String Theory

Exercise 7.1:

Apply the GSO projection to the open string states of mass $M^2 = 1/\alpha'$ in the R and NS sector constructed in exercise 6.7. How many states are there in each sector after the projection?

Exercise 7.2:

Construct the open string states at the next massive level ($M^2 = 2/\alpha'$). Show that after the GSO projection there are the same number of physical degrees of freedom in the NS and R sector.

Exercise 7.3:

What are the gauge transformations of B_2, C_1 and C_3 such that the field strengths $H_3 = dB_2$, $F_2 = dC_1$ and $\tilde{F}_4 = dC_3 + C_1 \wedge H_3$ are gauge invariant? Show that the Chern-Simons term appearing in the type IIA supergravity action

$$S_{CS} = -\frac{1}{4\kappa^2} \int B_2 \wedge F_4 \wedge F_4$$

changes by a total derivative under these gauge transformations.

Exercise 7.4:

Write explictly in indices the field equation $d(*F_2) = 0$ in four dimensions. Do you recognize this equation?

Exercise 7.5:

The bosonic field content of 11-dimensional supergravity is a metric G_{MN} and a 3-form potential A_{MNP}, with field strength $F_4 = dA_3$. The bosonic part of the action is

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{12\kappa_{11}^2} \int A_3 \wedge F_4 \wedge F_4$$

Consider $x^{11} \sim x^{11} + 2\pi R_{11}$ and the 11d metric to be

$$ds^2 = G_{MN}dx^M dx^N = e^{-\sigma} g_{\mu\nu} dx^\mu dx^\nu + e^{2\sigma}(dx^{11} + A_\mu dx^\mu)^2$$

(where the factor $e^{-\sigma}$ in front of the 10d metric is introduced for convenience). Do the Kaluza Klein reduction of the action on the circle, keeping only the zero modes. By doing certain identifications (show them), you should be able to recognise the dimensionally reduced action.

Exercise 7.6:

Consider the bosonic massless sector of type IIB string theory ($g_{\mu\nu}, B_{\mu\nu}, \phi, C_0, C_2, C_4$), whose action in Einstein frame is (recall $g_{E\mu\nu} = e^{-\phi/2} g^S_{\mu\nu}$ where g^S is the string frame metric used in class) that of type IIB supergravity:

$$S_{\rm E} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g_E} \left(R_E - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{-\phi}|H_3|^2 - \frac{1}{2}e^{2\phi}|F_1|^2 - \frac{1}{2}e^{\phi}|\tilde{F}_3|^2 - \frac{1}{4}|\tilde{F}_5|^2 \right)$$
$$-\frac{1}{4\kappa^2} \int C_4 \wedge H_3 \wedge F_3$$

where $\tilde{F}_3 = dC_2 + C_0 H_3$ and $\tilde{F}_5 = dC_4 + C_2 \wedge H_3$.

Combine the axion C_0 and the dilaton into a complex parameter $\tau = C_0 + ie^{-\phi}$, and the 3-form fluxes into a doublet $\mathbb{F}_3 = \begin{pmatrix} H_3 \\ F_3 \end{pmatrix}$ and

a) Show that the action can be written in the form

$$S_{\rm E} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g_E} \left(R_E + \frac{1}{4}{\rm tr}(\partial^\mu \mathcal{M} \partial_\mu \mathcal{M}^{-1}) - \frac{1}{12}\mathbb{F}^T_{\mu\nu\rho}\, \mathcal{M}\, \mathbb{F}^{\mu\nu\rho} - \frac{1}{4}|\tilde{F}_5|^2 \right)$$
$$-\frac{\epsilon_{ij}}{8\kappa^2} \int d^{10}x\, C_4 \wedge \mathbb{F}^i_3 \wedge \mathbb{F}^j_3$$

where $\mathcal{M} = \frac{1}{{\rm Im}\tau} \begin{pmatrix} |\tau|^2 & {\rm Re}\tau \\ {\rm Re}\tau & 1 \end{pmatrix}$

b) Show that the action is invariant under $SL(2,\mathbb{R})$ transformations that leave the Einstein frame metric g_E and $\tilde{F}_5$ invariant, while $\mathbb{F}$ transforms linearly as a doublet and τ nonlinearly:

$$\tau' = \frac{a\tau + b}{c\tau + d}\ , \quad \mathbb{F}' = \Lambda \mathbb{F}\ , \quad \text{where } \Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \in SL(2,\mathbb{R})$$

(i.e. $ad - bc = 1$). (Hint, first check that $\mathcal{M}' = (\Lambda^{-1})^T \mathcal{M} \Lambda^{-1}$.)

Exercise 7.7:

Consider the $SL(2,\mathbb{R})$ transformation in type IIB that takes $\tau \to \tau' = -1/\tau$. How does this act on the 3-form fluxes? For the particular case $C_0 = 0$, how does it act on the dilaton? What does this mean in terms of the string coupling?

Exercise 7.8:

Perform the same transformation of the previous exercise to the type I supergravity action, assuming the gauge field A_μ is invariant. The action you obtain is that of "heterotic supergravity" (to be discussed next class).