TASI Lectures on Critical Exponents and Effective Field Theories

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Contents

1. Introduction and Motivations

In quantum mechanics and quantum field theory we sum over trajectories weighted by the action

$$\langle \Phi_f(\vec{x}, t_f) | \Phi_i(\vec{x}, t_i) \rangle = \int [d\Phi]_{\Phi_i}^{\Phi_f} e^{\frac{i}{\hbar} S[\Phi]} \,. \tag{1}$$

The notation $[d\Phi]_{\Phi_i}^{\Phi_f}$ stands for a sum over all the field configurations that interpolate between $\Phi_i(\vec{x}, t_i)$ and $\Phi_f(\vec{x}, t_f)$. Classical equations emerge if we can take $\hbar \to 0$, where only the classical path survives — i.e. that path for which the action is stationary $\delta_\Phi S[\Phi] = 0$.

Unfortunately the most interesting quantum systems do not allow us to concentrate on the classical paths. That means that $\hbar$ is effectively large and, especially, that the interactions are strong.

The most interesting phenomena in modern physics are plagued with large quantum corrections. For instance, the vacuum configurations that are relevant in Yang-Mills theory are subject to very large quantum corrections and indeed we observe highly non-classical phenomena such as the formation of color flux tubes. Similarly, the field configurations that dominate the macroscopic properties of the Φ^4 model

$$S[\Phi] = \int d^2x dt \left(\frac{1}{2} (\dot{\Phi})^2 - \frac{1}{2} (\nabla \Phi)^2 - \lambda \Phi^4 \right) \,. \tag{2}$$

are susceptible to large quantum corrections and we have essentially no analytic control. There are many other, similar, examples in condensed matter physics – we will delve into some of them in detail later.

Here we will explore the applications of a very general idea: *In states with large quantum numbers N, quantum effects can be substantially reduced*

$$\boxed{\hbar_{eff} = \hbar/N^{\#}} \tag{3}$$

This is a version of Bohr's correspondence principle. Therefore, while the macroscopic properties in the vacuum may be out of analytic control, some excited states appear to be semi-classical.

After demonstrating how this general idea works in some relatively simple quantum systems, we will use this general ideology to tackle questions about conformal field theory (CFT). CFTs describe quantum and classical second order phase transitions and have countless applications.

CFTs epitomize the problem of strong interactions. The strong interactions in CFTs never shut down, not even on very large distance scales! By contrast, in Yang-Mills theory, the strong coupling dynamics is pretty much limited to distance scales around the string tension.

CFTs are characterized scaling exponents and operator product expansion (OPE) coefficients $\{\Delta_i, C_{ijk}\}$. The smallest scaling exponents Δ_i determine the response of the system to simple external fields. They are very well measured in many cases.

Here we will be interested in "heavy operators" with

$$\Delta_i \gg 1 \ . \tag{4}$$

Recently, some of these large scaling exponents were extracted in simulations and therefore we will be able to make contact with "experimental" data.

Our aim here is to apply the theory of semi-classical techniques at large quantum numbers to unveil the physics of some of the simplest heavy operators. Therefore, despite CFTs being the strongest interacting systems imaginable, we will be still able to utilize semi-classical techniques and make concrete predictions about some heavy operators.

On the way, we will be led to surprising new qualitative insights. To whet the appetite we mention some of the main points below and then jump straight into the details

- In an appropriate sense, strongly coupled theories become *asymptotically free* at small angles! This will lead to many predictions about heavy operators.

- The $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence holds "exactly" for any CFT_d, as long as one focuses on the physics of certain heavy operators.
- Heavy operators exhibit scaling laws that are familiar from Black-Hole physics, and if there is particle number symmetry, heavy operators also contain the physics of vortices familiar from superfluid theory.
- The methods are applicable for conformal impurities and boundaries, which would allow us a fresh insight into the Kondo problem and into higher dimensional extensions of the Kondo problem.
- Remarkably, the predictions for heavy operators work well when extrapolated deep into the strongly interacting regime of light operators with $\Delta \sim 1$.
- In some cases (3), can be understood as a double scaling limit where $\hbar \to 0$, $N \to \infty$ with fixed $\hbar_{eff}$. This is a compelling picture where heavy operators are made of many quanta that are assembled in a random matrix and such heavy operators are classical for the same reason that large random matrices are classical.

2. An Invitation to Large Quantum Numbers

In this section we will discuss a quantum mechanical problem of identical non-relativistic particles interacting with each other via a two-body δ function repulsive potential. This is just a Schrödinger equation for N particles.

For the spectrum to be discrete and to prevent the particles from flying out to infinity we will put the particles in some symmetric trap with potential $V_{tr}(\vec{r})$.

For example the Hamilotnian for two particles is

$$H = -\frac{\hbar^2}{2M}\nabla_1^2 - \frac{\hbar^2}{2M}\nabla_2^2 + V_{tr}(\vec{r}_1) + V_{tr}(\vec{r}_2) + V_{short-range}(\vec{r}_1 - \vec{r}_2)\ . \quad (5)$$

We have to specify the short range interaction. We will take it to have scattering length a, for instance, we could take it to be

$$V_{short-range}(\vec{r}_1 - \vec{r}_2) = \begin{cases} \infty & \text{if } |\vec{r}_{12}| < a \\ 0 & \text{if } |\vec{r}_{12}| \geq a \end{cases} \quad (6)$$

For a harmonic trap, the problem (5) can be easily solved analytically by the usual transformation to the center of mass.

A further simplification on top of (11) comes from considering the phase shift at small momentum $\delta_\ell = -\frac{(ka)^{2\ell+1}}{(2\ell+1)!!(2\ell-1)!!} + \cdots$. We see that this quickly shuts off at large ℓ and small momentum due to the centrifugal barrier that does not allow to probe the short range potential. The physics is dominated by s-wave scattering with $\delta_0 = -ka$.

Therefore what we will really do is to imagine that $V_{short-range}(\vec{r}_1 - \vec{r}_2)$ is such that there is scattering with scattering length a in the s-wave but no scattering whatsoever for $\ell \geq 1$. The analysis has a large degree of universality, and would be valid for vast variety of potentials as long as the gas is dilute enough etc. – as we will soon discuss. An actual short range "potential" which only leads to a contact interaction in the s-wave is the Fermi pseudo-potential, which is widely used to model short range potentials in nuclear physics and the physics of cold atoms.

Instead of solving the Schrödinger problem for 2 particles and then 3 particles etc, we switch to the second quantization formalism of this non-relativistic problem and consider the action

$$S = \int d^3x dt \left[i\hbar \Psi^\dagger \frac{\partial}{\partial t}\Psi - \Psi^\dagger(-\frac{\hbar^2}{2M}\nabla^2 + V_{tr})\Psi - \frac{1}{2} g \Psi^\dagger\Psi^\dagger\Psi\Psi \right] . \quad (7)$$

Since particle number is given by

$$N = \int d^3x \Psi^\dagger \Psi , \quad (8)$$

we see that the dimension of Ψ is $length^{-3/2}$. The coupling constant that appears in the action is $g = \frac{4\pi\hbar^2 a}{M}$ has the dimension of length. (The story in 2+1 dimensions is more subtle since there are logarithmic corrections to g, which is dimensionless – we won't discuss it here.)

The symmetries of the problem (7) in the absence of a trapping potential is the Galilean symmetry and particle number symmetry $\Psi \to e^{i\alpha}\Psi$.

An important questions to understand is what are the eigenstates at large N, and in particular, what is the ground state and its lowest-lying excitations. To analyze this question we will use the insight that states with large quantum numbers are nearly classical. To see that we rescale the wave function by $\sqrt{N}$, and rescale $g' = gN/\hbar = \frac{4\pi\hbar N a}{M}$ to obtain

$$\frac{1}{\hbar} S = N \int d^3x dt \left[i \Psi^\dagger \frac{\partial}{\partial t}\Psi - \Psi^\dagger(-\frac{\hbar}{2M}\nabla^2 + \frac{1}{\hbar}V_{tr})\Psi - \frac{1}{2} g' \Psi^\dagger\Psi^\dagger\Psi\Psi \right] , \quad (9)$$

and the constraint is $\int d^3x \Psi^\dagger \Psi = 1$. This is our first example of (3), and we see that $\hbar_{eff} = \hbar/N$ because N appears outside of the integral in (9).

Due to the factor of N outside of (9), the action would be dominated by classical trajectories. This scaling of the action and g' is not obviously physical since g' clearly scales with N. The point is that the wave function is order of $length^{-3/2}$ and hence the quartic term is of order $g'/volume$ and that is of order $Na/volume \sim na$ where n is the density. We will work in the limit where na, which is essentially a measure of diluteness in units of the interaction length, is not large. This is why the formalism we develop is appropriate for dilute gases. We can define

$$\xi = (an)^{-1/2} \tag{10}$$

which is an important length scale which is called the healing length. This is the only length scale in the absence of a trapping potential and it signifies the typical size of a vortex. The semi-classical limit is therefore rigorously justified at fixed healing length and $N \to \infty$. This is the sort of a double scaling limit which leads to classical physics.

The simplest trapping potential is a spherical box of radius L.

$$V_{tr} = \begin{cases} 0 & \text{if } |\vec{r}| < L \\ \infty & \text{if } |\vec{r}| \geq L \end{cases} \tag{11}$$

A classical solution is $\Psi = \frac{1}{\sqrt{V}} e^{-i\mu t}$, with $\mu = g'/V$. The energy of this solution is linear in the density $E \sim n$. The pressure is obtained by varying the volume with a fixed number of particle and we find $P = -\frac{\partial E}{\partial V} = \frac{1}{2} n^2 g$. And the speed of sound is given from $c_s^2 = \frac{1}{M} \frac{\partial P}{\partial n} = \frac{1}{M} ng$. This is why the repulsive interaction is crucial – without it, the gas has no sound.

Representative values are as follows, for sodium-23 (^{23}Na)

$$a_s \sim 2.75{\times}10^{-9}\,\text{m}, \quad n \sim 1.0{\times}10^{20}\,\text{m}^{-3}, \quad \xi \sim 3{\times}10^{-7}\,\text{m} \quad m_{^{23}\text{Na}} \sim 3.8{\times}10^{-26}\,\text{Kg}\,.$$

To understand the validity of the classical solution we now expand in fluctuations, with the following ansatz

$$\Psi = \frac{1}{\sqrt{V}} e^{-i\mu t} + e^{-i\mu t} \sum_k m_k(t) e^{ikx}\ . \tag{12}$$

Since we are expanding around a background with $\langle \Psi \rangle \neq 0$, we should already anticipate that $m_k(t)$ will contain both creation and annihilation operators, even though the original field Ψ had well defined particle number charge. This is standard when we expand around a solution that breaks the symmetry spontaneously.

Plugging in our mode expansion (12), the quadratic theory for $m_k(t)$ is

$$\left(m_k^* \; m_{-k}\right) \mathcal{M} \begin{pmatrix} m_k \\ m_{-k}^* \end{pmatrix}\ , \quad \mathcal{M} = \begin{pmatrix} i\frac{d}{dt} - \frac{\hbar}{2M}k^2 - \frac{g'}{V} & -\frac{g'}{V} \\ -\frac{g'}{V} & -i\frac{d}{dt} - \frac{\hbar}{2M}k^2 - \frac{g'}{V} \end{pmatrix} \tag{13}$$

The off diagonal terms $\sim m_{-k}m_k + c.c.$ clearly break particle number symmetry since we are expanding around a solution that breaks the symmetry spontaneously. The insight of how to quantize such a system is due to Nikolay Bogolyubov, who observed that the commutation relations $[m^*_{k'}(t), m_k(t)] = \delta_{kk'}$ allow to write each of the modes m_k as

$$m_k(t) = v_k a_k e^{i\omega t} + u_{-k} a^\dagger_{-k} e^{-i\omega t} \ ,$$

with $|v_k|^2 - |u_{-k}|^2 = 1$ and with $a, a^\dagger$ canonically normalized creation and annihilation operators. We get

$$\omega(k) = \sqrt{\frac{g'\hbar}{VM}k^2 + \frac{\hbar^2}{4M^2}k^4} \longrightarrow_{|k|\to 0} \sqrt{\frac{g'\hbar}{VM}}|k| = \sqrt{4\pi}\frac{\hbar}{M\xi}|k| \ . \quad (14)$$

This is a sound mode and the speed of sound that we can read from it agrees with the computation using thermodynamics. On the other hand the particles do not see the background condensate and we have the dispersion relation of ordinary non-relativistic free particles $\omega \to_{|k|\to\infty} \frac{\hbar k^2}{2M}$. Also, at large momentum, m_k becomes almost purely a destruction operator with a very small contribution from creation operators.

We have just described how a dilute gas becomes a superfluid at large particle number. We have also discussed the excitations around the ground state and found their dispersion relation. That dilute gases with repulsion indeed behave as superfluids at low temperatures was the subject of the 2001 Nobel Prize in Physics.

Now we cover a few more topics concerning superfluid theory and dilute gases that would be important later.

2.1. *Effective Theory and the Galilean Symmetry*

We observe in (9) that V_{tr} has a very interesting interpretation – it is the A_0 background gauge field component for particle number symmetry.

We have seen in (14) that we have a sound mode at low energies and it is useful to write an effective theory for it. Since we are expanding around some nonzero $\langle\Psi\rangle$, the low-energy mode is the angle variable φ which transforms under particle number symmetry inhomogenously $\varphi \to \varphi + \alpha$.

Using the Galilei symmetry and the fact that the trap potential is A_0, we see that an interesting combination is $X = \partial_t\varphi - V_{tr} - \frac{(\nabla\varphi)^2}{2M}$ and an allowed term in the effective theory is

$$S = \int d^3x dt P(X) \ , \quad (15)$$

with P the pressure. The simplest equation of state is $P \sim X^2$ and that is what happens in our dilute gas model at leading order.

The effective action has higher derivative terms in addition to (15). But the pressure term is the leading one. The fact that we can determine the dependence on V_{tr} in the low energy theory for the sound mode is crucial in thinking about general superfluid theory.

2.2. *Vortex Solutions*

Now let us consider the vortex solution of superfluid theory. The ansatz is of the form $\Psi = e^{-i\mu t} e^{i\theta} \xi(r_\perp)$ with a normalized $\int dV \xi^2 = 1$. The formula for the angular momentum reads

$$L_z = \int dV r_\perp^2 n M v^\phi = \frac{\hbar N}{2i} \int dV (\Psi^\dagger \partial_\theta \Psi - c.c.) = \hbar N \ , \tag{16}$$

where we have used that the current density is $j = \frac{\hbar}{2Mi}(\Psi^\dagger \nabla \Psi - c.c.)$. Note that the angular velocity behaves like $v^\phi \sim 1/r^2$, in contrast to a rigid body, where it is constant. The velocity of rotation near the core is very large and decays at long distances.

To estimate the energy of this configuration we will take into account only the contribution from the kinetic term, and specifically the centrifugal term, which reads $N \int dV \frac{\hbar^2}{2M} \frac{1}{r_\perp^2} \chi^2 \sim \frac{\hbar^2 N L}{MV} \log(R/\xi) \sim \frac{\hbar^2 n L}{M} \log(R)$. R is the transverse size to the vortex and L is the length of the vortex. This shows that the vortex tension is divergent as the volume of the system goes to infinity. For finite systems the vortices are finite energy and very important in dynamics.

2.3. *Harmonic Trap*

Let us consider a harmonic trap $V_{tr} = \frac{1}{2} M \omega^2 \sum \alpha_i^2 x_i^2$ with $\prod_i \alpha_i^2 = 1$. Starting from the our usual action (9)

$$\frac{1}{\hbar} S = N \int dV dt \left[i\Psi^\dagger \frac{\partial}{\partial t} \Psi - \frac{1}{2} \Psi^\dagger (-\frac{\hbar}{M} \nabla^2 + \frac{M\omega^2}{\hbar} \sum \alpha_i^2 x_i^2) \Psi - \frac{1}{2} g' \Psi^\dagger \Psi^\dagger \Psi \Psi \right] , \tag{17}$$

we now define dimensionless units for time and space via along with another redefinition of the coupling constant

$$x = \sqrt{\frac{\hbar}{\omega M}} \tilde{x} \ , \quad t = \omega^{-1} \tilde{t} \ , \quad g' = \omega g'' \ ,$$

to find

$$\frac{1}{\hbar}S = N\left(\frac{\hbar}{\omega M}\right)^{3/2}\int d\tilde{V}d\tilde{t}\left[i\Psi^\dagger\frac{\partial}{\partial\tilde{t}}\Psi - \frac{1}{2}\Psi^\dagger(-\tilde{\nabla}^2 + \sum\alpha_i^2\tilde{x}_i^2)\Psi - \frac{1}{2}g''\Psi^\dagger\Psi^\dagger\Psi\Psi\right] , \tag{18}$$

Finally since the normalization of the wave function is now inconvenient, $\int d\tilde{V}\Psi^\dagger\Psi = \left(\frac{\hbar}{\omega M}\right)^{-3/2}$, we rescale the wave function $\Psi \to \left(\frac{\hbar}{\omega M}\right)^{-3/4}\Psi$ to a dimensionless wave function to obtain the final form

$$\frac{1}{\hbar}S = N\int d\tilde{V}d\tilde{t}\left[i\Psi^\dagger\frac{\partial}{\partial\tilde{t}}\Psi - \frac{1}{2}\Psi^\dagger(-\tilde{\nabla}^2 + \sum\alpha_i^2\tilde{x}_i^2)\Psi - \frac{1}{2}g_{eff}\Psi^\dagger\Psi^\dagger\Psi\Psi\right] , \tag{19}$$

$$\int d\tilde{V}\Psi^\dagger\Psi = 1 \ . \tag{20}$$

The important coupling constant in the action is $g_{eff} = g''\left(\frac{\hbar}{\omega M}\right)^{-3/2}$, given by $g_{eff} \sim \left(\frac{\hbar}{M\omega}\right)^{-1/2}Na$. Identifying $D_{har} = \left(\frac{\hbar}{M\omega}\right)^{1/2}$ as the usual length scale of the harmonic oscillator we find $g_{eff} \sim Na/D$. For large g_{eff}, we are in the so-called Thomas Fermi regime, since the repulsion between the bosons is large and the condensates spreads over a large distance with small derivatives. Then we just need to minimize the scalar potential $V = -\mu n + \frac{1}{2}\sum\alpha_i^2\tilde{x}_i^2 n + \frac{1}{2}g_{eff}n^2$, with n the dimensionless density. This leads to a solid ellipsoid inside which there is a BEC, given by $n = \frac{1}{g_{eff}}\left(\mu - \frac{1}{2}\sum\alpha_i^2\tilde{x}_i^2\right)\Theta\left(\mu - \frac{1}{2}\sum\alpha_i^2\tilde{x}_i^2\right)$, with Θ the usual Heavyside step function. An equivalent expression is $n = \left(n(0) - \frac{1}{2g_{eff}}\sum\alpha_i^2\tilde{x}_i^2\right)\Theta\left(n(0) - \frac{1}{2g_{eff}}\sum\alpha_i^2\tilde{x}_i^2\right)$. The constant $n(0)$ should be fixed by demanding that the density integrates to 1, however, assuming $\alpha \sim 1$ and estimating the size of the ellipsoid by $\tilde{x} \sim \sqrt{n(0)g_{eff}}$ and hence the volume being $\sim (n(0)g_{eff})^{3/2}$, we see that $1 = \int n \sim n^{5/2}(0)g_{eff}^{3/2}$ and hence $n(0) \sim g_{eff}^{-3/5}$. This makes sense and simply says that inside a box of the size that is dictated by the harmonic oscillator there are few bosons in the Thomas Fermi regime.

The opposite regime with $g_{eff} \ll 1$ is the near-ideal regime since the repulsion provides just a small perturbation. In this case the bosons are simply in the ground state of the harmonic oscillator.

2.4. *The Landau Criterion*

The Landau criterion is a beautiful application of the Galilean symmetry. We explain it here since it is important for some later discussions.

Let us start from the Galilean algebra with K_i the boost. Then $[K_i, E] = i\hbar P_i$, $[K_i, P_j] = i\hbar\delta_{ij}M$, among the other, obvious commutation relations. More precisely this is a central extension of the Galilean algebra and M is the central term. It is a general fact that in quantum mechanics the Galilean algebra has this central extension. The mass M can be therefore interpreted as an anomaly. The centrally extended algebra is sometimes referred to as the Bargmann algebra.

A Casimir is $I = ME - \frac{\vec{P}^2}{2}$. Take a state $|\Psi_0\rangle$ with energy E_0 and vanishing momentum. We can boost this state to momentum P_0 and find from the Casimir $E = E_0 + \frac{P_0^2}{2M}$. This is just the obvious dispersion relation of Galilean theories.

Now we add an excitation on top of the original state $|\Psi_0, \epsilon(p)\rangle$. The state has now momentum p and energy $E_0 + \epsilon(p)$. Let us boost to total momentum $\vec{P}_0 + \vec{p}$, to find

$$E = E_0 + \frac{P_0^2}{2M} + \epsilon(p) + \vec{v}_0 \cdot \vec{p} \, ,$$

where $\vec{v}_0 = \frac{\vec{P}_0}{M}$. We can thus say that the dispersion relation of the original excitation on top of the moving fluid is $\boxed{\epsilon(p) + \vec{v}_0 \cdot \vec{p}}$. If this can be made negative this leads to a problem since we can lower the energy by adding localized excitations and that leads to a critical Landau velocity $v_L = \min_p \frac{\epsilon(p)}{p}$.

We cannot directly use the excitations to lower the energy since momentum is conserved in a closed system. We have to allow for a large stationary reservoir through which the superfluid flows. Above v_L friction with the reservoir appears since momentum can be transferred to it while lowering the energy of the flowing superfluid and emitting heat. (The kinetic energy transferred to the reservoir $p^2/2M_{reservoir}$ is negligible so the energy has to escape as heating of the reservoir as usual when there is friction.)

Note that the discussion above can be interpreted by saying that the Hamiltonian is $H + v_0 \cdot P$, which is as if we have added a chemical potential v_0 to the momentum charge. Above the landau velocity the Hamiltonian $H + v_0 \cdot P$ has negative eigenvalues and is, within the leading order effective theory, unbounded from below since we can construct phonon states with large occupation numbers.

For a rotating frame the considerations are similar. The Hamiltonian becomes $H - \Omega L$ and leads to the creation of vortices above a certain threshold rotation frequency, as we will now investigate.

2.5. *Rotating Solutions*

We will make some simple comments about a superfluid in a hard trap, i.e. a cylinder of radius R. The action is

$$\frac{1}{\hbar}S = N\int dV dt\left[i\Psi^\dagger\frac{\partial}{\partial t}\Psi + \Psi^\dagger\frac{\hbar}{2M}\nabla^2\Psi - \frac{1}{2}g'\Psi^\dagger\Psi^\dagger\Psi\Psi\right] , \qquad \Psi\big|_{r_\perp=R} = 0 . \tag{21}$$

We assume that the healing length $\xi \ll R$. We recall that the energy of a vortex scales like $\frac{\hbar^2 nL}{M}\log R$ and the angular momentum is $N\hbar$. If our trap is rotating then we must consider the Hamiltonian

$$H - \Omega L_z \tag{22}$$

and this leads to the critical rotation frequency $\Omega_c \sim \frac{\hbar\log(R/\xi)}{MR^2}$ for the creation of a vortex. Recall that the speed of sound is $c_s \sim \hbar/M\xi$ and hence the velocity at the edge of the bucket rotating with frequency Ω_c is $\Omega_c R = \frac{\hbar\log(R/\xi)}{MR} \ll \frac{\hbar}{M\xi} = c_s$. Therefore, locally near the boundary, we are still well below the Landau criterion and no heat is generated. The system is still a superfluid and it has a vortex at the center.

If we increase the frequency then the number of vortices grows. The vortices are not on top of each other due to the repulsion between them; the approximate energy of s vortices, neglecting the logarithms, is $\frac{\hbar^2 ns^2 L}{M}$ while the angular momentum is linear in s leading to a critical frequency

$$\Omega_c(s) \sim \frac{\hbar s}{MR^2} \tag{23}$$

This leads to velocities of order of the speed of sound near the edge when $s \sim R/\xi$, in which case the number of vortices per unit area is $1/\xi R$. The vortices are still sparse in this regime and far from their maximal density$\sim 1/\xi^2$. The famously form the triangular lattice.

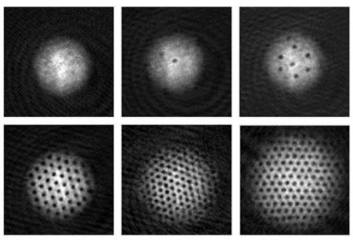


Fig. 1. In this figure we see how the number of vortices grows as the rotation frequency increases and eventually we get a beautiful triangular lattice.

2.6. *The Giant Vortex*

We now study a qualitatively new solution with angular momentum. The ansatz is identical to the one we already employed for the fundamental vortex

$$\Psi = e^{-i\mu t+i\ell\varphi}\chi(r_\perp) \ . \tag{24}$$

While fundamental vortices repel and the solution with, $\ell = 2$, for instance, should be multi-centered, we will find a stable solution with azimuthal symmetry of the above form at large ℓ!

As always we require that there are N particles in the trap $\int dV \chi^2 = 1$. We find the equation of motion

$$-\frac{\hbar}{2M}\left(\frac{\partial^2}{\partial r_\perp^2}+\frac{1}{r_\perp}\frac{\partial}{\partial r_\perp}-\frac{\ell^2}{r_\perp^2}\right)\chi(r_\perp)+g'\chi(r_\perp)^3=\mu\chi(r_\perp) \ . \tag{25}$$

As for the elementary vortex, the superfluid density far away from the center of the cylinder is constant and quickly goes to zero on the cylinder itself. The length scale for the density to plunge to zero is ξ for the fundamental vortex. Now we have the centrifugal term which is of order $\frac{\ell^2}{R^2}$ which must be compared to $\frac{1}{\xi^2}$. We therefore have to assume that

$$1 \ll \ell \ll R/\xi \tag{26}$$

so that the centrifugal term is negligible asymptotically near the edge of the bucket. The velocity near the edge of the bucket is $\frac{\hbar\ell}{MR}$. This approaches the speed of sound when $\ell \to R/\xi$ but in our limit (26) we do not approach the speed of sound at the edge.

The main feature of (25) is that the centrifugal barrier term grows quickly to compete with the GP term and they become comparable for $r \sim \ell\xi \gg \xi$. Therefore for large ℓ the centrifugal term can create a hole due to the repulsion from the origin. Note that the velocity of the fluid at that point is $\frac{\hbar\ell}{M\ell\xi}$ which is larger than near the edge and in fact parametrically equal to the speed of sound in the stationary fluid.

We treat the equation (25) in Thomas Fermi approximation where we neglect the kinetic term altogether and assume that the various domains we find are glued to each other on distances of order ξ. Then we have

$$\frac{\hbar}{2M}\frac{\ell^2}{r_\perp^2}\chi(r_\perp)+g'\chi(r_\perp)^3=\mu\chi(r_\perp) \ . \tag{27}$$

The solution is

$$\chi(r_\perp) = \begin{cases} \sqrt{\frac{\mu}{g'}-\frac{\hbar}{2Mg'}\frac{\ell^2}{r_\perp^2}} & R > r > r_a \\ 0 & r < r_a \end{cases} , \qquad r_a^2 = \frac{\hbar\ell^2}{2M\mu}$$

Finally we must require that the density is appropriately normalized,

$$2\pi L \int_{r_a}^{R} dr_\perp \left(\frac{\mu}{g'} r_\perp - \frac{\hbar}{2Mg'} \frac{\ell^2}{r_\perp} \right) = 1$$

which leads to $\pi L \frac{\mu}{g'} R^2 - \pi L \frac{\hbar \ell^2}{Mg'} \left(\log(R/r_a) + \frac{1}{2}\right) = 1$. The second term evaluates to $L\frac{\hbar\ell^2}{Mg'} \sim \ell^2\xi^2/R^2$ and since we assume $\ell^2\xi^2/R^2 \ll 1$ indeed we can neglect the second term and find $\mu = g'/V$ to a good approximation. This also means that $r_a \sim \xi\ell$ which is the radius of the hole.

We will make an important point about the fluctuations around the giant vortex.

We will consider the regime where $r_a = R(1 - \frac{1}{2}\delta)$, with $\delta \ll 1$. That is when the hole of the giant vortex grows large and consumes almost the whole trap. This is what happens as the rotation is increased. We still require that $R\delta \gg \xi$ so that we can neglect the higher derivatives and the effective theory for the angular mode is valid.

The modes in this regime are approximately, to leading order in δ, given by

$$\omega_{m,n} = \frac{\hbar\ell m}{MR^2} + \frac{x_n \hbar\ell}{\sqrt{2\delta}MR^2} \ , \quad m \in \mathbb{Z}, n \in \mathbb{N} \tag{28}$$

with $J_0'(x_n) = 0$, starting from $x_0 = 0$. The next one is at $x_1 = 3.8317$ etc. For instance, the density fluctuations associated with the modes $\omega_{m,n=0}$ is $\sim \sin(\omega t - m\varphi)$.

Roughly, the n modes correspond to excitations along the radial direction while the m modes are azimuthal waves

It is very important that there is no absolute value in m, and the energy of these modes can be positive or negative. The origin of this linear term is from the term ΩL in (22). Physically, the giant vortex represents *ultrasonic* rotation and that is why the dispersion is dominated by the second term. Indeed, the density profile $\sim \sin(\omega t - m\varphi)$ indicates that the waves are just dragged along with the rotating fluid and do not have any intrinsic propagation, to leading order. See figure 2. It is as if the waves are frozen in the rotating frame (and indeed there is Carrollian symmetry to leading order in δ).

If we put this superfluid in a trap then there would be friction according to the Landau criterion and the rotation would slow down. But in the absence of a medium to which the momentum can be transferred, the rotation continues forever. The ground state is infinitely degenerate since states in the Fock space with $\sum_i m_i = 0$ have the same energy and angular

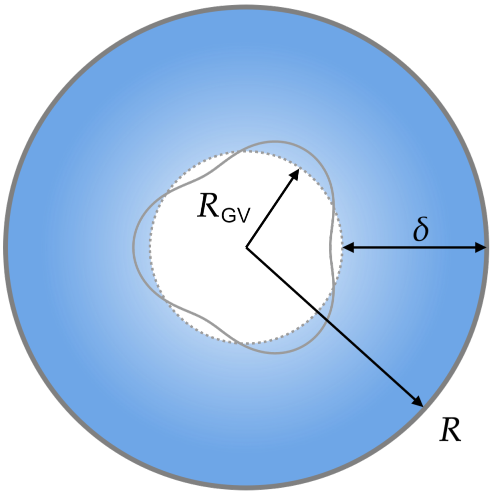
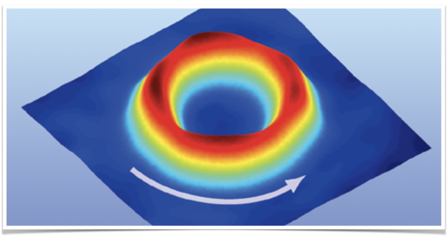
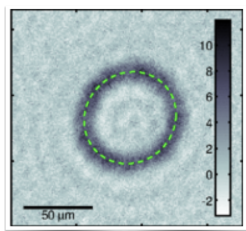


Fig. 2. A density wave corresponding to $m = 3$ rotating supersonically with the ambient superfluid. The white region is empty and the blue region contains the superfluid.

momentum as the original giant vortex. Small corrections turn out to lift this infinite degeneracy and lead to a small gap. The details of how this infinite degeneracy is lifted depend on the trap. For the hard trap there is a correction of order $\sqrt{\delta}|m|$ to the dispersion relation. This lifts the degeneracy and the originally giant vortex is thus indeed stable!

These classical solutions and their excitations reflect the spectrum of excited states of the quantum dilute gas in a trap. There are indications that experiments have either already reached or are close to reaching this supersonic regime with a giant vortex [].





3. CFT

There are many outstanding reviews of the basics of Conformal Field Theory. We will only summarize some notation and a few facts that are necessary to emphasize. In particular, we review radial quantization.

3.1. *Why Conformal Theories?*

The renormalization group (RG) is a powerful conceptual and computational framework for understanding the behavior of physical systems near critical points, where correlation lengths diverge and scale invariance emerges. The renormalization group is a central tool in many branches of physics (and in adjacent fields, too).

A *critical point* is a point in the phase diagram of a system where the correlation length ξ diverges. It is a fixed point of the renormalization group. Near such a point, the system becomes scale-invariant: fluctuations occur on all length scales. A classic example is the liquid-gas critical point, or the Curie point in a ferromagnet.

Physical quantities exhibit power-law behavior near criticality. For instance, the correlation length diverges as

$$\xi \sim |t|^{-\nu}, \tag{29}$$

where t is a coupling constant at zero temperature, or it could be related to temperature $t = (T - T_c)/T_c$ (depending on the context in which the criticality arises). ν is one of the critical exponents.

Similarly, other observables behave as:

$$C \sim |t|^{-\alpha}, \quad \text{(specific heat)} \tag{30}$$
$$\chi \sim |t|^{-\gamma}, \quad \text{(susceptibility)} \tag{31}$$
$$M \sim |t|^{\beta}, \quad \text{(order parameter)} \tag{32}$$
$$M(h) \sim h^{1/\delta}, \quad \text{(field response at } t = 0), \tag{33}$$

where α, β, γ, δ are various other scaling exponents.

This zoo of critical exponents can be related to the scaling dimensions of local operators in the theory. Indeed, we can turn on t and the external field h by deforming the scale invariant fixed point with local operators $\delta S = \int d^dx(h\mathcal{O}_h(x)+t\mathcal{O}_t(x))$ and thus, if $\Delta_{h,t}$ are the scaling dimensions of the operators $\mathcal{O}_{t,h}$, respectively, then the scaling dimension of the coupling constants h, t are $d - \Delta_{h,t}$. Since the partition function in a Euclidean box of volume V can be be written as $Z = e^{-VF(h,t)}$, the free energy density $F(t,h)$ can be written due to scaling symmetry as

$$F(t,h) = t^{\frac{d}{d-\Delta_t}} f(th^{\frac{\Delta_t - d}{d-\Delta_h}}) \ .$$

We have included the operators $\mathcal{O}_{h,t}$ since they are the most relevant ones with scaling dimensions $\Delta < d$. In general, to capture sub-leading effects we need to add infinitely many operators and there are infinitely many scaling dimensions Δ_i that can be extracted from precision measurements. But only the relevant operators influence the leading order effects which affect the macroscopic properties of the system. We are imagining a critical point with one knob which is a symmetry breaking field h (e.g. a magnetic field) and one knob which is a symmetric field (e.g. temperature) – of course general critical points could have many other important relevant operators or none at all.

This logic allows us to relate $\alpha, \beta, \gamma, \delta, \eta$ to the scaling dimensions $\Delta_{h,t}$. For instance,

$$\nu = \frac{1}{d - \Delta_t} \ , \tag{34}$$

$$\beta = \frac{\Delta_h}{d - \Delta_t} \ , \tag{35}$$

$$\gamma = \frac{d - 2\Delta_h}{d - \Delta_t} \ . \tag{36}$$

As an exercise the reader can determine α, δ in terms of the scaling dimensions $\Delta_{h,t}$.

Universality refers to the observation that systems with very different microscopic details exhibit identical critical exponents. This remarkable property stems from the fact that near the fixed point, the RG flow forgets the microscopic details, and only a few relevant directions (relevant operators) dictate the behavior.

It is important that so far we have only used scaling symmetry. A surprising fact is that many critical points are invariant under the full conformal group. The conformal group leads to stringent consistency conditions on which values of Δ_i furnish sensible conformal field theories, so it is inevitable that microscopically different systems end up sharing the same critical exponents.

3.2. *Examples: The 3d Ising and XY Models*

The 3D Ising model is defined on a three-dimensional lattice where each site i hosts a spin variable $s_i = \pm 1$. The Hamiltonian is given by:

$$H = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i, \tag{37}$$

where the first term sums over nearest-neighbor pairs and h is an external magnetic field. The interaction strength $J > 0$ favors alignment of neighboring spins, leading to ferromagnetic ordering at low temperatures.

The model with $h = 0$ has a global $\mathbb{Z}_2$ symmetry: it is invariant under the transformation $s_i \to -s_i$ for all i.

At zero external field $h = 0$, the model exhibits a continuous (second-order) phase transition at a critical temperature T_c. Below T_c, the system develops spontaneous magnetization $\langle s_i \rangle \neq 0$, while above T_c, thermal fluctuations restore the symmetry and $\langle s_i \rangle = 0$.

Near the critical point, the system shows power-law behavior and scale invariance, characterized by the divergence of the correlation length and non-analytic behavior of thermodynamic observables. The relevant critical exponents include:

$$\begin{aligned}
\xi &\sim |t|^{-\nu}, && \text{(correlation length)} \\
M &\sim |t|^{\beta}, && \text{(spontaneous magnetization)} \\
\chi &\sim |t|^{-\gamma}, && \text{(magnetic susceptibility)} \\
C &\sim |t|^{-\alpha}, && \text{(specific heat)} \\
M(h) &\sim h^{1/\delta}, && \text{(magnetization at } t = 0\text{)},
\end{aligned}$$

where $t = (T - T_c)/T_c$ is the reduced temperature.

The critical exponents are known with high precision (through Monte Carlo simulations, high-temperature expansions, the conformal bootstrap, and experiments). Representative values are:

$$\begin{aligned}
&\nu \approx 0.630, \\
&\alpha = 2 - d\nu \approx 0.110, \\
&\beta \approx 0.326, \\
&\gamma \approx 1.237, \\
&\delta \approx 4.789, \\
&\eta \approx 0.036.
\end{aligned}$$

These exponents are universal, meaning they apply not only to the lattice Ising model but also to many physical systems such as uniaxial magnets, binary fluid mixtures, liquid-vapor transitions near criticality, and the confinement-deconfinement transition in $SU(2)$ gauge theory in 3+1 dimensions.

From the above one can infer the scaling dimensions of the operator $\mathcal{O}_h$ which is the relevant operator corresponding to turning on an external magnetic field and $\mathcal{O}_t$ which corresponds to the de-tuning the temperature away from T_c

$$\Delta_h = 0.51.. \ , \quad \Delta_t = 1.41... \ . \tag{38}$$

Another interesting model in the same 3D Ising universality class is the interacting field theory

$$S = \int d^3x \left(\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2}m^2\Phi^2 + \lambda\Phi^4 \right) \ . \tag{39}$$

Φ is a real field. The $\mathbb{Z}_2$ symmetry acts by $\Phi \to -\Phi$. For large positive m^2 the theory is trivially gapped while for large negative m^2 the $\mathbb{Z}_2$ is spontaneously broken. The phase transition is second order in the Ising universality class. The scaling dimensions in the infrared CFT are as above (38) $\Delta(\Phi) = \Delta_h$ and $\Delta(\Phi^2) = \Delta_t$.

The XY model is the paradigmatic system for studying continuous symmetries and phase transitions. It exhibits a second-order phase transition characterized by nontrivial critical exponents and belongs to the *3D XY universality class.*

The XY model is defined on a three-dimensional lattice. At each lattice site i, there is a two-component unit vector $\vec{s}_i = (\cos\theta_i, \sin\theta_i)$. The Hamiltonian is given by:

$$H = -J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j), \tag{40}$$

where the sum runs over nearest neighbors and $J > 0$ favors alignment of neighboring vectors.

This model has a global $U(1)$ symmetry $\theta_i \to \theta_i + \alpha$. Thermal fluctuations compete with this ordering tendency, and at a critical temperature T_c, the order disappears.

A symmetry breaking field $\vec{h}$ could be turned on by adding a term to the Hamiltonian $\delta H = -\vec{h} \cdot \sum_i \vec{s}_i$. We can then repeat the usual scaling analysis and investigate the same critical exponents as above.

Extensive Monte Carlo simulations, field-theoretic methods (such as the epsilon expansion and conformal bootstrap) and experiments, have yielded

$$\Delta_{\vec{h}} = 0.51... \ , \quad \Delta_t = 1.50... \ . \tag{41}$$

Famously, the 3D XY model describes the critical behavior of the superfluid transitions in ^{4}He (lambda transition) and it also appears in magnetic systems with easy-plane anisotropy.

As above, another interesting model in the same XY universality class is the interacting field theory

$$S = \int d^3x \left(|\partial\Phi|^2 + m^2|\Phi|^2 + \lambda|\Phi|^4 \right) \ . \tag{42}$$

Φ is now a complex field. The $U(1)$ symmetry acts by $\Phi \to e^{i\alpha}\Phi$. For large positive m^2 the theory is trivially gapped while for large negative m^2 the $U(1)$ is spontaneously broken and we have a superfluid phase. The phase transition is second order in the Ising universality class. The scaling dimensions in the infrared CFT are as above (41), $\Delta(\Phi) = \Delta_h$ and $\Delta(|\Phi|^2) = \Delta_t$.

3.3. *The Basics*

In a conformal theory, the energy-momentum tensor is traceless,

$$T^\mu_\mu(x) = 0 \ . \tag{43}$$

In addition to boosts and rotations $M_{\mu\nu}$, and translations P_μ, the traceless energy-momentum tensor gives rise to dilatations and special conformal transformations, which act on space-time by $x^\mu\partial_\mu$ and $2x_\mu(x\cdot\partial) - x^2\partial_\mu$, respectively. The conserved charges are usually denoted by D and K_μ, respectively.[a] The full conformal algebra is

$$[M_{\mu\nu}, P_\rho] = \delta_{\nu\rho}P_\mu - \delta_{\mu\rho}P_\nu, \tag{44}$$
$$[M_{\mu\nu}, K_\rho] = \delta_{\nu\rho}K_\mu - \delta_{\mu\rho}K_\nu, \tag{45}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\nu\rho}M_{\mu\sigma} - \delta_{\mu\rho}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\rho\mu} - \delta_{\mu\sigma}M_{\rho\nu}, \tag{46}$$
$$[D, P_\mu] = P_\mu, \tag{47}$$
$$[D, K_\mu] = -K_\mu, \tag{48}$$
$$[K_\mu, P_\nu] = 2\delta_{\mu\nu}D - 2M_{\mu\nu} \ . \tag{49}$$

[a] In 1+1 dimensions, the traceless energy-momentum tensor leads to an infinite set of additional conserved charges.

3.4. *Representations of the Conformal Group*

As always, local operators are in representations of the algebra of the conserved charges since we can surround local operators with $d-1$ dimensional surfaces and by locality we must obtain a new local operator in this way, independent of the shape of the surrounding surface as in Fig 3.

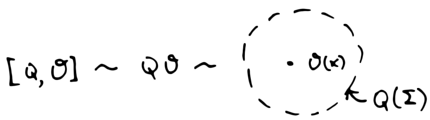


Fig. 3. The action of a conserved charge Q on a local operator $\mathcal{O}$. The result is independent of the shape of the surrounding surface.. This figure is taken from [?].

Now that we have our conserved charges P, M, D, K, we can classify operators into representations of those charges. Since M, D, K all fix the origin, it is most convenient to represent the charges on local operators at the origin. Then for operators at $\vec{x} \neq 0$ we can obtain the representation of $M.D, K$ by conjugating the operators with P.

The action of P on local operators is simply by an infinitesimal translation

$$[P_\mu, \mathcal{O}(0)] = \partial_\mu \mathcal{O}(0) \ . \tag{50}$$

Next, we take the local operators $\mathcal{O}$ have a well defined spin under $SO(d)$ rotations

$$[M_{\mu\nu}, \mathcal{O}^a(0)] = (\mathcal{S}_{\mu\nu})_b{}^a \mathcal{O}^b(0), \tag{51}$$

where $\mathcal{S}_{\mu\nu}$ are matrices satisfying the same algebra as $M_{\mu\nu}$, and a, b are indices for the SO(d) representation of $\mathcal{O}$. In our convention, the matrices $\mathcal{S}$ are *anti*-hermitian, $\mathcal{S}^\dagger = -\mathcal{S}$.

We can choose D to act by rescaling the operators $\mathcal{O}$ according to their scaling dimension Δ.[b]

$$[D, \mathcal{O}(0)] = \Delta \mathcal{O}(0). \tag{52}$$

Note that the scaling dimension of $\partial_\mu \mathcal{O}$ is $\Delta + 1$ etc. Therefore, acting with P_μ raises the scaling dimension. An important physical requirement in local models is that

$$\Delta \geq 0 \ . \tag{53}$$

[b]In logarithmic theories, D cannot be diagonalized. But we will not discuss such models here.

This comes from the following argument: considering the two-point correlation function and applying the scaling symmetry, we find that $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{x^{2\Delta}}$. It would be unacceptable for correlations to grow with distance indefinitely and hence we must impose (53).

The representation of K_μ on local operators is interesting. While P_μ raises the scaling dimension, K_μ lowers it. It is like an "inverse derivative." Indeed,

$$[D, [K_\mu, \mathcal{O}(0)] = (\Delta - 1)[K_\mu, \mathcal{O}(0)]. \tag{54}$$

Thus, repeatedly acting we lower the scaling dimension, but we know that this cannot continue indefinitely due to (53). Therefore there must be special operators annihilated by K_μ,

$$[K_\mu, \mathcal{O}(0)] = 0\ . \tag{55}$$

Such operators are called primary operators.

Acting with P_μ on the primary operators we generate a tower of descendants

$$[P_{\mu_1}, \cdots [P_{\mu_n}, \mathcal{O}(0)] \cdots] \tag{56}$$

To summarize, a primary operator satisfies

$$\begin{aligned} [D, \mathcal{O}(0)] &= \Delta \mathcal{O}(0) \\ [M_{\mu\nu}, \mathcal{O}(0)] &= \mathcal{S}_{\mu\nu}\mathcal{O}(0) \\ [K_\mu, \mathcal{O}(0)] &= 0. \end{aligned} \tag{57}$$

The representation of the conformal group that we have constructed is a lowest-weight representation. These are the only representations that are relevant in local, unitary, conformal field theories.

3.5. *Radial Quantization and the State-Operator Map*

We can study CFTs in Minkowski space $\mathbb{R}^{d-1,1}$, with the usual time evolution. The spectrum is gapless due to scale invariance and the charges are realized on these states as anti-Hermitian operators.

A central idea in the field is to quantize the system differently, so that the spectrum is discrete. In particular, we will describe a quantization in which the spectrum of energies is the spectrum of scaling dimensions Δ.

To understand the basic idea and utility of this idea it is worth considering a toy model example. Consider the following Hamiltonian in quantum mechanics:

$$H = \frac{p^2}{2} + \frac{g}{2x^2}. \tag{58}$$

This is a very interesting, and very rich model. It has conformal symmetry due to the charges

$$D = \tfrac{1}{2}(px + xp) \qquad\qquad C = \tfrac{1}{2}x^2. \tag{59}$$

As before, D is the dilatations charge while K is the special conformal charge. However, note that we have taken them to be Hermitian now. The algebra obeyed by these operators is

$$[D, H] = 2iH\ , \tag{60a}$$

$$[D, C] = -2iC\ , \tag{60b}$$

$$[H, C] = -iD\ . \tag{60c}$$

The factor of 2 in these commutation relations is due to the fact that it is a Schrödinger conformal theory where space and time scale with dynamical exponent $z = 2$, unlike the relativstic theories which have $z = 1$. The charges D, K are clearly not conserved, as usual for the conformal symmetries. (They are "spectrum generating.")

The model (58) is subtle due to the small x singularity of the potential. The behavior of the wave function at small x is given by $\Psi \sim x^\lambda$ with $\lambda_\pm = \frac{1}{2} \pm \sqrt{g + \frac{1}{4}}$. There are three important cases

$$\begin{cases} \lambda_+ \checkmark, \lambda_- ✗ & g > \frac{3}{4}\ , \\ \lambda_+ \checkmark, \lambda_- \checkmark & \frac{3}{4} > g > -\frac{1}{4}\ , \\ \textit{unbounded} & -\frac{1}{4} > g\ . \end{cases} \tag{61}$$

- For $g > \frac{3}{4}$ where the interaction is repulsive, the system can be quantized while preserving H, D, K and the wave function at small x decays as x^{λ_+}.
- In the window $\frac{3}{4} > g > -\frac{1}{4}$ both $x^{\lambda_\pm}$ are possible normalizable wave functions. We can choose that at distance $x = \epsilon \ll 1$ from the origin we have $\Psi = c_+ x^{\lambda_+} + c_- x^{\lambda_-}$. This breaks conformal invariance in general. But if we set $c_- = 0$ or $c_+ = 0$ then we preserve conformal invariance and we can then take $\epsilon \to 0$ to get conformal quantum mechanics. The conformal quantum mechanics with $c_+ = 0$ is fine tuned (since if c_+ is very small but nonzero at some ϵ, it will grow quickly and overtake the c_- mode) while the

one with $c_- = 0$ is the infrared stable fixed point. One can express all these facts using a beta function for the parameter c_+/c_- which in effect evolves as we change the cutoff surface $x = \epsilon$. Interestingly, $g = 0$, is just the usual free particle. The unstable fixed point with $c_+ = 0$, corresponds to a Neumann boundary conditions at $x = 0$. The stable fixed point $c_- = 0$ corresponds to a Dirichlet boundary condition at the origin $x = 0$. Indeed there is an RG flow between these two fixed points if we add a potential at $x = 0$, $u\delta(x)$, where u has dimensions of mass.

- The attractive interaction is too singular, in a way. We see that the wave function oscillates as $\Psi \sim x^{1/2} e^{\pm i\sqrt{-g-\frac{1}{4}}\log x}$. If we do not cut off at some $x = \epsilon \ll 1$ the wave function would have oscillated infinitely many times near the origin. This means that if we do not impose a cutoff ϵ the energy is not bounded from below. Since we cannot remove the cutoff, conformal symmetry is broken by choosing some boundary conditions at the cutoff scale. (There is still approximate discrete conformal invariance that can be discussed at long distances after regularizing the singular behavior near the origin.)

Henceforth we concentrate on $g > -\frac{1}{4}$. In the domain with two possible quantizations $\frac{3}{4} > g > -\frac{1}{4}$ we will refer to the two as the stable and unstable fixed points.

With the usual time evolution generated by H the spectrum is continuous. This can be seen directly from the conformal symmetry of the problem since for any state $|\Psi\rangle$ with eigenvalue E, we can consider the state $e^{i\alpha D}|\Psi\rangle$ whose energy can be calculated from $e^{-i\alpha D} H e^{i\alpha D} = e^{2\alpha} H$ as $e^{2\alpha} E$. The continuum wave functions are as usual delta function normalizable. The model does not have a ground state.

An interesting alternative way to think about the model is as follows. We can redefine time as $\omega\tau = \tan^{-1}\omega t$ and hence $\frac{d}{d\tau} = (1+\omega^2 t^2)\frac{d}{dt}$, which corresponds to evolving in time with

$$\tilde{H} = H + \omega^2 C \ . \tag{62}$$

The Hamiltonian in this frame is thus

$$\tilde{H} = \frac{p^2}{2} + \frac{g}{2x^2} + \frac{\omega^2}{2} x^2 \ . \tag{63}$$

Now there is a unique ground state for $g > -\frac{1}{4}$ and the energy is given by

$$E^{\pm}_{g.s.} = \omega(1 \pm \sqrt{g + \frac{1}{4}}) \ . \tag{64}$$

$$\begin{cases} E^{+}_{g.s.} \checkmark, E^{-}_{g.s.} ✗ & g > \frac{3}{4} \ , \\ E^{+}_{g.s.} \checkmark, E^{-}_{g.s.} \checkmark & \frac{3}{4} > g > -\frac{1}{4} \ , \\ \textit{unbounded} & -\frac{1}{4} > g \ . \end{cases} \tag{65}$$

For $g > \frac{3}{4}$, $E^{+}_{g.s.}$ is the energy of the ground state in the trap. In the window $\frac{3}{4} > g > -\frac{1}{4}$ we have two versions of the theory in the trap, with a difference in the $x = 0$ boundary condition corresponding to the stable and unstable fixed point. The ground state energy in the stable fixed point is $E^{+}_{g.s.}$ while in the unstable fixed point it is $E^{-}_{g.s.}$. Again, setting $g = 0$, this is the usual particle in a (half) harmonic oscillator with Neumann boundary conditions corresponding to $E^{-}_{g.s.} = \omega/2$, and Dirichlet boundary conditions corresponding to the wave function which vanishes at $x = 0$ and its energy being $E^{+}_{g.s.} = 3\omega/2$.

All the higher energy levels in the trap are given by $E^{\pm}_{g.s.} + 2n\omega$, with a positive integer n (and no degeneracy). These states are obtained by acting with the ladder operators

$$L_{\pm} = H - \omega^2 C \mp i\omega D \ , \tag{66}$$

which obey $[L_{\pm}, \tilde{H}] = [H - \omega^2 C \mp i\omega D, H + \omega^2 C] = \pm 2\omega L_{\pm}$. ($L_-$ is the energy-increasing operator, with this somewhat confusing notation originating from the usual conventions for $sl(2)$ algebra, where L_{-1} is usually taken to be the energy-increasing operator.)

We see that the evolution with $\tilde{H}$ corresponds to a particle in a trap, leading to a nice discrete spectrum. But what is the meaning of all of this? Our final goal is to answer this question. Let us return to the time foliation with H and consider the vacuum $|VAC\rangle$. Act on this vacuum with a local operator $\mathcal{O}(t = 0)|VAC\rangle$, next we evolve with H for one unit of Euclidean time

$$|\mathcal{O}\rangle \equiv e^{-H/\omega}\mathcal{O}(t = 0)|VAC\rangle \tag{67}$$

A short calculation, using that $e^{H/\omega} C e^{-H/\omega} = C + \frac{1}{\omega}[H, C] + \frac{1}{2\omega^2}[H[H, C]] = C - \frac{i}{\omega}D - \frac{1}{\omega^2}H$. Therefore $e^{H/\omega}(H + \omega^2 C)e^{-H/\omega} = \omega^2 C - i\omega D$. This shows that the state $|\mathcal{O}\rangle$ corresponding to a *primary* operator $\mathcal{O}$, satisfies

$$\boxed{\tilde{H}|\mathcal{O}\rangle = \omega\Delta|\mathcal{O}\rangle} \ . \tag{68}$$

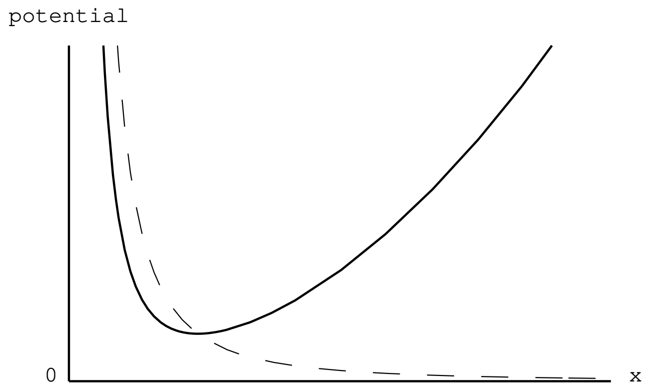


Fig. 4. A comparison between the potentials for H and $\tilde{H}$. The dashed line is the potential energy part of H and the solid line is that for $\tilde{H}$. Figure borrowed from [9911066]

This allows us to interpret the ground state of $\tilde{H}$ as corresponding to a primary operator in the conformal quantum mechanics. **explain that there is no state operator map in our model since no state is conformally invariant**

3.6. *Relativistic State-Operator Map*

Our next step is to find a "trap" that will host a relativistic CFT and the spectrum in the trap will coincide with the spectrum of scaling dimensions of local operators Δ.

As in the QM example, it is useful to switch to Euclidean time and then we perform a Weyl rescaling from $\mathbb{R}^d$ to the cylinder $\mathbb{R}\times S^{d-1}$,

$$\begin{aligned} ds^2_{\mathbb{R}^d} &= dr^2 + r^2 ds^2_{S^{d-1}} \\ &= r^2\partial\frac{dr^2}{r^2} + ds^2_{S^{d-1}} \\ &= e^{2\tau}(d\tau^2 + ds^2_{S^{d-1}}) = e^{2\tau} ds^2_{\mathbb{R}\times S^{d-1}} \sim ds^2_{\mathbb{R}\times S^{d-1}} \end{aligned} \tag{69}$$

where $r = e^\tau$ and in the last step we have removed the overall Weyl factor which does not influence the dynamics of conformal field theories.

Dilatations $r \to \lambda r$ become shifts of radial time $\tau \to \tau + \log\lambda$. Now we finally go back to the physical Minkowski time and discuss the Lorentzian cylinder with metric

$$ds^2 = -dt^2_{cyl} + ds^2_{S^{d-1}} \ . \tag{70}$$

The quantum states on the cylinder are in correspondence with local operators in ordinary flat space. P and K are, respectively, raising the lowering operators on the cylinder, corresponding to the tower of descendants we discussed in (56).

We can also act on states on the cylinder with local operators on the cylinder. The local operators on the cylinder are obtained from those in flat space by a conformal transformation

The operators on the cylinder and flat space are related by

$$\mathcal{O}_{\mathrm{cyl.}}(\tau, \hat{x}) \equiv e^{\Delta\tau}\mathcal{O}_{\mathrm{flat}}(\vec{x} = e^{\tau}\hat{x}) \ , \tag{71}$$

with $\hat{x} \in S^{d-1}$. As a consistency we can calculate the two point function on the cylinder, separated in time but at the same point on S^{d-1} in the vacuum

$$\langle \mathcal{O}_{\mathrm{cyl.}}(\tau_1, \hat{x})\mathcal{O}_{\mathrm{cyl.}}(\tau_2, \hat{x})\rangle = \frac{e^{\Delta(\tau_1+\tau_2)}}{(e^{\tau_1} - e^{\tau_2})^{2\Delta}} = \frac{1}{(e^{\frac{\tau_1-\tau_2}{2}} - e^{\frac{\tau_2-\tau_1}{2}})^{2\Delta}} \to_{|\tau_{12}|\to\infty} e^{-\Delta|\tau_1-\tau_2|} \ .$$

This makes sense for a quantum theory on the Cylinder since hitting the cylinder vacuum with the operator $\mathcal{O}$ creates an excited state with energy $\Delta + n$ with nonnegative integer n. The one that propagates the longest is the $n = 0$ mode which precisely leads to the exponential damping, as for any other gapped system.

4. Heavy Charged Operators in the 3D XY Model

Let us discuss the problem of local operators $\mathcal{O}$ in the XY model with charge Q under the $U(1)$ symmetry.

We will take $Q \gg 1$. By the state-operator map we ought to seek states on S^2 with charge Q and the dictionary is

$$E = \Delta/R \ .$$

(In fact the radius of the S^2 can be set to 1 without loss of generality but we will keep it for clarity.) There are many such states with charge Q. We will concentrate on the ground states at fixed Q,

We make some assumptions about the properties of such states:

- These states are assumed to have a nice thermodynamic limit, namely, we can keep the density Q/R^2 fixed and take Q and R to infinity in order to obtain an interesting state in flat space with nonzero charge density. We assume that this limit makes sense.

- Regarding the above state in infinite volume, we assume that the ground state at fixed charge density is a superfluid, breaking the $U(1)$ particle number symmetry spontaneously. Because this is a finite energy state we anticipate that the boost symmetry is broken as well. The superfluid sound mode would simultaneously serve as the NGB for particle number symmetry and for the boost symmetry.

If we start from the Landau-Ginzburg description (42) we could phrase the problem as trying to compute the scaling dimensions of the operators Φ^Q, for $Q \gg 1$. Roughly speaking, these are the lowest lying primaries of charge Q.

While the first assumption, that a thermodynamic limit exists, is certainly hard to prove from first principles, it seems pretty much guaranteed from what we know about many body quantum systems. The second assumption, that a superfluid phase is the thermodynamic limit, needs to be justified better.

We can start from the Landau-Ginzburg description (42), and study finite density states. This system has several scales, the density, the scale of the quartic interaction, and the quadratic term m^2. The analysis of finite density states is valid only at weak coupling, and one can brazenly assume that no phase transition happens as we study finite density states at strong coupling.

So we consider the relativistic LG model with a complex field Φ, and Lagrangian:

$$\mathcal{L} = |\partial\Phi|^2 - m^2|\Phi|^2 - \lambda|\Phi|^4 \ .$$

We add a chemical potential to reflect that we seek for the ground states at finite particle number density. This is put in by $D_\mu = \partial + i\mu\delta_{\mu 0}$. The Lagrangian becomes

$$\mathcal{L} = |\partial\Phi|^2 + i\mu(\Phi\partial_0\Phi^* - \partial_0\Phi\Phi^*) - (m^2 - \mu^2)|\Phi|^2 - \lambda|\Phi|^4 \ .$$

One could complain that the background gauge field $A_0 \sim \mu$ is flat and can be removed by a field transformation so we have not done anything, really. The point is that the charge density is now given by $j_0 \sim i\Phi\overleftrightarrow{\partial}\Phi^* + 2\mu|\Phi|^2$, so constant field configurations now carry charge density and so the sole purpose of introducing the flat gauge field $A_0 = \mu$ is so that we can seek and expand around constant field configurations, which is more convenient. (Also, once we introduce temperature and consider imaginary chemical potentials, it is no longer possible to remove it by a change of variables.)

Another point of view is in the Hamiltonian formulation. The change of variables required to remove μ affects the time evolution and so we can say that by introducing μ we have changed the Hamiltonian to

$$H - \mu Q \ . \tag{72}$$

With $m^2 > 0$, for $\mu^2 < m^2$ no condensate develops. The equations therefore describe waves

$$\Phi = \sum_k a_k^* e^{i\omega_- t + ikx} + b_k e^{-i\omega_+ t + ikx} \ ,$$

with dispersion $\omega_\pm = \sqrt{k^2 + m^2} \pm \mu$. All we did was to split the particle from the anti-particle due to the shift of the Hamiltonian as in (72).

When $\mu^2 = m^2$ we hit a phase transition since one of the modes becomes massless. We expand around a homogeneous condensate as $\Phi = v + \phi$. We immediately see that

$$2v^2 = \frac{\mu^2 - m^2}{\lambda} > 0 \ . \tag{73}$$

We now expand the action in $\phi_{1,2}$, eliminating total derivative terms and constants, to find a very similar story to what we had in (13).

After some work that the read can carry out as an exercise, we see two solutions, one where ω remains nonzero at small momentum and solves the equation $\omega^2 = 4\lambda v^2 + 4\mu^2 = 6\mu^2 - 2m^2$. This is a "Higgs" particle that does not participate in low energy dynamics. The second solution has vanishing energy at small momentum and solves the equation $-4\lambda v^2(\omega^2 - k^2) - 4\mu^2\omega^2 = 0$, and see that the solution behaves like $\omega = c_s|\vec{k}| + \cdots$ with $-4\lambda v^2(c_s^2 - 1) - 4\mu^2 c_s^2 = 0$ i.e. $c_s^2 = \frac{2\mu^2 - 2m^2}{6\mu^2 - 2m^2} < 1$.

What this exercise shows is that the weakly coupled Φ^4 model at large enough density is always in a superfluid phase. (One can compute loop corrections to the superfluid equation of state and try to reach closer to the infrared fixed point – these corrections are quite nontrivial and interesting [].) This does not prove that the XY critical point is in a superfluid phase at nonzero density since the limits of low energy and large density may or may not commute.

If we make the assumption that the critical point at nonzero density is in a superfluid phase, we can get pretty far, as we will see.

We denote as in (15) the angle variable of Φ by φ and write an effective action constrained by relativistic invariance, and conformal invariance, since it is a superfluid state in a conformal field theory. So the action must be a function of $(\partial_\mu\varphi)^2$,

$$S = \int d^2x dt P(\partial_\mu\varphi\partial^\mu\varphi) \ .$$

From scale invariance we can fix $P(x) = \alpha x^{3/2}$ so we obtain the conformal superfluid theory

$$S_0 = \alpha \int d^2x dt (\partial_\mu\varphi\partial^\mu\varphi)^{3/2} \ . \tag{74}$$

This fractional power looks alarming, however, we are only using this as an effective theory around nonzero density states $\varphi = \mu t + ...$ (with $\mu \sim \sqrt{Q}/R$) and the expansion around such states is perfectly fine as a low-energy expansion.

For general covariance we must also allow to put the superfluid on general curved spaces as

$$S_0 = \alpha \int d^2x dt \sqrt{g} (\partial_\mu\varphi\partial^\mu\varphi)^{3/2} \ . \tag{75}$$

In particular, we will be able to eventually go back to $S^2 \times \mathbb{R}$ to make contact with the spectrum of scaling dimensions.

Since it is an effective theory, we must contemplate various higher derivative terms suppressed by the energy or momentum in comparison with the density ∂/μ. Such terms must be Weyl invariant since these are states in a conformal field theory. we give examples of two such terms which arise at order $O(\partial^2/\mu^2)$:[c]

$$\begin{aligned} S \supset \alpha_2 \int d^dx \sqrt{g} |\partial\varphi|^3 & \left[\frac{\mathcal{R}}{|\partial\varphi|^2} - 8 \frac{\nabla^2 \left(|\partial\varphi|^{1/2}\right)}{|\partial\varphi|^{5/2}} \right] \\ + \alpha_3 \int d^dx \sqrt{g} |\partial\varphi|^3 & \left[\mathcal{R}_{\mu\nu} \frac{\partial^\mu\varphi\partial^\nu\varphi}{|\partial\varphi|^4} - \frac{\nabla^2 |\partial\varphi|}{|\partial\varphi|^3} + \frac{\partial^\mu\varphi\partial^\nu\varphi\nabla_\mu\nabla_\nu \left(|\partial\varphi|^{-1}\right)}{|\partial\varphi|^3} \right] , \end{aligned} \tag{76}$$

where $\mathcal{R}^\mu{}_{\nu\rho\sigma}$ is the Riemann tensor. The scale μ is therefore the natural cutoff for the EFT since $\partial\varphi$ appears in the denominator in higher order

[c] We chose the operators multiplied by α_2 and α_3 to be exactly Weyl invariant, and not only up to a boundary term; see [?] for a discussion of the constraints from unitarity on the values of α_2 and α_3.

terms in the effective field theory.[d]

To find the relationship between Q and μ and to find the energy of the configuration $\varphi = \mu t$ we write the current and energy-momentum tensor neglecting the higher derivative corrections to the effective field theory and considering only the term (75)

$$j_\mu = 3\alpha|\partial\varphi|\partial_\mu\varphi \ , \tag{77}$$

$$T_{\mu\nu} = \alpha\left(3|\partial\varphi|\partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}|\partial\varphi|^3\right) \ . \tag{78}$$

Note that the energy-momentum tensor is traceless. Remembering that the volume of the two-sphere is $4\pi R^2$ we see that the total charge is $Q = 12\pi R^2\alpha\mu^2$ and the total energy is $E = 8\pi R^2\alpha\mu^3$ and hence we find that

$$\boxed{\Delta = RE = \frac{1}{3^{3/2}\sqrt{\pi\alpha}}Q^{3/2} + \ldots \equiv c_1 Q^{3/2} + \cdots \ .} \tag{79}$$

with $c_1 = \frac{1}{3^{3/2}\sqrt{\pi\alpha}}$. Corrections to the scaling dimension arise from irrelevant operators such as the $\alpha_{2,3}$ terms as well as loop corrections. From the fluctuations around the homogeneous superfluid.

We have therefore determined the scaling dimension of the primary operator Φ^Q with $Q \gg 1$ in the XY model. We see that $\Delta = c_1 Q^{3/2} + \cdots$. This is a very interesting qualitative result.

Equation (79) is the same scaling as charge Q operators describing extremal black-hole states.[e] So which is the ground state of a holographic

[d]While in strongly coupled models the Wilson coefficients α_i are $O(1)$ and $\mu \sim \sqrt{Q}/R$, in weakly coupled models μ may be parametrically smaller than $\sqrt{Q}/R$ and Wilson coefficients may have nongeneric scaling. We refer to [?] for a general discussion of the derivative expansion in those cases and to [?, ?, ?] for some examples.

[e]The spherically symmetric Reissner–Nordström–AdS$_4$ black hole has

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \ , \quad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{L^2} \ , \tag{80}$$

where M is the mass, Q the electric charge, and L the AdS$_4$ radius.
The extremal condition corresponds to zero temperature:

$$f(r_+) = 0, \quad f'(r_+) = 0.$$

In the large Q limit,

$$r_+ = \left(\frac{L^2}{3}\right)^{1/4} Q^{1/2} \ ,$$

and the extremal mass is

$$M_{\rm ext} \approx \left(\frac{L^2}{3}\right)^{1/4} Q^{3/2} \quad \text{for large } Q \ .$$

.

CFT, the superfluid or the extremal black hole? They are certainly not identical states, the extremal black hole has a very small gap to the next primary while the superfluid has gap of $O(1)$, as we will see. Presumbaly, the black hole is the correct ground state at "intermediate Q", with $Q \sim c$, with c some central charge in the holographic theory, while for truly large Q, the correct answer is the superfluid.

More generally, to my knowledge, any known quantum system at large enough density becomes a superfluid.[f] This is famously also the case in QCD at large baryon density.

Let us now study the fluctuations about the superfluid, which will teach as about the gap to the next primary of charge Q. The fluctuations $\varphi = \mu t + \delta\varphi$ to second order have an action

$$S_{\text{fluctuations}} = 3\mu^2\alpha \int d^3x\sqrt{g}\,\partial_0\delta\varphi + 3\mu\alpha \int d^3x\sqrt{g}\left[(\partial_0\delta\varphi)^2 - \frac{1}{2}(\partial_i\delta\varphi)^2\right] . \tag{81}$$

The first term is a total derivative – it affects the zero mode dynamics, which is important for various questions that we will explore later on. In flat space the dispersion relation is obviously

$$\omega = \frac{1}{\sqrt{2}}|k| . \tag{82}$$

The speed of sound $c_s = 1/\sqrt{2}$ is of course the telltale sign of conformal matter at finite temperature in 2+1 dimensions.

On $S^2 \times \mathbb{R}$, expanding the fluctuations in spherical harmonic modes, they obey the dispersion relation ($J \neq 0$)

$$\omega^2 = \frac{1}{2R^2}J(J+1) . \tag{83}$$

The spectrum of states on S^2 has to contain states corresponding to the descendants of Φ^Q. These have to have protected energy. Indeed, the descendants correspond to states involving a number $n > 0$ of spin one quanta, each increasing the energy by $\omega(J=1) = 1/R$. The other phonons could receive corrections to their dispersion from corrections to the effective field theory, but the $J = 1$ phonon is special and protected by conformal symmetry.

[f] The free fermion is a counter-example, since it forms a Fermi gas. But the Fermi gas is unstable to interactions, which lead to a superfluid phase, so presumably in any interacting system the true ground state at sufficiently large density is a superfluid. There are also non-Fermi liquids which might be stable, but this remains to be seen.

The first excitation that corresponds to a primary is therefore $J = 2$ with $\omega = \sqrt{3}/R$ – this means that, while the ground state at fixed Q has $\Delta = c_1 Q^{3/2} + \cdots$, the first next primary has

$$\Delta_{1st} - \Delta_{0th} = \sqrt{3} + \cdots \ .$$

This is a parameter-free prediction for the gap between heavy operators. (This is qualitatively different than the primary gap in extremal black holes, which is extremely small [].)

Since the cutoff of our theory is μ, we cannot excite too many phonons – we see from the dispersion relation that we should only trust phonons below $J \sim \mu R$ which is why this description is valid up until $0 < J \ll \sqrt{Q}$. Note that $\omega(J)$ is a sub-additive function, i.e.

$$\omega(J_1 + J_2) \leq \omega(J_1) + \omega(J_2) \ .$$

This means that to minimize the energy for a given Q and J it is always beneficial to excite exactly one phonon of spin J.[g] The detailed map between the Fock space of phonons and the primary operators of charge Q (with various derivatives inserted) is discussed in [**?**] for some weakly coupled theories.

5. Charged and Spinning Operators in the 3D XY Model

We will now discuss spinning superfluid states which are important to understand the lowest lying operators with charge and spin.

Previously we used phonons to describe states with J not exceeding $\sqrt{Q}$.

But what happens if we search for the ground state with some large Q and $J > \sqrt{Q}$? Is there some new semiclassical description?

The important role of vortices in superfluids was recognized long ago [**?**, **?**]. See also [**?**, **?**, **?**] and in the context of the XY model see [**?**, **?**]. Vortices appear when the superfluid is stirred – as we have already seen in the analysis of dilute gases (23).

In the previous section that φ is compact was only important for the quantization of Q itself. Our spatial configurations were single valued, regular functions on $\mathbb{R}^2$ or S^2 (such configurations are called irrotational). In

[g] In general there is a contribution $\sim J^3/(R^2\mu^2)$ to the dispersion relation (83) which arises from the higher derivative operators (76). For $J \gtrsim (R\mu)^{2/3} \sim Q^{1/3}$ the subadditivity of $\omega(J)$ depends on the sign of this contribution and multi-phonon states might be preferred over single-particle ones [**?**].

this section we explore the configurations which utilze the compact nature of φ.

The action for a vortex moving along a prescribed trajectory has the usual kinetic term contributions with the physical mass of the vortex setting its rest energy – we expect the physical mass to be of order μ. But the vortex also receives a contribution from the linear term in (81). To analyze the contribution of the linear term $3\alpha\mu^2\int d^3x\sqrt{g}\partial_0\varphi$ to the action of a vortex moving on a trajectory $x_0(t)$, we observe that this term fails to be a total derivative only around the location of the vortex. Therefore, we pick normal coordinates y^i in the vicinity of the vortex, in terms of which $\varphi = \varphi(y^i - y_0^i(t))$ very close to the vortex core. The linear term in the action thus becomes

$$3\alpha\mu^2\int d^3x\sqrt{g}\partial_0\varphi = 3\alpha\mu^2\int dtd^2y\frac{\partial\varphi}{\partial y^i}\dot{y}_0^i(t)\ . \tag{84}$$

To evaluate this integral we use the definition of q units of vorticity

$$\mathcal{V}_{ij} = \partial_i\frac{\partial\varphi}{\partial y^j} - \partial_j\frac{\partial\varphi}{\partial y^i} = 2\pi q\epsilon_{ij}\delta^2(y - y_0(t))\,, \tag{85}$$

with ϵ_{ij} being the usual ± 1 symbol in any coordinate system. With simple manipulations we then find[h]

$$I_i = \int d^2y\frac{\partial\varphi}{\partial y^i} = \pi q\epsilon_{ik}y_0^k\ . \tag{86}$$

Plugging this result into the action of the vortex we therefore find the following term in the action

$$S \supset 3\alpha\mu^2\int dtd^2y\frac{\partial\varphi}{\partial y^i}\dot{y}_0^i(t) = 3\alpha\mu^2\pi q\int dt\epsilon_{ik}y_0^k\dot{y}_0^i(t)\,. \tag{87}$$

This action for the vortex core is very familiar from the motion of a classical particle in a transverse constant magnetic field (leading to the Lorentz force). To make the analogy explicit, we introduce the effective magnetic field $\partial_iA_j - \partial_jA_i = -6\alpha\mu^2\pi\epsilon_{ij}\sqrt{g}$. We therefore see that each vortex has the following term in its worldline action

$$S \supset q\int dtA_i(y(t))\dot{y}^i\ , \qquad \partial_iA_j - \partial_jA_i = -6\alpha\mu^2\pi\epsilon_{ij}\sqrt{g} = -\frac{Q}{2R^2}\epsilon_{ij}\sqrt{g}\ . \tag{88}$$

[h] To obtain eq. (86) we take another derivative with respect to y_0 and contract with the epsilon tensor: $\epsilon^{ki}\partial_{y_0^k}I_i = -\epsilon^{ki}\int d^2y\partial_k\frac{\partial\varphi}{\partial y^i}$. Now we use the property (85). Thus we find $\epsilon^{ki}\partial_{y_0^k}I_i = -2\pi q$ from which we infer that $I_i = \pi q\epsilon_{ij}y_0^j$.

Note that this is a properly quantized magnetic field on S^2, with exactly Q units of the minimal magnetic field.

Of course the fact that superfluid vortices effectively move in a constant magnetic field can be derived more easily by dualizing the superfluid mode to a gauge field. The constant magnetic field is essentially the Magnus effect that acts on vortices transversely to their direction of motion.

We will assume that the vortices move at velocity much smaller than the speed of light so the kinetic energy is just the rest energy.

The potential when more than one vortex is present was already argued to lead to a logarithmic interaction – let us compute it in detail now, starting from the action $-\frac{3\mu\alpha}{2}\int dtd^2x\sqrt{g}(\partial_i\varphi)^2$. For vortices with locations $x_\alpha(t)$ where the index α labels the vortices we have the equation of motion $\Delta_{S^2}\varphi = 0$ away from the vortices which is solved by

$$\partial_i\varphi = 2\pi\sqrt{g}\epsilon_{ij}\sum_\alpha q_\alpha \partial^j G(x, x_\alpha) \ , \tag{89}$$

with q_α the vorticity and G the standard Green's function on the sphere:

$$G(x, y) = -\frac{1}{4\pi}\log\left(\hat{n}_x - \hat{n}_y\right)^2 \ . \tag{90}$$

Here $\hat{n}$ is unit normalized three-vector describing the embedding of the 2-sphere in $\mathbb{R}^3$; $R^2(\hat{n}_x - \hat{n}_y)^2 \equiv L^2_{xy}$ is the *chordal* distance between the points x and y on S^2.

Using eq. (89) in the action and after a little bit of algebra we find the static component of the logarithmic interaction between the vortices

$$S \supset 3\pi\mu\alpha\int dt\left[\sum_{\alpha\neq\beta} q_\alpha q_\beta \log(\mu L_{\alpha\beta}) + \text{const.}\right] \ . \tag{91}$$

The cutoff μ appears explicitly in the static potential between vortices, which accounts for the self energy as well, due to the constraint that the total number of vortices vanishes. The constant term in eq. (91) is interpreted as the contribution of the vortex masses and it is an independent Wilson coefficient within EFT. (More precisely, this independent Wilson coefficient is due to the ratio of the vortex mass and the scale μ, which is a model dependent coefficient, expected to be ~ 1 in the XY model.)

We are now ready to summarize the action of the vortices:

$$\boxed{S_{vortices} = \int dt\left[\sum_\alpha q_\alpha A_i(x^i_\alpha)\dot{x}^i_\alpha(t) + 3\pi\mu\alpha\sum_{\alpha\neq\beta} q_\alpha q_\beta \log\left(\mu L_{\alpha\beta}\right)\right]} \ , \tag{92}$$

where the gauge field satisfies $\partial_i A_j - \partial_j A_i = -6\alpha\mu^2\pi\epsilon_{ij}\sqrt{g} = Q/(2R^2)\epsilon_{ij}\sqrt{g}$. We neglected the masses of the vortices and the kinetic energy. This is analogous to the dynamics in the lowest Landau level [?, ?, ?, ?, ?].

The energy stored by the vortices, on top of the ground state energy $8\pi R^2\alpha\mu^3$, is independent of the magnetic field and is just given by the electrostatic potential:

$$E_{vortices} = -3\pi\mu\alpha \sum_{\alpha\neq\beta} q_\alpha q_\beta \log\left(\mu L_{\alpha\beta}\right) \ . \tag{93}$$

The charge is not sensitive to the vortices and it is given as before by $Q = 12\pi R^2\alpha\mu^2$.

The angular momentum is sensitive to the vortices. Let us compute the angular momentum of a static q vortex at the north pole. From (78) we find $T_{0\phi} = 3\alpha\mu^2 q$, therefore a single vortex contributes to J_z by $\frac{1}{2}\int d^2x T_{0\phi} = 6\pi R^2\alpha\mu^2 q$. We divided by 2 since the configuration $\varphi = q\phi$ has a q vortex in the north pole and a q anti-vortex in the south pole, each of which is contributing a half to the total angular momentum. More generally, the angular momentum is given by

$$\vec{J} = 6\pi R^2\mu^2\alpha \sum_\alpha q_\alpha \hat{n}_\alpha \ , \tag{94}$$

where $\hat{n}_\alpha$ is the unit vector pointing from the center of the sphere to the location of the vortex. Of course, the angular momentum also receives contributions from the motion of vortices but those are again negligible for slowly moving vortices. The fact that a single vortex has angular momentum is of course analogous to the angular momentum in the monopole-charge system [?]. In the presence of many vortices, the angular momentum is additive (94) since the equation of motion for the fluctuations around $\chi = \mu t$ is linear to leading order and hence the profile of φ is a superposition over all the vortices (89). We have already seen this connection between the angular momentum and superfluid vortices in our discussion of dilute gases (16).

5.1. *Spinning Superfluid*

Consider a vortex anti-vortex pair, each with unit vorticity. When the vortices are antipodally placed, as in (95), they are static. More generally, we have to balance the Lorentz force, which is the first term in (92) with the electric potential, which is the second term in (92). Vortices therefore move via drift motion with velocity $v \sim 1/(\mu L)$ around the sphere, where L is the relative distance. The energy and angular momentum of the configuration are

$$\Delta/R = 8\pi R^2\alpha\mu^3 + 6\pi\mu\alpha\log(\mu L) + \dots , \qquad J_z = 6\pi R^2\mu^2\alpha L = \frac{Q}{2}\frac{L}{R} . \quad (95)$$

We can rewrite the scaling dimension corresponding to these operators as

$\Delta = c_1 Q^{3/2} + \frac{\sqrt{Q}}{6c_1}\log\frac{J(J+1)}{Q} + \cdots$.(94)The vortex mass and various other corrections are smaller than the terms we kept in (5.1).

By moving the vortices farther apart we can increase the angular momentum while keeping the charge fixed. In this way we can increase the angular momentum until we reach $J_z = Q$, when the vortices are antipodal. On the lower end of the regime of validity of the vortex anti-vortex system, for $J_z \lesssim \sqrt{Q}$, the vortices are nearby and they become relativistic so our EFT breaks down. Indeed, for $J_z \lesssim \sqrt{Q}$, phonons are sufficient. In summary, the vortex anti-vortex configuration is well suited to be the ground state of the system for $\sqrt{Q} \ll J \leq Q$.

The angular momentum can be increased seemingly as much as we like if we just keep increasing the vorticity q. But this is not the ground state since, for instance, a 2-vortex is generally going to be unstable towards breaking up to two single vortices.

Given these arguments we now look for configurations that can take us beyond $Q \sim J$. We first consider configurations that are made out elementary (winding ± 1) vortices only.

For sufficiently large J we may approximate the vortex distribution with a continuous function $\rho(x)$. The vortex contribution to the energy (93) thus reads:

$$E_{vortices} = 6\pi^2\mu\alpha \int d^2x\sqrt{g}d^2x'\sqrt{g'}\rho(x)\rho(x')G(x,x') , \qquad (95)$$

while the angular momentum is

$$\vec{J} = 6\pi\alpha\mu^2R^2 \int d^2x\sqrt{g}\rho(x)\hat{n}(x) . \qquad (96)$$

To minimize the energy at fixed angular momentum we assume $J_x = J_y = 0$ and consider the functional $E_{vortices} + \lambda J_z$, from which we obtain the following minimum condition

$$\int d^2x'\sqrt{g'}\rho(x')G(x',x) = \frac{\lambda}{2}\mu R^2\cos(\theta) , \qquad (97)$$

where λ is a Lagrange multiplier. The first term on the left hand side of (97) is the electric potential due to charges with density ρ and the second term on the right hand side states that this electric potential is proportional to $\cos\theta$. The equation (97) is of course solved by acting on both sides with Δ_{S^2} in the x coordinates. Expressing λ in terms of J_z, we find the following result for the vortex density

$$\rho = \frac{J_z}{8\pi^2\alpha\mu^2 R^4}\cos(\theta) = \frac{3J_z}{2\pi Q R^2}\cos\theta\,. \tag{98}$$

Eq. (98) corresponds to a velocity profile of a rigid body with a spherical shape, explicitly $v_\phi = j_\phi/j_0 \simeq \sin^2\theta J/(cQ^{3/2})$ (therefore v^ϕ is constant and hence the angular velocity is θ independent, as for a rigid body). The energy of the rigid body as a function of Q, J is inferred from (95):

$$E = \frac{1}{3^{3/2}R\sqrt{\pi\alpha}}Q^{3/2} + \frac{3^{3/2}\sqrt{\pi\alpha}}{2R}\frac{J^2}{Q^{3/2}}\ . \tag{99}$$

To remain within the non-relativistic regime we must require that $J \ll Q^{3/2}$, since as J comes close to $Q^{3/2}$ the rotation velocity becomes relativistic and our treatment of the problem needs to be revisited. Interestingly, the two terms in (99) become comparable for $J \sim Q^{3/2}$ which is another indication that the rigid body breaks down in that domain, as the energy of the background superfluid begins to be challenged by the potential of the vortices.

On general grounds, we expect that when the superfluid velocity exceeds the speed of sound $c_s = 1/\sqrt{2}$ the rigid body configuration becomes unstable [**?**], and the superfluid settles in a new ground state [**?**]. Remarkably, such a transition was experimentally observed for Bose-Einstein condensates in anharmonic traps [**?**].

We can continue and ask what happens when the angular momentum exceeds $Q^{3/2}$.

One can continue and push the superfluid description all the way to $J \ll Q^2$ - in fact there are two different proposals for what happens between $Q^{3/2} \ll J \ll Q^2$!

One is the giant vortex, which is very similar to the one that we encountered in the dilute gas discussion (24). The other is a relativistic configuration of elementary vortices []. Both of these almost certainly exist and it seems like the relativistic configuration of elementary vortices is slightly preferred. For $J > Q^2$ none of these descriptions makes sense and the superfluid theory breaks down. We will discuss this regime later.

Both of these phases have a similar formula for the scaling dimensions of the corresponding operators

$$\Delta = RE = J + \# \frac{Q^3}{J} + \cdots \ , \quad Q^{3/2} \ll J \ll Q^2 \ . \tag{100}$$

The notation # stands for a numerical coefficient that differs between between the giant vortex and relativistic rotating vortices. We won't go into details but outline some important qualitative points.

- To leading order, the relationship between the energy and the angular momentum is $E = \frac{1}{R} J$. That the coefficient in this formula is exactly 1 is very important – we will see that this is a general feature of the Regge limit.
- The subleading corrections of order Q^3/J becomes of order Q near the boundary of the regime of validity of the giant vortex $J \sim Q^2$. This agrees parametrically with the sub-leading correction in the Regge limit, as we will see later.

6. Regge Physics in Conformal Field Theory

Our discussion has brought us to the point that we need to understand ultrafast spinning operators, with J the biggest parameter.

In theories with a $U(1)$ charge, we have seen that we need a new description for $J \gtrsim Q^2$. In theories like the 3D Ising model or other CFTs we can similarly ask about the large spin operators with the smallest scaling dimension possible.

The large spin limit is more universal than the large charge limit because the large spin limit always exists and also because the properties of generic CFTs in the large spin limit have been determined without any assumptions about the existence of a thermodynamic limit or about the symmetry breaking patterns.

Interestingly, the large spin limit of *any* CFT_d happens to be conveniently captured by a systematic expansion in AdS_{d+1}.

That the large spin limit of any CFT has the properties below can be proven (quite rigorously) without any assumptions. The answer is not quite a classical effective theory, instead, it is a weakly coupled quantum problem with coupling constant which is an inverse power of J, as anticipated in (3).

We present the solution for the large J limit but not the general proof for why it is correct. We need to find highly spinning states on S^2. Just take any state on S^2 and boost it by acting with the generator P which raises the energy on the sphere as in (56). The corresponding operators are descendants. A more interesting question is about spinning primary operators.

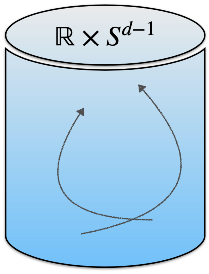


Fig. 5. A two particle state on the cylinder.

The central idea here is to consider any two primary local operators $\mathcal{O}_1$ and $\mathcal{O}_2$ and consider their product with J derivatives inserted in between, written schematically as $\mathcal{O}_1 \partial^J \mathcal{O}_2$. For $J = 0$ or generic small values of J it is not even a priori clear what is meant by this expression since such operators either do not exist or just ambiguous to identify. However, for $J \gg 1$ it turns out that one can arrange the derivatives in such a way that these are primary operators and furtthermore, their scaling dimension is almost as if there are no interactions:

$$\Delta(\mathcal{O}_1 \partial^J \mathcal{O}_2) = J + \Delta(\mathcal{O}_1) + \Delta(\mathcal{O}_2) + \cdots \ , \tag{101}$$

where the $\cdots$ stand for terms suppressed in the large J limit. Each of the $\mathcal{O}_1$ can be interpreted to create a "parton" on the sphere and the linear dispersion with J signifies that they are moving very fast (basically at the speed of light). The fact that there are no corrections to this free parton picture is reminiscent of asymptotic freedom, except that here it arises in generic strongly interacting systems at large angular momentum. More generally, one can define a family of operators schematically written as $\mathcal{O}_1 \Box^n \partial^J \mathcal{O}_2$ whose scaling dimension is $J + 2n + \Delta(\mathcal{O}_1) + \Delta(\mathcal{O}_2)$ up to small corrections. The parameter n will be mapped to some eccentricity of the orbits and J is the angular momentum of the two-parton system.

A convenient notation for these special primary operators is $[\mathcal{O}_1 \mathcal{O}_2]_{n,J}$. These are called the double twist operators. They minimize the scaling dimension for fixed angular momentum as for asymptotically large J,

$$\tau \equiv \Delta - J = 2n + \Delta(\mathcal{O}_1) + \Delta(\mathcal{O}_2) + \cdots \ . \tag{102}$$

Choosing the "partons" $\mathcal{O}_1, \mathcal{O}_2$ appropriately one can then find the true global minimum of the scaling dimension at fixed J. But we will study this whole family of "double twist" primaries and learn how to calculate their properties even in strongly coupled theories.

The existence of these operators and the "asymptotic freedom" of the partons has been by now well established as we can see in figures 6 and 7.

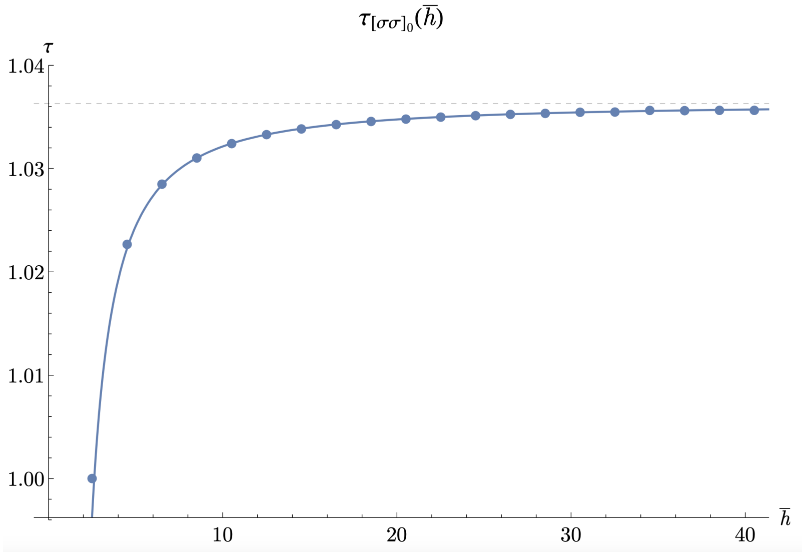


Fig. 6. from DSD. Here $\bar{h} = J$ and as in (102), $\tau = \Delta - J$. In this plot we see the double twist operators corresponding to $[\Phi\Phi]_{n=0,h}$ in the 3D Ising model. We observe a clear horizontal asymptote at $2\Delta(\Phi)$ as predicted from the asymptotically free parton picture.

In the remaining we will explain how to precisely construct these spin J, eccentricity n primary operators, how to calculate the corrections in (102), and finally, we will return to the super fast spinning charged states with $J > Q^2$.

We will study AdS_{d+1} in global coordinates, see figure 8, with metric

$$ds^2 = \frac{L^2}{\cos^2\rho}\left(dt^2 - d\rho^2 - \sin^2\rho\, d\Omega^2\right). \tag{103}$$

The Dilatation operator generates t-translations, so that bulk energies correspond to CFT dimensions via

$$\Delta_{\mathrm{CFT}} = E_{\mathrm{AdS}} L. \tag{104}$$

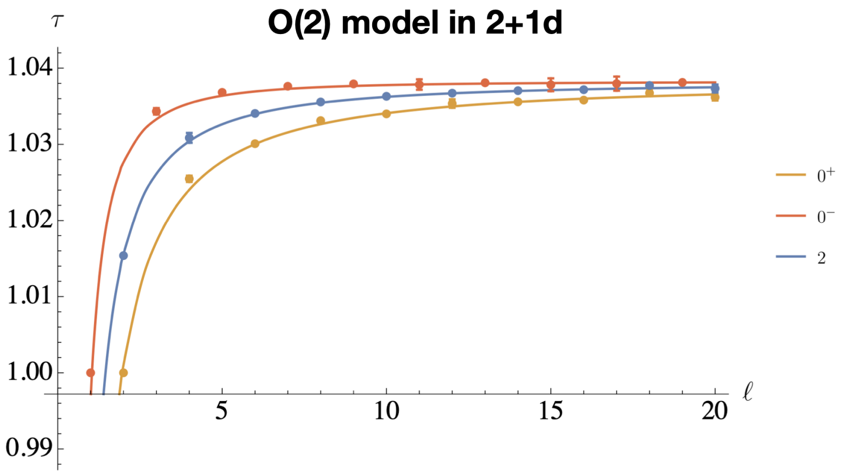


Fig. 7. From Liu, Meltzer, Poland, Simmons-Duffin. Here we see the three $O(2)$ representations corrresponding to the double-twist operators $\left[\Phi\Phi^\dagger\right]_{n=0,\ell}$, and $[\Phi\Phi]_{n=0,\ell}$ in the XY model. We observe a clear horizontal asymptote at $2\Delta(\Phi) = \Delta(\Phi) + \Delta(\Phi^\dagger)$ as predicted from the asymptotically free parton picture.

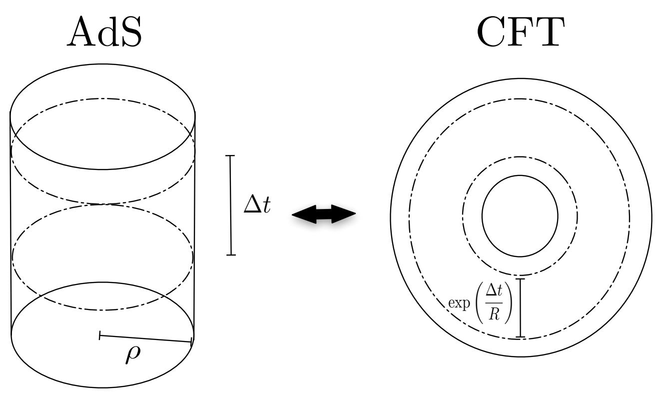


Fig. 8. AdS_{d+1} in global coordinates allows to geometrize the physics of CFT_d for some particular questions, even if there is no concrete holographic duality or weak coupling or sub-cosmological scale locality in the bulk.

We can consider a massive, relativistic particle in AdS_{d+1} and ask about its wave function. For scalar fields this just corresponds to the solutions of the Klein-Gordon equation.

A wave function can be then acted upon with the AdS isometries to create new interesting wave functions. This is analogous to boosting wave functions in flat space, for instance.

Anti-de Sitter space AdS_{d+1} can be embedded in $\mathbb{R}^{2,d}$ as the hyperboloid:

$$-X_{-1}^2 - X_0^2 + \sum_{i=1}^{d} X_i^2 = -L^2.$$

The global coordinates (t, ρ, Ω_i), where $\rho \in [0, \pi/2)$ is a compactified radial coordinate and Ω_i parametrize the S^{d-1}, are related to the embedding coordinates by:

$$\begin{aligned} X_{-1} &= L \sec\rho \cos t, \\ X_0 &= L \sec\rho \sin t, \\ X_i &= L \tan\rho\, \Omega_i, \qquad \text{with } \sum_{i=1}^{d} \Omega_i^2 = 1. \end{aligned}$$

This change of variables gives rise to the global AdS metric:

$$ds^2 = \frac{L^2}{\cos^2\rho}\left(-dt^2 + d\theta^2 + \sin^2\theta\, d\Omega_{d-1}^2\right) \ ,$$

where $t \in \mathbb{R}$ is global time and $d\Omega_{d-1}^2$ is the line element on the unit S^{d-1}.

The isometry group of AdS_{d+1} is $SO(2, d)$. The generators of this group in the embedding space are given by:

$$J_{AB} = X_A \partial_B - X_B \partial_A, \qquad A, B \in \{-1, 0, 1, \ldots, d\}.$$

They map to various Killing vectors in AdS as follows

$$J_{-1,0} = \partial_t.$$

For $i, j = 1, \ldots, d$, the spatial rotation generators are:

$$J_{ij} = -i\left(\Omega_i \frac{\partial}{\partial \Omega_j} - \Omega_j \frac{\partial}{\partial \Omega_i}\right),$$

where Ω_i are coordinates on the unit S^{d-1}. These generate the $SO(d)$ subgroup.

Finally, we have boost-like transformations in AdS, which are the remaining $2d$ generators P_i and K_i.

The wave operator (Laplacian) in AdS is related to the quadratic Casimir of $SO(2,d)$ as:

$$\mathcal{C}_2 = \frac{1}{2} J^{AB} J_{AB}, \qquad \Box = -\frac{1}{L^2}\mathcal{C}_2.$$

Hence, the Klein–Gordon equation:

$$\left(\Box - m^2\right)\Psi = 0$$

is equivalent to:

$$\mathcal{C}_2 \Psi = m^2 L^2 \Psi.$$

A special wave function is that which is annihilated by $K_{+,i}$ – corresponding to a primary operator. To find this wave function we just solve the corresponding first order differential equation $K_{+,i}\Psi = 0$. We are seeking a scalar wave function and hence we simply obtain, after a short calculation, $(\sin\rho\,\partial_t + i\cos\rho\,\partial_\rho)\Psi = 0$,

$$\Psi(t,\rho,\Omega) = e^{i\Delta t}\cos^\Delta\rho. \tag{105}$$

By "corresponding to a primary operator" we do not mean to suggest a holographic correspondence. So far it is just a statement about representation theory; the conformal isometries in CFT_d are realized as true isometries in AdS_{d+1} and this wave function (105) is invariant under the K_μ isometry. It describes a particle deep inside AdS and a rapidly decaying support towards the boundary.

The Laplacian is given by

$$\Box = \frac{1}{L^2}\left[-\cos^2\rho\,\frac{\partial^2}{\partial t^2} + \cos^2\rho\,\frac{\partial^2}{\partial\rho^2} + (d-1)\cot\rho\,\frac{\partial}{\partial\rho} + \frac{\cos^2\rho}{\sin^2\rho}\nabla^2_{S^{d-1}}\right],$$

where $\nabla^2_{S^{d-1}}$ denotes the Laplacian on the unit $(d-1)$-dimensional sphere. Evaluating on our wave function (105) we find the mass of the corresponding particle

$$m^2 = \Delta(\Delta - d)\ .$$

We can act on the wave function with the raising operator P_μ as in (56). Then the particle, instead of sitting at the origin, begins to toss and turn in AdS.

Following [] we now write the wave functions of these descendants. These are obtained by acting with P_μ on the primary wave function (105). The general descendant has the scalar combination P^2 n times and the rest of the P are in a symmetric tensor of ℓ indices. The projection of the angular momentum on some given axis (say the z axis) is given by J.

The corresponding wave function is (see e.g. [**?**, **?**])

$$\psi_{n,\ell J}(t,\rho,\Omega) = \frac{1}{N_{\Delta n\ell}} e^{-iE_{n,\ell}t} Y_{\ell J}(\Omega) \left[\sin^\ell \rho \cos^\Delta \rho \, {}_2F_1\left(-n, \Delta+\ell+n, \ell+\frac{d}{2}, \sin^2\rho\right)\right] \tag{106}$$

with normalizations

$$N_{\Delta n\ell} = (-1)^n \sqrt{\frac{n!\Gamma^2(\ell+\frac{d}{2})\Gamma(\Delta+n-\frac{d-2}{2})}{\Gamma(n+\ell+\frac{d}{2})\Gamma(\Delta+n+\ell)}}, \tag{107}$$

where

$$E_{n,\ell} = \Delta + 2n + \ell \ .$$

While the primary wave function (105) is situation inside AdS and does not toss and turn very much, the wave functions (108) do. For instance, the $n=0$, large ℓ wave function is swirling around the center of AdS – to see that we note that for $n=0$ the wave function simplifies to

$$\psi_{0,\ell J}(t,\rho,\Omega) = \frac{1}{N_{\Delta 0\ell}} e^{-iE_{0,\ell}t} Y_{\ell J}(\Omega) \sin^\ell \rho \cos^\Delta \rho \ , \tag{108}$$

which means that it is centered around $\tan^2\rho_* \sim \ell/\Delta$, which corresponds to physical radial distance

$$D \sim \int^{\rho_*} \frac{d\rho}{\cos\rho} = \coth^{-1}(\sin\rho_*) \sim \coth^{-1}(1-\frac{\Delta}{2\ell}) \sim \frac{L}{2}\log(\frac{2\ell}{\Delta}) \tag{109}$$

These wave functions are descendants with minimal scaling dimension at fixed spin. They just describe a particle in an orbit around the center of AdS.

As we said our actual interest is in constructing primary representations of the conformal ground with minimal scaling dimension at fixed angular momentum, while so far we just discussed descendants.

The idea is to consider a two particle state whose center of mass is essentially at the origin of AdS, while they are rotating around this common center of mass exactly antipodally to each other.

It is important for them to rotate anti-podally to each other since otherwise some small forces between the particles, e.g. electric or gravitational forces, would cause the system to evolve.

If the particles have total angular momentum ℓ and radial quantum number n we denote this state $[\mathcal{O}_1\mathcal{O}_2]_{n,\ell}$ as in (102).

This particular configuration is a primary because the center of mass is at the center of AdS. It is depicted in figure 5. We can then act on this configuration with P_μ and find descendants by having the center of mass rotate around the center of AdS.

If we neglect interactions in AdS it is clear that the scaling dimension of this primary is $\Delta(\mathcal{O}_1)+\Delta(\mathcal{O}_2)+2n+\ell$. Now comes the crucial point. Generic theories are far from being holographic. But for large enough ℓ, since the particles are very separated in the radial coordinate (109), the physics in AdS does correctly reproduce the CFT properties, and furthermore, AdS interactions allow to systematically calculate corrections to the scaling dimensions of $[\mathcal{O}_1\mathcal{O}_2]_{n,\ell}$.

One must also ask what precisely does the state $[\mathcal{O}_1\mathcal{O}_2]_{n,\ell}$ mean in terms of the field theory – i.e. which exact combination of derivatives in (101) leads to a primary. The strategy is to write the two particle state, which is ought to be a primary, as a sum of products of single particle wave functions, each of which could be a primary or descendant. This exercise was done in [**?**, **?**]. In the case of $n=0$ one finds

$$[\mathcal{O}_1\mathcal{O}_2]_\ell = \sum_{\ell_1+\ell_2=\ell} s_{\ell_1,\ell_2}\left(\partial_{\mu_1}\cdots\partial_{\mu_{\ell_1}}\mathcal{O}_1\right)\left(\partial_{\nu_1}\cdots\partial_{\nu_{\ell_2}}\mathcal{O}_2\right) \tag{110}$$

with coefficients

$$s_{\ell_1,\ell_2} = \frac{(-1)^{\ell_1}}{\ell_1!\ell_2!\Gamma(\Delta_1+\ell_1)\Gamma(\Delta_2+\ell_2)}. \tag{111}$$

Exercise: find the $\ell_{1,2}$ that dominate the sum in the large ℓ limit.

The above describes large spin primaries to leading order in the large spin expansion in *any* CFT.

Let us parameterize the corrections to this large spin picture as

$$\Delta(\mathcal{O}_1)+\Delta(\mathcal{O}_2)+2n+\ell+\gamma(n,\ell)\ . \tag{112}$$

The corrections to the free partons picture in the large ℓ picture arise from interactions between the two partons. They may be very far from each other due to (109) but there is still some small force between them.

For instance there is always some small gravitational attraction and if these AdS particles are charged there may be Coulomb forces as well.

We can estimate $\gamma(n,\ell)$ from the gravitational force as

$$\gamma(n,\ell) = \gamma_n/\ell^{d-2}\ . \tag{113}$$

The gravitational force in AdS arises from the energy momentum tensor on the boundary and $d-2$ is the twist $\Delta - J$ of the energy-momentum tensor. More generally, operators of twist τ lead to corrections at large ℓ that scale like $1/\ell^\tau$. Therefore the corrections decay at large ℓ if the twist spectrum is strictly positive (for nontrivial operators). In any unitary CFT in $d > 2$ the twist spectrum is strictly positive so the large ℓ expansion works. But for CFT_2 the story is more complicated since the twist of the energy-momentum tensor vanishes and it needs to be treated carefully. In the AdS_3 picture we see that gravitational interactions cannot be neglected since particles create conical singularities.

Now we sketch the calculation of $\gamma(0, \ell)$ by computing the gravitational interaction between separated bodies in AdS. Since the actual primary state is a complicated linear combination (110) the computation is quite arduous.

For completeness we will now quote the result for $\gamma(0, \ell)$ due to Coulomb forces and gravitational forces (i.e. due to conserved currents and the energy-momentum tensor in the field theory language):

$$\Delta([\Phi\Phi]_{0,\ell}) = 2\Delta(\Phi) + \ell + \frac{\Gamma(d)\Gamma^2(\Delta)}{\Gamma^2(d/2)\Gamma^2(\Delta - \frac{d-2}{2})} \left(\frac{1}{c_J} - \frac{2d(d+1)\Delta^2}{(d-1)^2 c_T} \right) \frac{1}{\ell^{d-2}} \ .$$

$$\Delta([\Phi^\dagger\Phi]_{0,\ell}) = 2\Delta(\Phi) + \ell - \frac{\Gamma(d)\Gamma^2(\Delta)}{\Gamma^2(d/2)\Gamma^2(\Delta - \frac{d-2}{2})} \left(\frac{1}{c_J} + \frac{2d(d+1)\Delta^2}{(d-1)^2 c_T} \right) \frac{1}{\ell^{d-2}} \ .$$

In many theories, including in the XY model, these are the leading corrections at large ℓ. c_J and c_T stand for the two-point functions of the current and energy-momentum tensor. In AdS, these are essentially the inverse fine structure constant and the inverse gravitational constant. By inspection of the above results for $\Delta([\Phi\Phi]_{0,\ell})$ and $\Delta([\Phi^\dagger\Phi]_{0,\ell})$, we see that the gravitational interaction always reduces the scaling dimension of the double twist operators while the Coulomb interaction increases the energy for like charges and decreases the energy for opposite charges. These of course are the sensible outcomes.

As we said the derivation of this result in full fledged form is quite technically demanding so we will do something *much* simpler and discuss the corrections to the energy of non-relativistic particles in the Newtonian approximation.

It is most convenient to switch to the AdS_{d+1} coordinates

$$ds^2 = L^2 \left[-(1 + \frac{r^2}{L^2})dt^2 + (1 + \frac{r^2}{L^2})^{-1} dr^2 + r^2 d\Omega_{d-1}^2 \right] \ .$$

We specialize to $d = 4$, i.e. AdS_5. We find the gravitational potential due to nonrelativistic point sources

$$V(r) = -\frac{8\pi G_N M_1 M_2}{3\Omega_3}\frac{1}{r^2}$$

G_N is the Newton constant in 5 dimensions, and the exponential decay reflects the confining nature of AdS space. $\Omega_3 = 2\pi^2$ the surface area of a unit 3 sphere. (Of course here we see another reason that $d = 2$ is very subtle – the gravitational potential is actually logarithmic and does not decay. The same is true for the Coulomb potential.)

If we are very far from the AdS center we can trade r for the geodesic distance $D = L\log r$, which is approximately correct at large r. Then we find that the potential is

$$V(r) = -\frac{4G_N M_1 M_2}{3\pi}e^{-2D/L}$$

The exponential decay of the gravitational force with the geodesic distance is a reflection of the fact that AdS is a box.

For identical particles of mass M in the particular state (110) each parton carries spin $\sim \ell/2$ and is geodesic distance $\frac{L}{2}\log(\ell/\sqrt{M})$ from the origin and hence the geodesic distance between the particles is $D \sim L\log(\ell/M)$, which leads to

$$V(r) = -\frac{4G_N}{3\pi}\frac{M^2}{\ell^2} \ .$$

This approximation makes sense for very heavy particles for which $M \sim \Delta$. In addition, we use the relation between the Newton constant and the two-point function of the energy-momentum tensor $c = \frac{\pi}{8G_N}$, which finally gives

$$\gamma(0,\ell) = -\frac{1}{6}\frac{(\Delta_1\Delta_2)^2}{c}\left(\frac{1}{\ell}\right)^2 , \qquad (114)$$

Remarkably, this agrees precisely with the exact bootstrap results in the large Δ limit. As expected, the gravitational interactions, i.e. energy momentum tensor exchanges, make the double twist operators with $n = 0$ approach their asymptotic scaling dimensions from below.

6.1. *Back to the XY Model at $J \gg Q^2$*

Here we discuss the regim $J \gg Q^2$ using the ideas from the large spin expansion that we have just explained. The lowest dimension operator of charge Q is schematically Φ^Q. We will seek the lowest lying operators with

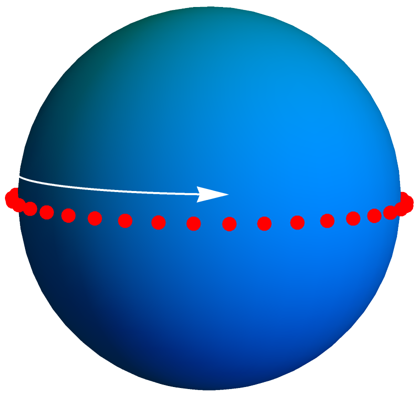


Fig. 9. In the large spin limit, the minimal dimension operator(s) can be thought of as Q partons rotating on the equator of the sphere. When $J/Q^2 \sim 1$ these partons should collapse to the giant vortex.

charge Q and spin J. For very large J these are Q-twist operators of the schematic type

$$\overleftrightarrow{\partial}^{a_1}\Phi\overleftrightarrow{\partial}^{a_2}\Phi\cdots\overleftrightarrow{\partial}^{a_Q}\Phi \ , \tag{115}$$

i.e. we insert many derivatives between any two Φ "partons" such that $\sum_{i=1}^{Q} a_i = J$. If J and each a_i are large enough (we will soon compute how large they need to be) then the large-spin expansion applies and the leading and subleading order terms in the formula for the scaling dimension of the operator (115) are

$$\Delta = J + Q\Delta_\Phi + \cdots \ . \tag{116}$$

One can loosely think about the corresponding state on S^2 in terms of Q partons rotating on the equator of the sphere, see fig. 9

Note that (116) implies a large degeneracy due to the insensitivity of Δ to rearrangements of the derivatives, to leading order. This is analogous to the large ground state degeneracy we encountered at leading order in δ in our analysis of the giant vortex. Next, derivatives with contracted indices can be added to the operator (115) without changing the angular momentum. In fact, there are two ways to add contracted derivatives – either in the form $\partial^\mu\Phi\cdots\partial_\mu\Phi$ which corresponds to $n = 2$ since we have added two derivatives without changing the angular momentum or via $\partial^\nu\Phi\cdots\epsilon_{\mu\nu\rho}\partial^\rho\Phi$

which corresponds to $n = 1$ since we have added one derivative without changing the angular momentum. More generally, we can obtain all integer positive n in this way, exactly matching the giant vortex fluctuations!

It is time to discuss the corrections to (116), as this will allow us to determine when the large-spin expansion breaks down. The corrections to the scaling dimension (116) come from interactions between the $Q = 1$ constitutes. The exchanged "forces" are due to $Q = 0$ lowest-twist operators, where from the unit operator we obtain (116). The next lowest twist operators in the $O(2)$ model are the energy momentum tensor and the $O(2)$ current, both of which have twist 1. A qualitative estimate for the interaction between the Q partons comes from thinking about the problem in the analog AdS_4 configuration of Q partons spinning at radial distance $d \sim L \log(J/Q)$ from the center of AdS_4. The gravitational interaction between all the $\sim Q^2$ pairs thus scales as $\sim Q^3/J$.

We therefore see that the large-spin expansion breaks down at $J \sim Q^2$ and it is only valid for $J \gg Q^2$. This nicely matches with the boundary of the regime of validity of the giant vortex description and further reinforces the connection between the giant vortex theory and the large spin expansion.

Conjecturally, the quantum theory of Q partons with the $\sim Q^3/J$ interactions (mediated by the exchange of the energy momentum tensor and current) settles in a new ground state, which is essentially the giant vortex state.

Qualitatively, since the interaction per parton scales like Q^2/J, once $J \sim Q^2$ the interaction provides sufficient energy for partons to climb over the $O(1)$ repulsive barrier of parton recombination. Notice that this picture is nicely consistent with the interaction between the partons being attractive.[i]

Our discussion so far assumed that we break up the original operator Φ^Q into Q constitutes separated by $\sim J/Q$ derivatives from each other. This is how one arrives at (116).

Here it is important to mention the results of [] which showed that the large spin ground state due to the slight attraction is made out of repulsive molecules.

[i]Let us check that the interaction between the partons is attractive. Denote the two-point functions of the $U(1)$ current and energy-momentum tensor by

$$\langle J_\mu J_\nu \rangle = \frac{\tau}{16\pi^2}\frac{I_{\mu\nu}}{x^4} \;, \qquad \langle T_{\mu\nu} T_{\rho\sigma} \rangle = \frac{C_T}{16\pi^2}\frac{I_{\mu\nu;\sigma\rho}}{x^6} \tag{117}$$

where $I_{\mu\nu} = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}$ and $I_{\mu\nu;\sigma\rho} = \frac{1}{2}\left(I_{\mu\sigma}I_{\nu\rho} + I_{\mu\rho}I_{\nu\sigma}\right) - \frac{1}{3}\delta_{\mu\nu}\delta_{\sigma\rho}$ (see [?]). For a free complex scalar field in 2+1 dimensions of charge 1 we have $\tau = 2$ and $C_T = 3$. In the interacting theory, we have approximately [?] that $\tau = 1.809$ and $C_T = 2.832$. Finally we recall that the parton-parton interaction is attractive if [?]

$$\frac{\Delta_\Phi}{\sqrt{C_T}} \geq \frac{1}{\sqrt{6}}\frac{1}{\sqrt{\tau}} \;. \tag{118}$$

Using the above values of C_T, τ, and $\Delta_\Phi = 0.519$ we find that the left hand side evaluates to 0.308 while the right hand side to 0.304. Since the inequality (118) is indeed satisfied the force between the partons is attractive. This is consistent with our physical picture. We expect however that a weakly repulsive force might also be compatible with a superfluid state; it is indeed known that bosons with repulsive interactions still form a superfluid state for generic values of the angular velocity [?]. Note that, along the lines of our intuition from AdS, if we could increase C_T indefinitely and thereby decrease the Newton constant, we would never have binding gravitational interactions. Notice that inequalities similar to eq. (118) were analyzed in [?] in relation with the weak gravity conjecture [?] in holography.