

Topological Order and Quantum Computation

Anyons

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February 13, 2014

1 Species of anyons (topological charges) in the toric code:

- $\mathbb{I}$: vacuum or identity (it is always present).
- e : it is created by Z strings.
- m : it is created by X strings.
- ψ : it is the simultaneous presence of e and m .

2 We can write down the **fusion rules**:

- e and m are their own conjugate particles: if I change twice an A or B stabilizer I go back to the vacuum state:

$$e \times e = m \times m = \mathbb{I}$$

- Definition of ψ :

$$e \times m = \psi$$

- It follows:

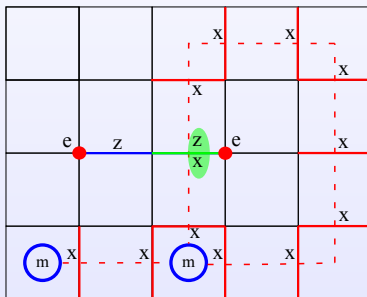
$$e \times \psi = m; \quad m \times \psi = e, \quad \psi \times \psi = \mathbb{I}$$

Braidings in the toric code

- e and m singularly behave as bosons.

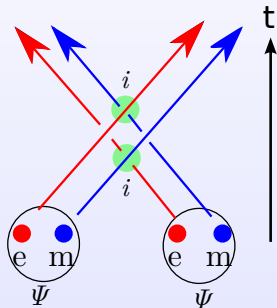
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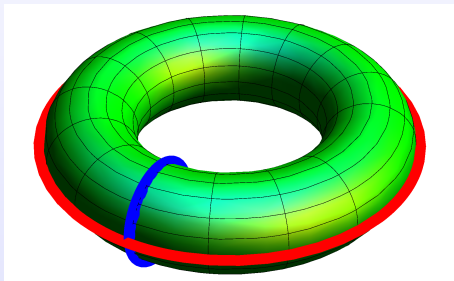
- e and m singularly behave as bosons.
- The mutual statistics of e and m is given by $R_{em} = e^{i\frac{\pi}{2}}$.
- ψ is a fermion.



Braiding and degeneracy of the ground states

Consider a generic topologically ordered system on a torus. For each kind of anyon a in the system, we can define two string **symmetries**, T_1 and T_2 , that correspond to:

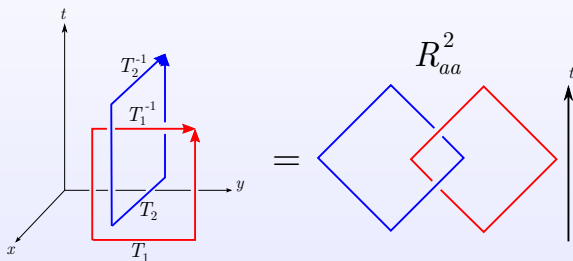
- 1 Create a pair of anyons.
- 2 Wind them around one non-trivial loop.
- 3 We reannihilate them.



Braiding and degeneracy of the ground states

The commutation relation between T_1 and T_2 is related to the braiding statistics R_{aa} of the anyon a :

$$T_1 T_2 T_1^{-1} T_2^{-1} :$$



- If $R_{aa}^2 = 1$, then $[T_1, T_2] = 0$, so there is no degeneracy (bosons or fermions).
- If $R_{aa}^2 \neq 1$, then $[T_1, T_2] \neq 0$, thus there are two non-commuting symmetries and the ground state of the system is degenerate.

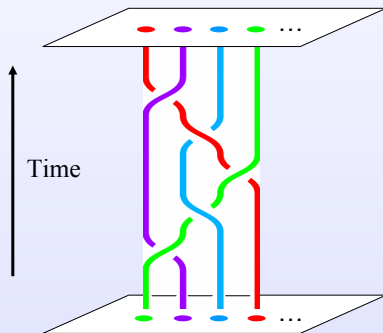
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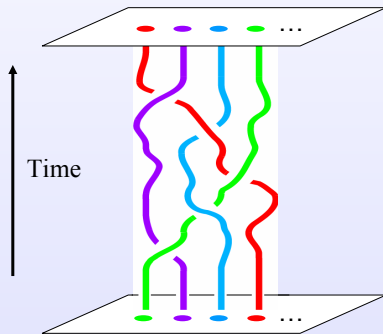
- These unitary operators describe the adiabatic evolution of the system and may be represented in terms of world lines.
- The result of the exchanges does not depend on the detail of the path the anyons undergo.



Anyons

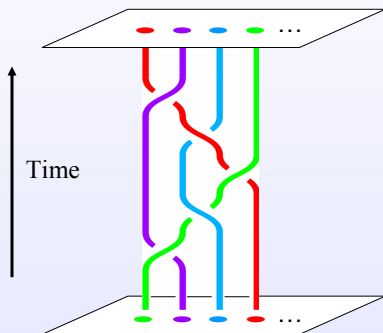
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Braid Group

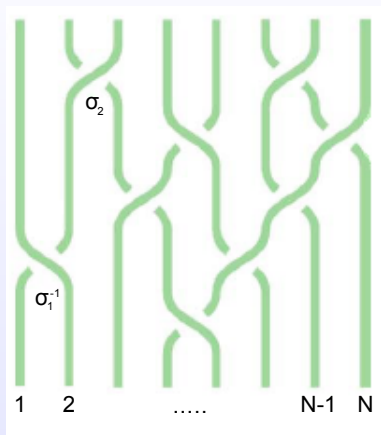
- For fermions or bosons, the wavefunction of a set of indistinguishable particles at fixed position depends only on their permutation.
- For anyons, instead, we must keep track of their time evolution, since $R \neq R^{-1}$.
- The anyon world lines in $2 + 1D$ are self-avoiding strands.
- Their exchange statistics is defined by the **braid group**.



Braid Group (Oktoberfest definition)



Braid Group



- The braid group is generated by the counterclockwise and clockwise exchanges of neighboring anyons $\sigma_i, \sigma_i^\dagger$.
- Disjoint operators commute:

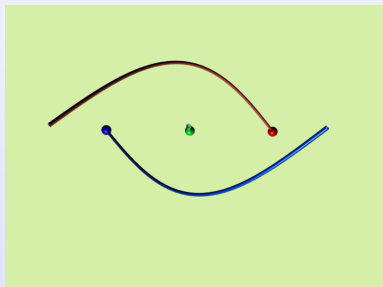
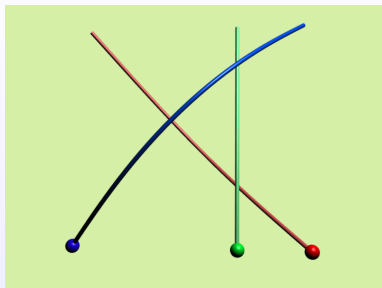
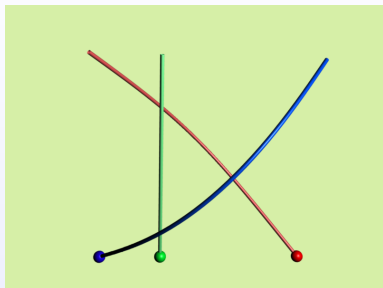
$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i - j| > 1$$

- Neighboring operators obey the Yang-Baxter relation:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

- For $\sigma_i^2 = 1$ we recover the permutations.

Yang Baxter Braiding



Algebra relations

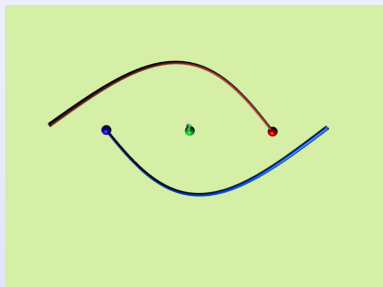
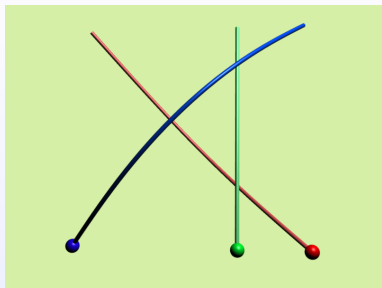
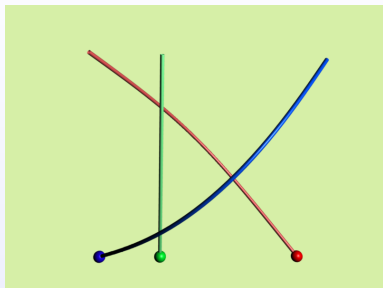
For non-adjacent operators:

$$[\sigma_i, \sigma_k] = 0 \quad \text{if} \quad |i - k| \geq 2$$

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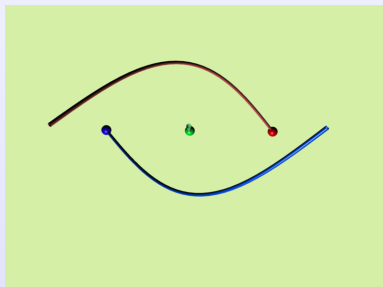
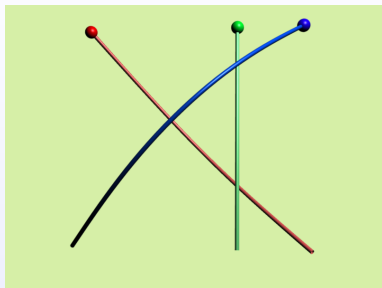
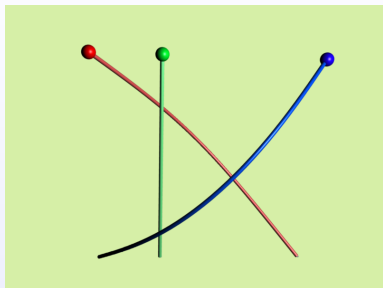
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Abelian anyons

- The easiest non-trivial representation of the braid group is provided by Abelian anyons:

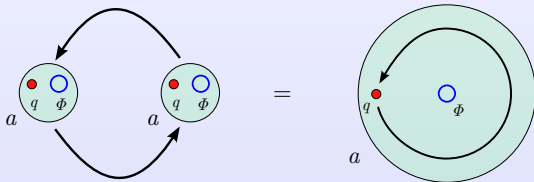
$$\sigma \rightarrow R_{aa} = e^{i\theta_a}.$$

- Abelian anyons can be described in terms of charge-flux composite objects where:

$$\theta_a = q_a \Phi_a$$

- **Spin-Statistics:** The exchange of two Abelian anyons a gives the same phase as a 2π rotation of q_a around Φ_a :

$$\frac{1}{2}q_a\Phi_a + \frac{1}{2}q_a\Phi_a = q_a\Phi_a$$

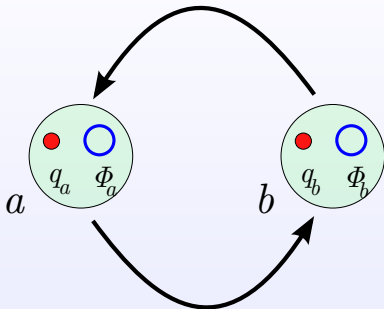


For two Abelian anyons:

$$a \times b = c$$

Then:

$$R_{ab}^c = \exp[i(\theta_c - \theta_a - \theta_b)/2]$$

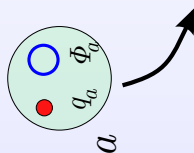
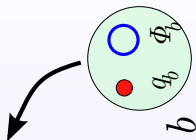


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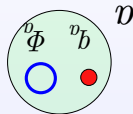
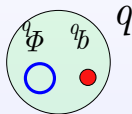


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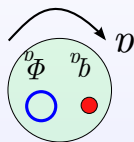
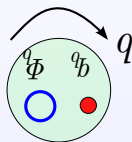


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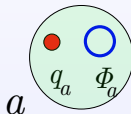
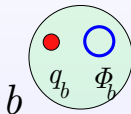


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We could also write:

$$\phi_a(z_1) \phi_b(z_2) = \frac{1}{(z_1 - z_2)^{\Delta_a + \Delta_b - \Delta_c}} \phi_c(z_1)$$



- **Non-Abelian Anyons** correspond to higher dimensional representations of the Braid group.

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- To obtain these higher dimensions we need to introduce a new degeneracy.
- A pair of non-Abelian anyons may assume different states, characterized by different topological charges:

$$a \times b = c \oplus d \oplus e \oplus \dots$$

- Each pair define a Hilbert space, and the braidings are unitary operators on these spaces.
- Braidings of neighboring pairs of non-Abelian anyons, in general, do not commute.

Fusion Rules

Let's consider the simple case of spin $\frac{1}{2}$ (Qubit):

$$\frac{1}{2} \times \frac{1}{2} = 0 + 1 \quad \longrightarrow \quad 2 \otimes 2 = 1 \oplus 3$$

- A particle with spin $1/2$ is described by a two-dimensional Hilbert space
- When two of them fuse, they give rise to a singlet or to a triplet. This is a **Fusion Rule**

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Non-Abelian anyons are characterized by non trivial fusion rules.

Ising Anyons / Majorana modes:

$$\sigma \times \sigma = \mathbb{I} + \varepsilon$$

$$\gamma \times \gamma = \mathbb{I} + \psi$$

Fibonacci Anyons:

$$\tau \times \tau = \mathbb{I} + \tau$$

In general one writes:

$$a \times b = \sum_c N_{ab}^c c$$

Non Abelian Anyons: main ‘ingredients’

To describe a non-Abelian anyon model we need a theory characterized by the following elements:

- **Fusion Rules:** N_{ab}^c
- **Associativity Rules:** $(F_d^{abc})_{xy}$
- **Braiding Rules:** $\sigma \rightarrow R_{ab}^c$

These rules must have a coherent structure and must obey several constraints.

A non-Abelian anyonic model is defined starting from a **finite set of particles** (*Topological charges*).

These particles are linked by the fusion rules:

$$a \times b = \sum_c N_{ab}^c c \quad \longrightarrow \quad V_a \otimes V_b = \bigoplus_c N_{ab}^c V_c \quad \longrightarrow \quad d_a d_b = \sum_c N_{ab}^c d_c$$

where $N_{ab}^c = 0, 1$; $V_{ab}^c = V_c$ are Hilbert spaces and d_i are their quantum dimension.

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where $N_{ab}^c = 0, 1$; $V_{ab}^c = V_c$ are Hilbert spaces and d_i are their quantum dimension.

a is a non-Abelian anyon if $\sum_c N_{aa}^c \geq 2$.

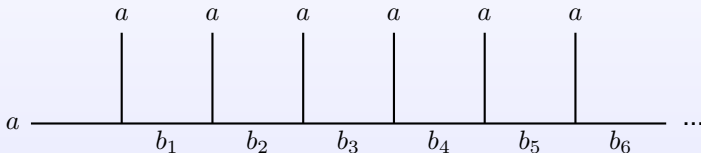
This means that a pair of a anyons may be found in at least two degenerate states.

Fusion Rules

- N_{ab}^c can be understood as a (transfer) matrix: $(N_a)_{b_i}^{b_{i+1}}$.
- Starting from the anyon b_i , N_a defines the possible states b_{i+1} that can be obtained adding a .

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- Consider a chain of a anyons:

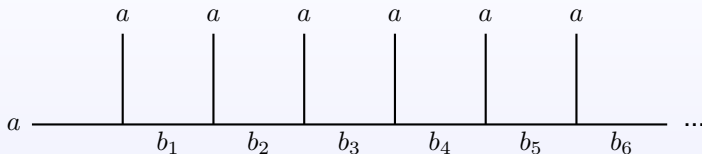


- A state in this chain is defined by the string $\{b_i\}$ and lives in the space:

$$V_{a_1 \dots a_n}^{b_n} = \bigoplus_{b_1, \dots, b_{n-1}} V_{a_1 a_2}^{b_1} \otimes V_{b_1 a_3}^{b_2} \otimes V_{b_2 a_4}^{b_3} \otimes \dots \otimes V_{b_{n-2} a_n}^c.$$

Anyon chains

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- A state in this chain is defined by the string $\{b_i\}$ and lives in the space:

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- The number of total orthogonal states (strings) is:

$$\dim(V_{a_1 \dots a_n}^{b_n}) = (N_{a_2} N_{a_3} \dots N_{a_n})_{a_1}^{b_n} = \left[(N_a)^{n-1} \right]_a^{b_n} \approx d_a^{n-1}$$

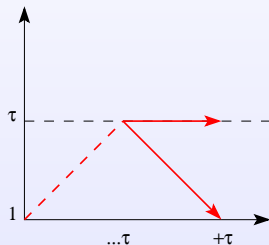
- d_a is the highest eigenvalue of N_a , it is called quantum dimension of a .

Fibonacci anyons

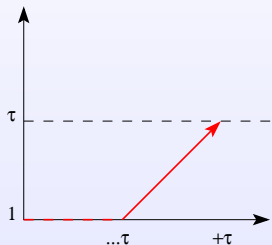
Fusion rules

- The model is characterized by two sectors: the **Vacuum** $\mathbb{I}$ and the **Fibonacci anyon** τ .
- **Fusion Rules:**

$$\tau \times \tau = \mathbb{I} + \tau$$



$$\mathbb{I} \times \tau = \tau$$



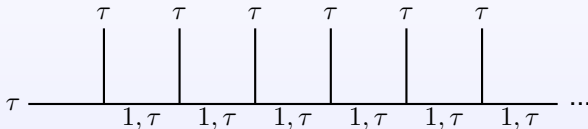
These fusion rules correspond to:

$$N_{\tau} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \implies d_{\tau}^2 - d_{\tau} - 1 = 0 \implies d_{\tau} = \frac{1 + \sqrt{5}}{2} \equiv \phi$$

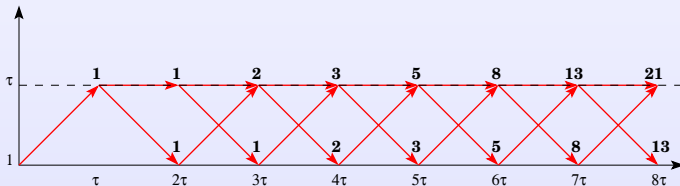
Bratteli diagram

Fibonacci chain

$$\tau \times \tau = \mathbb{I} + \tau, \quad N_\tau = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$



Constraint: there cannot be two consecutive vacua 1.



The number of states grows like the Fibonacci numbers.

$d_\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio!

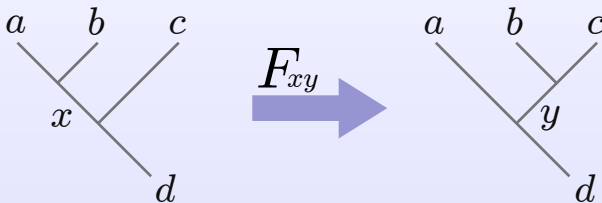
Associativity Rules

F-Matrices

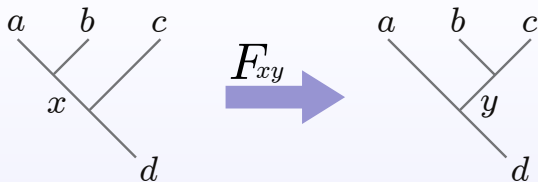
- For an anyonic theory to be consistent the fusion rules N must be associative:

$$\sum_x N_{ab}^x N_{xc}^d = \sum_y N_{ay}^d N_{bc}^y$$

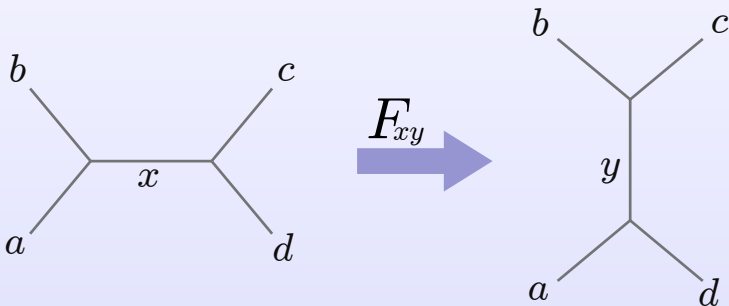
- These relations characterize the fusion process $abc \rightarrow d$ in the fusion space $V_{abc}^d = V_{(ab)c}^d = V_{a(bc)}^d$.
- The two descriptions of the space V_{abc}^d correspond to different orthogonal bases
- There must be a unitary operator that relates these bases:

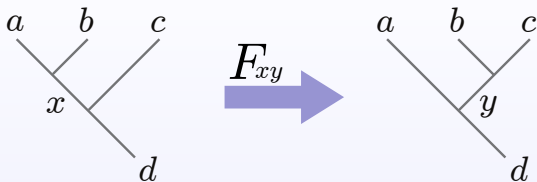


- $(F_d^{abc})_{xy}$ is this transformation.

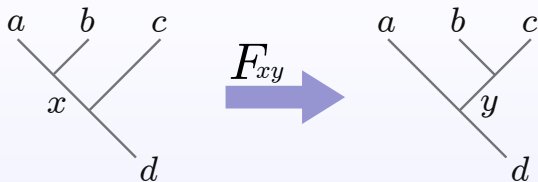


Topologically equivalent to:

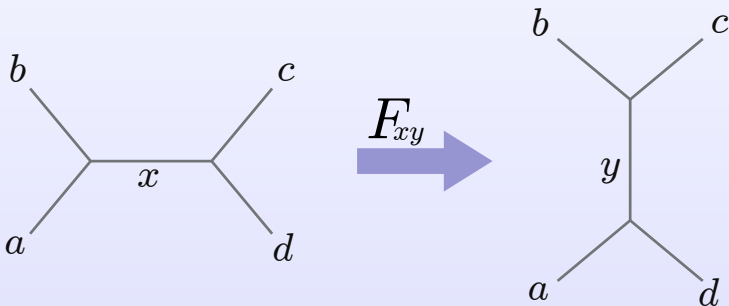




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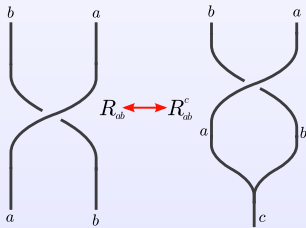


Braidings R_{ab}

A couple of anyons $a \times b$ can be in a superposition of states V_{ab}^k defined by the fusion rules:

$$\phi_a(z_1)\phi_b(z_2) = \frac{\phi_c(z_2)}{(z_1 - z_2)^{\Delta_a + \Delta_b - \Delta_c}} + \frac{\phi_d(z_2)}{(z_1 - z_2)^{\Delta_a + \Delta_b - \Delta_d}} + \dots$$

The clockwise exchange R_{ab} does not affect their total charge:



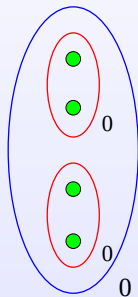
$$R_{ab} = \begin{pmatrix} R_{ab}^a & 0 & 0 & 0 \\ 0 & R_{ab}^b & 0 & 0 \\ 0 & 0 & R_{ab}^c & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}$$

where: $(R_{ab}^c)^2 = e^{-2\pi i(\Delta_a + \Delta_b - \Delta_c)}$

The representations of the braid generators σ_i are given by combinations of F and R .

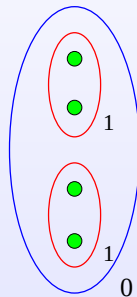
Fibonacci anyons and qubits

- Differently from Ising anyons and Majorana modes, Fibonacci anyons allow for universal quantum computation with braidings only.
- To encode a qubit we use a system of 4 anyons whose total charge is trivial:



$|0\rangle$

Each pair annihilates.

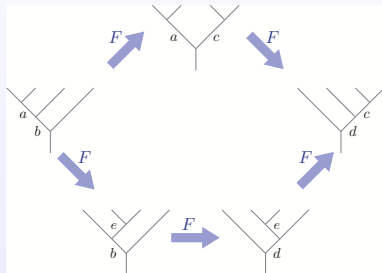


$|1\rangle$

Each pair gives a single τ

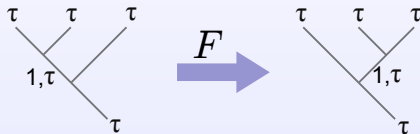
Fibonacci F - Matrix

The unitary matrix $F_{\tau}^{\tau\tau\tau}$ can be calculated from a particular constraint called pentagon equation:



$$F_{11} = F_{1\tau} F_{\tau 1}$$

$$F_{11} + F_{\tau\tau}^2 = 1$$



The resulting matrix is:

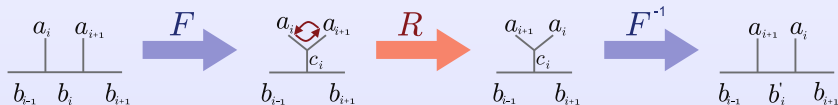
$$F = \begin{pmatrix} \varphi & \sqrt{\varphi} \\ \sqrt{\varphi} & -\varphi \end{pmatrix} \quad \text{with} \quad \varphi = d_{\tau}^{-1} = \frac{1 - \sqrt{5}}{2}$$

Fibonacci Braiding

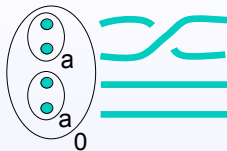
- To process a single qubit we must find the operators σ that defines the braidings.
- From the Yang-Baxter eq. (or the hexagon equation) one finds out the R matrix:

$$R = \begin{pmatrix} e^{\frac{4}{5}\pi i} & 0 \\ 0 & -e^{\frac{2}{5}\pi i} \end{pmatrix}$$

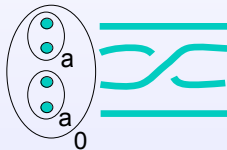
- In a Fibonacci chain, to find the representations of σ 's, we need to make a basis transformation in order to apply the R - matrix:



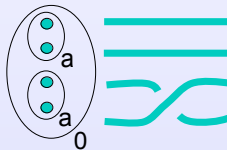
Fibonacci Braiding



$$\sigma_3 = \sigma_1 = R^{-1} = \begin{pmatrix} e^{-\frac{4}{5}\pi i} & 0 \\ 0 & -e^{-\frac{2}{5}\pi i} \end{pmatrix}$$



$$\sigma_2 = F\sigma_1 F = \begin{pmatrix} -\varphi e^{-i\frac{\pi}{5}} & -\sqrt{\varphi} e^{i\frac{2\pi}{5}} \\ -\sqrt{\varphi} e^{i\frac{2\pi}{5}} & -\varphi \end{pmatrix}$$

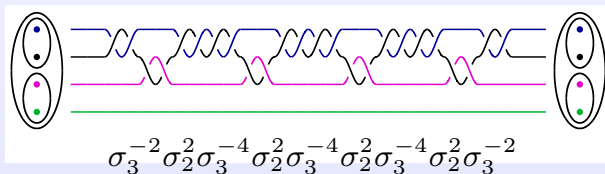


$$\sigma_1 = \sigma_3 = R^{-1} = \begin{pmatrix} e^{-\frac{4}{5}\pi i} & 0 \\ 0 & -e^{-\frac{2}{5}\pi i} \end{pmatrix}$$

Single-Qubit Gate Compiling

Bonesteel et al.

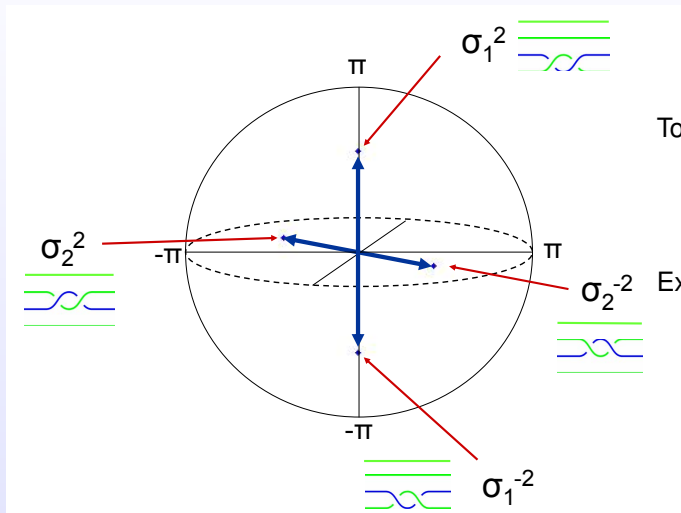
- To the purpose of Universal Quantum Computation we want to approximate, at any give accuracy, any single-qubit gate using as generators the braidings σ_1 and σ_2
- For Fibonacci anyons the elementary braidings generate an **infinite group**, dense in $SU(2)$



$$\cong -iX \pm 0.0031$$

Brute Force search

Bonesteel et al.



Total weaves:

$$B_N \cong 3^N$$

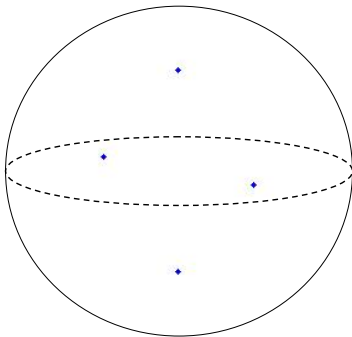
Expected error:

$$\varepsilon_N \cong \frac{1}{3^{N/3}}$$

Brute Force search

Bonesteel et al.

$N = 1$



Total weaves:

$$B_N \cong 3^N$$

Expected error:

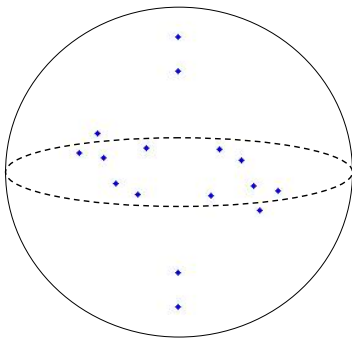
$$\varepsilon_N \cong \frac{1}{3^{N/3}}$$



Brute Force search

Bonesteel et al.

$N = 2$



Total weaves:

$$B_N \cong 3^N$$

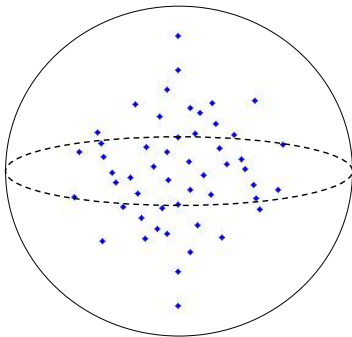
Expected error:

$$\varepsilon_N \cong \frac{1}{3^{N/3}}$$

Brute Force search

Bonesteel et al.

$N = 3$



Total weaves:

$$B_N \cong 3^N$$

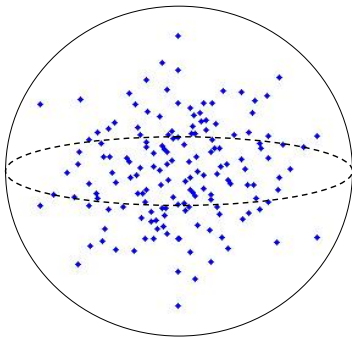
Expected error:

$$\varepsilon_N \cong \frac{1}{3^{N/3}}$$

Brute Force search

Bonesteel et al.

$N = 4$



Total weaves:

$$B_N \cong 3^N$$

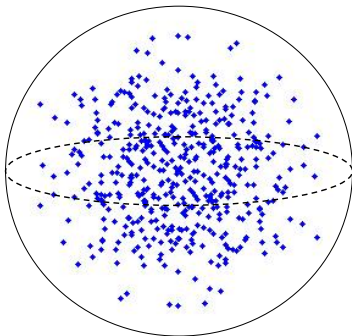
Expected error:

$$\varepsilon_N \cong \frac{1}{3^{N/3}}$$

Brute Force search

Bonesteel et al.

$N = 5$



Total weaves:

$$B_N \cong 3^N$$

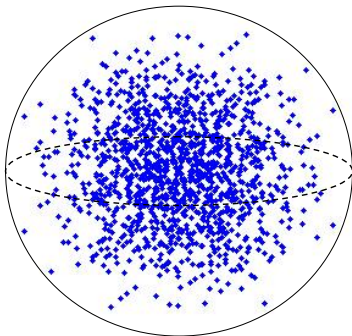
Expected error:

$$\varepsilon_N \cong \frac{1}{3^{N/3}}$$

Brute Force search

Bonesteel et al.

$N = 6$



Total weaves:

$$B_N \cong 3^N$$

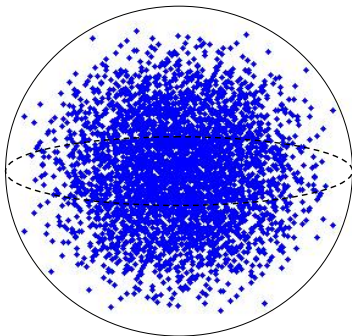
Expected error:

$$\varepsilon_N \cong \frac{1}{3^{N/3}}$$

Brute Force search

Bonesteel et al.

$N = 7$



Total weaves:

$$B_N \cong 3^N$$

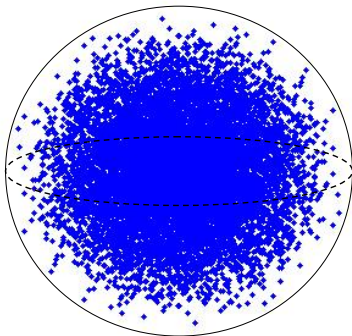
Expected error:

$$\varepsilon_N \cong \frac{1}{3^{N/3}}$$

Brute Force search

Bonesteel et al.

$N = 8$



Total weaves:

$$B_N \cong 3^N$$

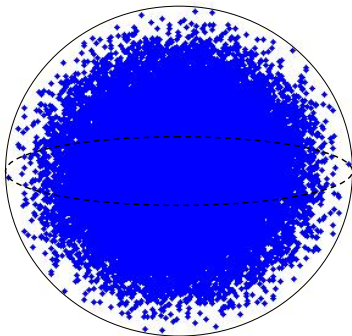
Expected error:

$$\varepsilon_N \cong \frac{1}{3^{N/3}}$$

Brute Force search

Bonesteel et al.

$N = 9$



Total weaves:

$$B_N \cong 3^N$$

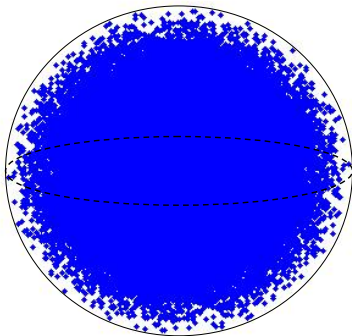
Expected error:

$$\varepsilon_N \cong \frac{1}{3^{N/3}}$$

Brute Force search

Bonesteel et al.

$N = 10$



Total weaves:

$$B_N \cong 3^N$$

Expected error:

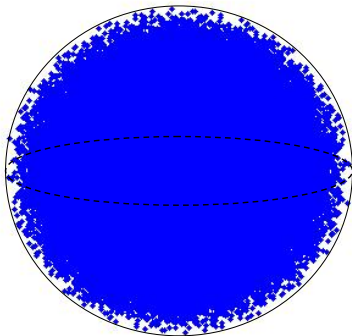
$$\varepsilon_N \cong \frac{1}{3^{N/3}}$$



Brute Force search

Bonesteel et al.

$N = 11$



Total weaves:

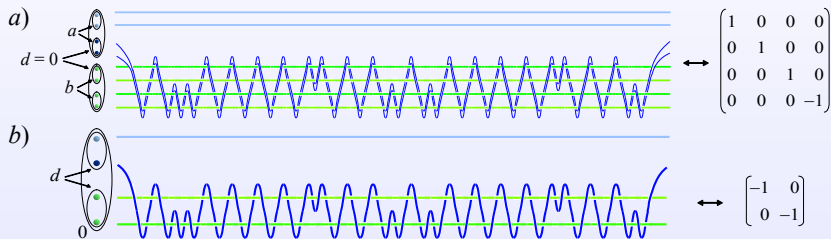
$$B_N \cong 3^N$$

Expected error:

$$\varepsilon_N \cong \frac{1}{3^{N/3}}$$

Two-qubit operators

Hormozi, Bonesteel and Simon, PRL 103 (2009)



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