String Theory

Exercise 2.1:

Compute the mode expansion and mass of an open string with Neumann boundary conditions for the coordinates $X^0, ..., X^{24}$, while the remaining coordinate satisfies the following boundary conditions:

a)Dirichlet boundary conditions at both ends

$$X^{25}(\tau, 0) = y_0 \qquad \text{and} \qquad X^{25}(\tau, \pi) = y_1$$

with y_0, y_1 constant. What is the interpretation of such a solution? Compute the integrated momenta $\int_0^\pi P^\mu(\tau, \sigma) d\sigma$. Are they conserved for all 26 directions?

b) Dirichlet boundary conditions on one end and Newmann boundary conditions on the other

$$X^{25}(\tau, 0) = 0 \qquad \text{and} \qquad \partial_\sigma X^{25}(\tau, \pi) = 0$$

What is the interpretation of this solution?

Exercise 2.2:

Compute the mode expansion and mass spectrum of a closed string with periodic boundary conditions for the coordinates $X^0, ..., X^{24}$ and:

$$X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma) + 2\pi m R$$

where m is integer. When can we make sense of this solution? What condition does one get from the constraint $\int_0^\pi \dot{X} \cdot X' = 0$?

Exercise 2.3:

Compute $[L_m, \alpha_n^\mu]$

Exercise 2.4:

Show that the state

$$|\phi >= e_\mu \alpha_{-1}^\mu |0>$$

has positive norm for e^μ spacelike. Does is satisfy $L_m|\phi >= 0 \ \forall m > 0$? Is it an eigenvector of the operator N? What does the condition $L_0|\phi >= a|\phi >$ require?

Exercise 2.5:

a) Are the following open string states eigenvectors of the operator N?. What is their mass squared?

$$|\phi_1 >= \alpha_{-1}^\mu |0>, \qquad |\phi_2 >= \alpha_{-1}^\mu \alpha_{-1}^\nu |0>$$

$$|\phi_3 >= \alpha^\mu_{-3}|0>, \qquad |\phi_4 >= \alpha^\mu_{-1}\alpha^\nu_{-1}\alpha^\rho_{-2}|0>$$

b) Find the mass squared of the following closed-string states

$$|\phi_1 >= \alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0>, \qquad |\phi_2 >= \alpha^\mu_{-1}\alpha^\nu_{-1}\tilde{\alpha}^\rho_{-2}|0>$$

c) What can you say about the following closed-string state?

$$|\phi_3 > \alpha^\mu_{-1}\tilde{\alpha}^\nu_{-2}|0>$$

Exercise 2.6:

Using the mode expansion for the open string with Neumann boundary conditions, the commutation relations

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\eta^{\mu\nu}\delta_{m+n,0}$$

and the following hint

$$\sum_n \cos(n\sigma)cos(n\sigma') = \pi\delta(\sigma - \sigma')$$

show the following equal time commutators

$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = [P^\mu(\sigma,\tau), P^\mu(\sigma',\tau)] = 0\,, \qquad \text{and} \quad [\mathrm{X}^\mu(\sigma,\tau), \mathrm{P}^\nu(\sigma',\tau)] = \mathrm{i}\eta^{\mu\nu}\delta(\sigma-\sigma')$$

Exercise 2.7:

Write the solution for $X^\mu(\tau,\sigma)$ for the closed string in terms of the world-sheet complex coordinate $z = e^{2i\sigma^+} = e^{2\tau_e + 2i\sigma}$, where $\tau_e \equiv i\tau$. Derive an expression for the oscillators α^μ_{-m} and $\tilde{\alpha}^\mu_{-m}$ in terms of a contour integral in the complex z plane of $\partial_z X^\mu$ and $\partial_{\bar{z}} X^\mu$.

Exercise 2.8:

Using the commutation relations for the oscillator modes and the identity

$$[AB, CD] = A[B,C]D + AC[B,D] + [A,C]DB + C[A,D]B$$

a) Show that

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0}$$

where the second term on the right hand side comes from ordering ambiguities.

b) Show that if $A(1) \neq 0$ it is possible to change the definition of L_0 by adding a constant, so that $A(1) = 0$.

Exercise 2.9: (optional)

Using the Jacobi identity

$$[[L_m, L_n], L_p] + [[L_p, L_m], L_n] + [[L_n, L_p], L_m] = 0$$

derive an equation for the coefficients $A(m)$ in the previous exercise. Assuming $A(1) = 0$, prove that $A(m) = (m^3 - m)A(2)/6$.