masses through these boundary conditions and the scalars get masses through loops of fermions. For $R_0 \to 0$ all fermions and

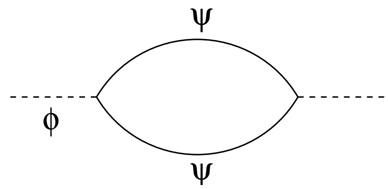

Figure 22: Scalars get mass through Yukawa couplings.

scalars decouple and we are left with pure YM in three-dimensions. Restricting to zero modes we have

$$\int d^4x \frac{1}{g_4^2} F_{\mu\nu}^2 = \int d^3x \frac{2\pi R_0}{g_4^2} F_{\mu\nu}^2 \,. \tag{4.15}$$

We see that the three-dimensional coupling is given by $\frac{1}{g_3^2} = \frac{2\pi R_0}{g_4^2}$. In order to keep the three-dimensional coupling finite in the decoupling limit, we need to send $R_0 \to 0$, and $g_4 \to 0$ with g_3 fixed. In this limit, one obtains a non-supersymmetric and non-conformal YM theory in three-dimensions. Such a theory confines, has a mass gap and a discrete spectrum of massive glueballs.

This model can be studied using a weakly coupled supergravity dual given again by the Euclidean black three-brane (4.1) with boundary topology $\mathbb{R}^3 \times S^1$. There is a refined and more complicated version of this construction that gives pure YM in four-dimensions [2, 4].

- **Confinement:** As discussed in section 2.8.2 , the criterion for confinement is the following: the warp factor e^{2A} multiplying the space-time part of the metric must be bounded above zero. In the black brane example, the warp factor reaches its minimum $e^{2A_0} = \sqrt{4\pi g_s N} u_0^2$ at the horizon u_0. The theory has then stable strings with finite tension; they live in the region of the solution where the warp factor has its minimum value e^{2A_0} and their tension will be given by $\frac{e^{2A_0}}{2\pi\alpha'}$.

- **Glueball Spectrum:** The masses of bound states can be extracted from correlation functions of gauge invariant operators evaluated at large distance,

$$\langle O(x)O(y)\rangle \sim \sum a_i e^{-M_i|x-y|}, \qquad |x-y| \gg 1 \tag{4.16}$$

More efficiently, as discussed in section 2.8.2, we can extract M_i from the normalizable solutions of the equations of motion. We expand a massive field in Fourier modes both on S^1 and on the three-dimensional space-time

$$\phi(x,\tau,u) = \phi(u) e^{in\tau} e^{ikx} \tag{4.17}$$

For simplicity, we consider the lowest states and we keep only the zero-mode on S^1 (n=0). The action in the black three-brane background is

$$S = \int_{u_0}^{\infty} du\, u^3 \left\{ u^2 \left(1 - \frac{u_0^4}{u^4}\right) \left(\frac{\partial \phi}{\partial u}\right)^2 + m^2 \phi^2 + \frac{k^2}{u^2}\phi^2 \right\}$$

The glueball spectrum is obtained by finding the normalizable solutions of this equation which are regular in the IR. The corresponding values $M^2 = -k^2$ give the masses of the bound states. With a redefinition of fields and coordinates $(u \to z, \phi \to \psi)$, similar to the one discussed in section 2.8.2, we can reduce the problem to a Schroedinger equation

$$-\psi'' + V(z)\psi = -k^2\psi, \qquad E = -k^2 = M^2 \tag{4.18}$$

The potential $V(z)$ behaves as $1/z^2$ near the boundary and goes to a finite value at $z_0 = z(u_0)$. Regularity and normalizability of the solution requires vanishing conditions at the two boundaries. We then face a standard Schroedinger problem for a well with impenetrable barriers. We find a discrete spectrum $E_i = -K_i^2 = M_i^2$ of glueball bounded from below.

Exercise: Show that the right change of coordinates is $z = \int_u^{\infty} \frac{du}{\sqrt{u^4 - u_0^4}}$ and study the resulting potential $V(z)$. Examine closely the boundary conditions. Recall that u is a polar coordinate: a zero mode, independent of the angle τ, is a smooth function in the origin only if it expands in even powers of u. Check that all normalizable solutions correspond to negative k^2 and thus positive masses.

- **Decoupling YM$_3$:** The theory on the black three-brane is $\mathcal{N} = 4$ SYM in the UV and flows to pure YM in three dimensions in the IR. We would like to take a limit where the low energy YM theory decouples from the CFT. Since the YM coupling constant

$$\frac{1}{g_3^2 N} = \frac{2\pi R_0}{g_4^2 N}\,. \tag{4.19}$$

should remain finite in the decoupling limit, we need to take $R_0 \to 0$ with $x = N g_4^2 \to 0$. Unfortunately, this is a limit where we can not trust supergravity. In fact, we have a weakly coupled gravitational background only when $x = N g_4^2 \gg 1$ (recall that the coupling constant on a D3 brane is related to the string coupling by $g_4^2 = 4\pi g_s$). In order to describe the pure YM theory in three-dimensions we need to use a strongly coupled string background. The supergravity solution describes a theory that reduces in the IR to pure three dimensional YM theory but has an UV completion which is a strongly coupled four-dimensional conformal gauge theory. The indications that the supergravity solution is not really describing a pure YM theory are strong. In a YM theory we expect a spectrum of bound states of arbitrary spin with masses of the order of the dynamically generated scale Λ which also fixes the scale of the string tension. In the supergravity approximation, however, the masses of bound states computed in supergravity are of order $m^2 \sim u_0^2$ which is much smaller than the string tension $\tau \sim \sqrt{4\pi g_s N} u_0^2$; moreover, there is a large separation of scale between the masses of the light bound states with maximum spin two, described by the supergravity fields, and the ones with greater spin, which are described by string states.

We might consider the supergravity solution as a description of pure YM with a finite cut-off $\Lambda \sim M$. The situation is similar, in spirit, to a lattice computation at strong coupling. To remove the cut-off, it would be sufficient to re-sum all world-sheet α' corrections in the string background. World-sheet corrections are, in principle, more tractable than loop corrections. In flat space, for example, all the α' corrections are computable. In the AdS case, the analogous computation is made difficult by the presence of RR-fields and at the moment it is outside our technical abilities.

Quite surprisingly, although the theory is not really pure YM, a numerical evaluation of the ratio of glueball masses for the black D3 brane gives results in good quantitative agreement with the lattice results for the three dimensional YM theory [54].

4.3 Examples of non conformal theories with gravitational dual

Various methods have been used in order to construct string duals of non-conformal gauge theories. Here we list the main ones and we refer to the reviews [11, 13] and references therein for a detailed discussion. There are very few solutions dual to non conformal theories which are completely regular.

- Finite temperature: Historically this was the first example of non-conformal gauge/gravity duals. We already briefly discussed the case of $\mathcal{N} = 4$ SYM at finite temperature. This method necessarily implies a dimensional reduction from an higher dimensional theory and the resulting theory is typically not supersymmetric.

- Deformations: we can break conformal invariance by deforming the AdS background. The gauge theory becomes conformal in the UV and the gravity background will asymptote to $AdS_5 \times H$. Most of the massive deformations of $\mathcal{N}=4$ SYM can be studied in this way. One notable example is the $\mathcal{N}=1$ massive deformation of $\mathcal{N}=4$ SYM, which is called the $\mathcal{N}=1^*$ theory. We still do not have a complete supergravity description of this theory. A solution can be obtained as a RG flow in five-dimensional gauged supergravity [55] but it is singular even when lifted to 10 dimensions [56]. A 10 dimensional approach has been pursuit in [57]; only the UV and IR behaviours of the solution are known.

- Fractional and Wrapped Branes: this method applies to branes wrapping vanishing cycles in singular internal geometries (fractional branes) or non trivial cycles in regular ones (wrapped branes). It has been the most successful way of obtaining regular solutions dual to $\mathcal{N}=1$ confining YM theories. The Klebanov-Strassler solution [28] uses fractional branes and it is based on a $SU(N+M) \times SU(N)$ gauge theory with bi-fundamental fields in four dimensions. It is asymptotic in the UV to a logarithmic deformations of AdS_5. The Maldacena-Nunez solution [29] uses wrapped branes and decompactifies to higher dimensions in the UV. Both theories are believed to reduce to an $SU(M)$ SYM in the IR. The gravity description is based in both cases on a background with bounded warp factor which is geometrically $\mathbb{R}^7 \times S^3$ in the IR. The spontaneous breaking of chiral symmetry, the existence of domain walls and confining strings can be explicitly verified in these backgrounds and are related to the behaviours of various types of D-branes.

4.4 The decoupling problem

All the weakly coupled supergravity solutions considered so far describe YM theories with many non decoupled massive modes.

- For example, the YM_3 of the black-brane example becomes $\mathcal{N}=4$ SYM in the UV. As already discussed, the supergravity limit is valid only when the $\mathcal{N}=4$ SYM is strongly coupled; in the same regime we cannot decouple the UV and IR regimes.

- Analogously pure $\mathcal{N}=1$ SYM can be embedded in a massive deformation of $\mathcal{N}=4$ (the $\mathcal{N}=1^*$ theory) and has a supergravity description. Even in this case we cannot decouple the massive modes without leaving the supergravity regime.

A mass deformation m indeed induces a dimensional scale $\Lambda \sim me^{-1/Ng_{YM}^2}$. Once again the decoupling limit is $m \to \infty$, $x = Ng_{YM}^2 \to 0$, with Λ fixed, and this is the opposite of the supergravity limit.

- A similar argument, although more complicated, applies to the Klebanov-Strassler solution, where one can decouple the pure YM theory in the IR from its complicated UV completion only at the price of leaving the supergravity regime.

The general lesson is that supergravity describes gauge theories which are strongly coupled at all scales. We learned indeed that the curvature of the background is inversely proportional to the strength of the coupling constant. When the curvature is small the coupling constant should be large. Within this general picture, QCD at large N, which is asymptotically free and becomes weakly coupled in the UV, cannot have a supergravity description; the QCD dual is a string theory on a strongly curved background. This expectation is also strengthened by simple considerations about the spectrum of bound states. In QCD we expect Regge trajectories of bound states with masses linearly increasing with the spin. On the other hand, in any effective supergravity limit of string theory we only have states with maximum spin two; all the other stringy states are separated by a very large energy gap. This clearly does not fit QCD expectations. The path to the real large N QCD involves quantizing tree level string theory on a strongly coupled background. As we already mention, we are not yet able to solve string theory on curved backgrounds, but this is a technical problem that we might hope to overcome in the future.

However, by engineering gauge theories on D-branes, we can construct many stringy-inspired modifications of pure gauge theories and we have a large moduli space of QCD-like theories. Some of these theories correspond to gauge theories with weakly coupled dual backgrounds. These theories have field content different from QCD or modified couplings and Lagrangian. In particular, they substantially differ from QCD in the UV where they are strongly coupled. Typically new massive fields are added in such a way that the UV completion has a weakly coupled holographic dual. The theory may become conformal or decompactify in the UV. The IR behavior instead can be very similar to that of QCD. We can use these exactly solvable theories as extremely efficient toy models for studying the strong dynamics of non abelian gauge theories, from confinement to chiral symmetry breaking.

Acknowledgments

I would like to thank the audience of these lectures in EPFL and in particular R. Rattazzi and M. Redi for many interesting comments and creative interruptions. I also thank Claudius Klare for reading the manuscript. This work is supported in part by INFN and MIUR under contract 2007-5ATT78-002.

A Appendix: The superconformal group

In this Appendix we include few information about representations of the (super) conformal group in four dimensions. For more information the reader is referred to [58–60] and references therein.

The CFT fields and operators are usually classified by quantum numbers under the non-compact sub-group $SO(1,2) \times S(1,3) \subset SO(2,4) \sim SU(2,2)$. As one can see from the algebra, P_μ and K_μ act as raising and lowering operators for D. The lowest state of a representation will be annihilated by K. Every irreducible (infinite dimensional) representation is then specified by an irreducible representation of the Lorentz group with definite conformal dimension and annihilated by K_μ. In terms of the operator in $x = 0$, stabilized by the sub-algebra $K_\mu, D, M_{\mu\nu}$, we define a *primary* conformal field, in the (j_L, j_R) representation of the Lorentz group, by

$$\begin{aligned} [D, O_{(j_L,j_R)}(0)] &= i\Delta O_{(j_L,j_R)}(0) \\ \left[K_\mu, O_{(j_L,j_R)}(0)\right] &= 0. \end{aligned} \tag{A.1}$$

The descendants $\partial....\partial O_{(j_L,j_R)}(0)$ reconstruct the operator by Taylor expansion and fill an infinite dimensional representation specified by three numbers (Δ, j_L, j_R), corresponding to the conformal dimension and the Lorentz quantum numbers of the primary operator.

Since the operators $O_{(j_L,j_R)}(0)$ lead to non normalizable states when applied to the vacuum [22], it is useful to consider the maximal compact subgroup $SO(2) \times SO(4) \subset SO(2,4)$. Using this compact subgroup, we can study and classify unitary representations of the conformal group using states with finite norm. The choice

[22]This is necessary, since otherwise these states would realize a finite dimensional unitary representation of a non compact group (the stability group at $x = 0$) which is forbidden by group theory; this is also the reason why the necessary i in the D commutator in A.1 does not contradict the hermiticity of D.

of this subgroup is also natural in AdS_5. In this picture, we classify states though the eigenvalues of $H = (P_0 + K_0)/2$ and $SO(4) = SU(2) \times SU(2)$, identified with (Δ, j_L, j_R) respectively. This choice of quantum numbers is also useful when, upon Euclidean continuation, we radially quantize the theory on $S^3 \times \mathbb{R}$.

The superconformal group $SU(2,2|N)$ corresponding to a theory with $\mathcal{N}$ supersymmetries is obtained by adding $\mathcal{N}$ supercharges Q^a_α, $\mathcal{N}$ superconformal charges S^α_a and the generators of a $U(\mathcal{N})$ global symmetry R^a_b. Some of the relevant commutation relations are

$$\begin{aligned}
[D, Q^a_\alpha] &= \frac{i}{2} Q^a_\alpha \\
[D, S^\alpha_a] &= -\frac{i}{2} S^\alpha_a \\
[K_\mu, Q^a_\alpha] &= -(\sigma_\mu)_{\alpha\dot{\alpha}} \bar{S}^{\dot{\alpha}}_a \\
[P_\mu, S^\alpha_a] &= (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} \bar{Q}_{\dot{\alpha}\, a} \\
\left[Q^a_\alpha, \bar{Q}_{\dot{\alpha}\, b}\right] &= 2\delta^a_b (\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu \\
\left[\bar{S}^{\dot{\alpha}\, a}, S^\alpha_b\right] &= 2\delta^a_b (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} K_\mu \\
\left\{Q^a_\alpha, S^\beta_b\right\} &= -i\delta^a_b (\sigma^\mu \bar{\sigma}^\nu)^\beta_\alpha J_{\mu\nu} - 2i\delta^a_b \delta^\beta_\alpha D - 4\delta^\beta_\alpha R^a_b
\end{aligned} \tag{A.2}$$

where $(Q^a_\alpha)^\dagger = \bar{Q}_{a\,\dot{\alpha}}$, $(S^\alpha_a)^\dagger = \bar{S}^{a\dot{\alpha}}$. The first two lines specify the dimension of the charges; notice that Q is a raising operator while S is a lowering operators for D. The next two lines clarify why we need to introduce S to close the algebra. Next we have the standard commutation relation defining supersymmetry and after that its conformal partner. There are other equations that specify the R-symmetry quantum numbers of the various quantities,

$$[R^a_b, Q^c_\alpha] = \delta^c_b Q^a_\alpha - \frac{1}{4}\delta^a_b Q^c_\alpha \qquad [R^a_b, S^\alpha_c] = -\delta^a_c S^\alpha_b + \frac{1}{4}\delta^a_b S^\alpha_c \tag{A.3}$$

The R^a_b are generators of $U(\mathcal{N})$ and close the corresponding algebra. For $\mathcal{N} = 4$, the trace R^a_a decouples from the algebra and become an outer automorphism. Correspondingly the R-symmetry is $SU(4)$.

Superconformal representation have a lowest state which is annihilated by both the lowering operators K and S; it is identified by the quantum numbers Δ, j_L, j_R under the conformal group and $R, a_1, \cdots a_{N-1}$ under the R-symmetry $U(1) \times SU(N)$. The infinite dimensional representation is obtained by acting on the lowest state with an infinite numbers of P and Qs. Since the Q are fermionic charges, only a finite numbers of applications of Qs give a non-vanishing result. We can keep track of the action of the supercharges by introducing superfields $\Phi_{(j_l,j_R,R,a)}(x,\theta^a_\alpha,\bar\theta^a_{\dot\alpha})$ where the θ^a are superspace coordinates in the $\underline{\mathcal{N}}$ of $SU(\mathcal{N})$; the expansion in θ has generically $2^{4N}r$ components, where r is the dimension of the Lorentz and R-symmetry representation, with a generic spin range of $\Delta j_L = \Delta j_R = N/2$.

Unitarity imposes bounds on the representations. These are obtained by imposing that all the states in a representation have positive norm. The saturation of the bounds corresponds to representations with null states which can be removed leaving a shorter representation. The vanishing of a particular state implies some differential constraints on the primary field (it is annihilated by some polynomial in P and Q) which indeed make the representation shorter.

For the conformal group non trivial operators must satisfy,

$$\begin{aligned} &\Delta \geq 1 + j_R \qquad j_L = 0\,, \qquad \text{or}\,(j_R \rightarrow j_L) \\ &\Delta \geq 2 + j_L + j_R \qquad (j_L j_R \neq 0) \end{aligned} \tag{A.4}$$

The two unitarity thresholds are satisfied by free massless fields and conserved tensors field, respectively. The saturation of the bounds corresponds in fact to

$$\begin{aligned} &\partial^2 \Phi_{(0,j_R)} = 0 \\ &\partial^{\alpha_1\dot\alpha_1} O_{\alpha_1..\alpha_{2j_L},\dot\alpha_1..\dot\alpha_{2j_R}} = 0 \end{aligned} \tag{A.5}$$

The first bound for $j_R = 0$ say that the dimension of a scalar primary operator is greater than one and it can be one if and only if the field is free. The second bound for $j_L = j_R = 1/2$ says that a spin one operator J_μ has dimension greater than 3 and equal to 3 if and only if it is a conserved current, $\partial^\mu J_\mu = 0$. Analogously, a spin two operator $T_{\mu\nu}$ ($j_L = j_R = 1$) has dimension greater than 4 and equal to 4 if and only if it is conserved.

The superconformal representations are richer but similar in spirit. Shortening now may correspond to annihilation of the primary operator by some supersymmetry charge or some combinations of charges. An example that can be familiar to the reader is that of a chiral superfields in $\mathcal{N} = 1$ supersymmetry $\bar D_\alpha \Phi = 0$. Chiral superfields are annihilated by half of the supersymmetries and the corresponding multiplet are short, depending only by θ and not by $\bar\theta$ (modulo subtleties). This shortening corresponds to a saturation bound fixing the dimension of the operator in terms of its R-charge

$$\Delta = \frac{3}{2}R \tag{A.6}$$

where, as customary, we normalized the R charge of Q to one. While the generic scalar superfield has a spin range of 1, a chiral superfields has spin range 1/2.

Just for curiosity,we can report the full unitarity bounds for non -trivial $SU(2,2|1)$ representations,

$$
\begin{array}{llll}
a) & \Delta \geq 2+2j_R+\frac{3}{2}R \geq 2+2j_L-\frac{3}{2}R & & (\text{or } j_L \to j_R,\ R \to -R) \\
b) & \Delta = \frac{3}{2}R \geq 2+2j_L-\frac{3}{2}R\,, & j_R=0 & (\text{or } j_L=0,\ j_R=j,\ R \to -R)\,.
\end{array}
$$

where R is the $U(1)$ R-charge. We will not discuss in detail all the varieties of shortening. The chiral superfield is contained in case b) as well as some generalizations. Case a) contains conserved supercurrents

$$D^{\alpha_1}L_{\alpha_1\cdots\alpha_{2j_L}\dot\alpha_1\cdots\dot\alpha_{2j_R}} = \bar D^{\alpha_1}L_{\alpha_1\cdots\alpha_{2j_L}\dot\alpha_1\cdots\dot\alpha_{2j_R}} = 0 \tag{A.7}$$

with $\Delta = 2+j_L+j_R$ and various other shortenings (for example semiconserved superfields ($\bar D^{\alpha_1}L_{\alpha_1\cdots\alpha_{2j_L}\dot\alpha_1\cdots\dot\alpha_{2j_R}} = 0$)). All these short as well as some long representation appear in the KK spectrum in generic $\mathcal{N}=1$ AdS_5 background.

The $\mathcal{N}=4$ superalgebra $SU(2,2|4)$ has analogously a variety of representations with different unitary bounds. The lowest state is characterized by $(\Delta, j_L, j_R, p, k, q)$, where p,k,q are the Dynkin labels of an $SU(4)$ representation. They correspond to $SU(4)$ Young tableaux with rows of length $p, p+k, p+k+q$ starting from the bottom. With this notation, the fundamental $\underline{4}$ is $[1,0,0]$, the anti-fundamental $\underline{\bar 4}$ is $[0,0,1]$ and the $\underline{6}$ (antisymmetric of $SU(4)_R$ and vectorial for $SO(6)$) is $[0,1,0]$. A generic long representation has now spin range 4. Short multiplets of $\mathcal{N}=4$ have

$$\Delta = 2p+k\,, \qquad p=q\,, \qquad j_L=j_R=0 \tag{A.8}$$

Consider first the case $p=0$. These are the so-called *chiral primary* operators and generalize the chiral operators of the $\mathcal{N}=1$ case. They are $\frac{1}{2}$ BPS representations, with maximum spin two and range of dimensions $\Delta_{\rm max}-\Delta_{\rm min}=4$. The case $k=1$ has a further shortening and has maximum spin 1; it is called a singleton representation. It can be described by a superfield $W_{[AB]}(x,\theta,\bar\theta)$, $A=1,..,4$ which satisfies

$$
\begin{aligned}
W_{[AB]} &= \tfrac{1}{2}\epsilon_{ABCD}\bar W_{[CD]} \\
\mathcal{D}_{\alpha A}W_{[BC]} &= \mathcal{D}_{\alpha[A}W_{BC]}\,.
\end{aligned} \tag{A.9}
$$

In a suitable harmonic superspace it becomes a twisted chiral superfield. W contains, as first component, a set of six scalars $\phi_{[AB]}$ in the 6 of SU(4), which will be denoted also $\phi_l, l = 1, .., 6$. The superfield itself will be also denoted W_l. The full multiplet contains six scalars, four Weyl fermions and a gauge vector; this is the field content of the elementary fields in $\mathcal{N} = 4$ SYM. All other representations with $k \geq 2$ have maximum spin 2 and can be constructed by tensor products of singletons as $W^k = W_{\{l_1} ... W_{l_k\}} -$traces. One can show that these multiplets are short. Their dimension is then $\Delta = k$. In the main text, we called them A_k. The special case $k = 2$ has a further shortening with respect to $k > 2$ since it contains conserved currents. The series of multiplets A_k with $k \geq 2$ is in one-to-one correspondence with the KK spectrum on S^5. $k = 1$ does not appear indicating that the theory is $SU(N)$ instead of $U(N)$. We could have predicted from the very beginning that all the KK modes correspond to short multiplets because their maximal spin range is 2. Representations with $p \neq 0$ are $\frac{1}{4}$ BPS with $\Delta_{\max} - \Delta_{\min} = 6$ and appear when multi-trace operators in CFT (or multi-particle states in the bulk) are considered.

There are also more complicated semishort multiplets of the $\mathcal{N} = 4$ superconformal algebra, which play a role in the operator product expansion of chiral primary operators.

References

[1] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[2] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," Adv. Theor. Math. Phys. **2**, 505 (1998) [arXiv:hep-th/9803131].

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[4] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[5] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[6] J. M. Maldacena, "TASI Lectures on AdS/CFT," arXiv:hep-th/0309246.

[7] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS/CFT correspondence," arXiv:hep-th/0201253.

[8] K. Skenderis, "Lecture notes on holographic renormalization," Class. Quant. Grav. **19**, 5849 (2002) [arXiv:hep-th/0209067].

[9] C. P. Herzog, I. R. Klebanov and P. Ouyang, arXiv:hep-th/0205100.

[10] A. Zaffaroni, PoS **RTN2005**, 005 (2005).

[11] F. Bigazzi, A. L. Cotrone, M. Petrini and A. Zaffaroni, Riv. Nuovo Cim. **25N12**, 1 (2002) [arXiv:hep-th/0303191].

[12] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics," arXiv:0903.3246 [hep-th].

[13] C. P. Herzog, "Lectures on Holographic Superfluidity and Superconductivity," arXiv:0904.1975 [hep-th].

[14] J. McGreevy, "Holographic duality with a view toward many-body physics," Adv. High Energy Phys. **2010**, 723105 (2010) [arXiv:0909.0518 [hep-th]].

[15] M. Ammon and J. Erdmenger, 'Gauge/gravity duality: Foundations and applications," Cambridge University Press, 2015.

[16] P. Di Francesco, P. Mathieu and D. Senechal, "Conformal Field Theory", Springer.

[17] S. Coleman, '1/N' in "Aspects of Symmetry", Cambridge University Press.

[18] E. D'Hoker, D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, "Graviton Exchange and Complete 4-Point Functions in the AdS/CFT Correspondence," Nucl. Phys. B **562** (1999) 353 [arXiv:hep-th/9903196].

[19] P. Breitenlohner and D. Z. Freedman, Phys. Lett. B **115**, 197 (1982).

[20] I. R. Klebanov and E. Witten, "AdS/CFT correspondence and symmetry breaking," Nucl. Phys. B **556**, 89 (1999) [arXiv:hep-th/9905104].

[21] V. Balasubramanian, P. Kraus, A. E. Lawrence and S. P. Trivedi, "Holographic probes of anti-de Sitter space-times," Phys. Rev. D **59**, 104021 (1999) [arXiv:hep-th/9808017].

[22] M. Henningson and K. Skenderis, "The Holographic Weyl Anomaly," JHEP **9807** (1998) 023 [arXiv:hep-th/9806087];

[23] S. Ryu and T. Takayanagi, "Holographic derivation of entanglement entropy from AdS/CFT," Phys. Rev. Lett. **96** (2006), 181602 [arXiv:hep-th/0603001 [hep-th]].

[24] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity," Commun. Math. Phys. **208** (1999), 413-428 [arXiv:hep-th/9902121 [hep-th]].

[25] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, JHEP **9812**, 022 (1998) [arXiv:hep-th/9810126].

[26] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, Adv. Theor. Math. Phys. **3**, 363 (1999) [arXiv:hep-th/9904017].

[27] D. Anselmi, D. Z. Freedman, M. T. Grisaru and A. A. Johansen, Nucl. Phys. B **526**, 543 (1998) [arXiv:hep-th/9708042].

[28] I. R. Klebanov and M. J. Strassler, "Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities," JHEP **0008**, 052 (2000) [arXiv:hep-th/0007191].

[29] J. M. Maldacena and C. Nunez, "Towards the large N limit of pure N = 1 super Yang Mills," Phys. Rev. Lett. **86**, 588 (2001) [arXiv:hep-th/0008001].

[30] J. Polchinski, "String Theory", Vol I and II, Cambridge University press.

[31] B. de Wit, "Supergravity," arXiv:hep-th/0212245.

[32] K. S. Stelle, "BPS branes in supergravity," arXiv:hep-th/9803116.

[33] J. Polchinski, "Lectures on D-branes," arXiv:hep-th/9611050.

[34] H. J. Kim, L. J. Romans and P. van Nieuwenhuizen, Phys. Rev. D **32**, 389 (1985).

[35] M. Gunaydin and N. Marcus, Class. Quant. Grav. **2**, L11 (1985).

[36] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, Adv. Theor. Math. Phys. **2**, 697 (1998) [arXiv:hep-th/9806074].

[37] R. G. Leigh and M. J. Strassler, Nucl. Phys. B **447**, 95 (1995) [arXiv:hep-th/9503121].

[38] O. Aharony, B. Kol and S. Yankielowicz, JHEP **0206**, 039 (2002) [arXiv:hep-th/0205090].

[39] O. Lunin and J. M. Maldacena, JHEP **0505**, 033 (2005) [arXiv:hep-th/0502086].

[40] A. Khavaev, K. Pilch and N. P. Warner, Phys. Lett. B **487**, 14 (2000) [arXiv:hep-th/9812035].

[41] A. Karch, D. Lust and A. Miemiec, Phys. Lett. B **454**, 265 (1999) [arXiv:hep-th/9901041].

[42] K. Pilch and N. P. Warner, Phys. Lett. B **487**, 22 (2000) [arXiv:hep-th/0002192].

[43] S. Kachru and E. Silverstein, Phys. Rev. Lett. **80**, 4855 (1998) [arXiv:hep-th/9802183].

[44] I. R. Klebanov and E. Witten, Nucl. Phys. B **536**, 199 (1998) [arXiv:hep-th/9807080].

[45] D. R. Morrison and M. R. Plesser, Adv. Theor. Math. Phys. **3**, 1 (1999) [arXiv:hep-th/9810201].

[46] A. Kehagias, Phys. Lett. B **435**, 337 (1998) [arXiv:hep-th/9805131].

[47] S. Franco, A. Hanany, K. D. Kennaway, D. Vegh and B. Wecht, JHEP **0601**, 096 (2006) [arXiv:hep-th/0504110].

[48] A. Hanany and K. D. Kennaway, arXiv:hep-th/0503149.

[49] S. Franco and D. Vegh, JHEP **0611**, 054 (2006) [arXiv:hep-th/0601063].

[50] E. Barnes, K. A. Intriligator, B. Wecht and J. Wright, Nucl. Phys. B **702**, 131 (2004) [arXiv:hep-th/0408156].

[51] A. Butti and A. Zaffaroni, JHEP **0511**, 019 (2005) [arXiv:hep-th/0506232].

[52] S. S. Gubser and I. R. Klebanov, Phys. Rev. D **58**, 125025 (1998) [arXiv:hep-th/9808075].

[53] D. Gaiotto and J. Maldacena, arXiv:0904.4466 [hep-th].

[54] C. Csaki, H. Ooguri, Y. Oz and J. Terning, JHEP **9901**, 017 (1999) [arXiv:hep-th/9806021].

[55] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, Nucl. Phys. B **569**, 451 (2000) [arXiv:hep-th/9909047].

[56] K. Pilch and N. P. Warner, Adv. Theor. Math. Phys. **4**, 627 (2002) [arXiv:hep-th/0006066].

[57] J. Polchinski and M. J. Strassler, arXiv:hep-th/0003136.

[58] S. Minwalla, Adv. Theor. Math. Phys. **2**, 781 (1998) [arXiv:hep-th/9712074].

[59] S. Ferrara and A. Zaffaroni, arXiv:hep-th/9908163.

[60] S. Ferrara and A. Zaffaroni, arXiv:hep-th/9807090.