The 1DBG

$$H = \int dx\, \psi^\dagger(x)\left(-\frac{\hbar^2}{2m}\partial_x^2\right)\psi(x)$$

$$+ \frac{1}{2}\int dx_1 dx_2\, V(x_1 - x_2)\, \psi^\dagger(x_1)\psi^\dagger(x_2)$$

$$\times\, \psi(x_2)\psi(x_1)$$

Fermions: $V \sim \delta(x_1 - x_2)$: no effect

GS: Fermi sea

Lieb-Liniger

$$H = \sum_{j=1}^{N} -\partial_{x_j}^2 + 2c \sum_{j_1<j_2} \delta(x_{j_1}-x_{j_2})$$

$$H = \int dx \left\{ \partial_x \psi^\dagger \partial_x \psi + c\, \psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x) \right\}$$

EOM: $$i\partial_t \psi = [H,\psi] = -\partial_x^2 \psi + 2c\, \psi^\dagger \psi \psi$$

Trivial charges:

$$\hat{N} = \int dx\, \psi^\dagger \psi \qquad \hat{P} = -\frac{i}{2}\int dx \left[\psi^\dagger \partial_x \psi - (\partial_x \psi^\dagger)\psi \right]$$

1st & 2nd quantized N-b.: linked by

$$|\Psi_N\rangle = \int dx_1 \cdots dx_N \, \Phi_N(x_1, \ldots, x_N) \, \psi^\dagger(x_1) \cdots$$
$$\times \cdots \psi^\dagger(x_N)|0\rangle$$

Schröd.: $H|\Psi_N\rangle = E_N|\Psi_N\rangle$

equiv.: $H\,\Phi_N(\{x\}) = E_N\,\Phi_N$

Bogoliubov Theory

FT: $\psi(x) = \frac{1}{L}\sum_k e^{ikx}\psi_k \qquad [\psi_k, \psi^\dagger_{k'}] = L\,\delta_{kk'}$

$$H = \frac{1}{L}\sum_k k^2 \psi^\dagger_k \psi_k + \frac{c}{L^3}\sum_{k_1,k_2,q} \psi^\dagger_{k_1+q}\psi^\dagger_{k_2-q}\psi_{k_2}\psi_{k_1}$$

Idea: $\hat{N} = \hat{N}_0 + \frac{1}{L}\sum_{k\neq 0}\psi^\dagger_k\psi_k$

Expect: $\langle \hat{N}_0 \rangle = O(N)$

Results:

$$H \to E_0 + \frac{1}{L}\sum_{k\neq 0} \varepsilon_B(k)\, \tilde{\psi}^\dagger_k \tilde{\psi}_k$$

$$\varepsilon_B(k) = \left[k^4 + k^2 \cdot 4cn\right]^{1/2}$$

$n = N/L$

$$\frac{E_0}{L} = cn^2 - \frac{4}{3\pi}(cn)^{3/2} = n^3\gamma\left\{1 - \frac{4}{3\pi}\sqrt{\gamma}\right\}$$

$$\gamma \equiv c/n$$

Eigenstates of Lieb Lin.

Start with $N=2$

$$H^{(2)} = -\partial_{x_1}^2 - \partial_{x_2}^2 + 2c\,\delta(x_1 - x_2)$$

move to relative coordinates

$$x_+ = \frac{1}{2}(x_1 + x_2) \quad x_- = x_1 - x_2$$

Integrate x_- in interval $[-\varepsilon, \varepsilon]$

$$-\frac{1}{2}(\partial_1 - \partial_2)\psi_2\Big|_{x_1 - x_2 = -0^+}^{x_1 - x_2 = 0^+} + c\,\psi_2\Big|_{x_1 = x_2} = 0$$

Using symm: $\left(\partial_{x_2}-\partial_{x_1}-c\right)\psi_2\Big|_{x_2-x_1=0^+}=0$

In domain D_2: $x_1<x_2$,

SE: solved by $e^{ik_1x_1+ik_2x_2}$

or $e^{ik_2x_1+ik_1x_2}$

$E=k_1^2+k_2^2$ $\quad P=k_1+k_2$

Ansatz:

$$\Psi_2(x_1,x_2|k_1,k_2)\Big|_{x_1,x_2\in D_2} = S_1\, e^{ik_1x_1+ik_2x_2} + S_2\, e^{ik_2x_1+ik_1x_2}$$

Bdry: gives $\dfrac{S_2}{S_1} = -\dfrac{c+i(k_1-k_2)}{c-i(k_1-k_2)}$

Notⁿ: $\dfrac{S_2}{S_1} = -e^{i\phi(k_1-k_2)}$

$$\phi(k) = \frac{1}{i}\ln\frac{c+ik}{c-ik} = 2\,\mathrm{atan}\,\frac{k}{c}$$

So:

$$\Psi_2(x_1,x_2|k_1,k_2)\Big|_{D_2} = e^{ik_1x_1+ik_2x_2-\frac{i}{2}\phi(k_1-k_2)} - e^{ik_2x_1+ik_1x_2+\frac{i}{2}\phi(k_1-k_2)}$$

extending to whole of $\mathbb{R}^2$:

$$\Psi_2 = \operatorname{sgn}(x_2-x_1)\left\{e^{ik_1x_1+ik_2x_2-\frac{i}{2}\operatorname{sgn}(x_2-x_1)\phi(k_{12})} - e^{ik_2x_1+ik_1x_2+\frac{i}{2}\operatorname{sgn}(x_2-x_1)\phi(k_{12})}\right\}$$

or

$$\Psi_2 = \operatorname{sgn}(x_2-x_1)\sum_{P\in\pi_2}(-1)^P e^{ik_{P_1}x_1+ik_{P_2}x_2} \times$$
$$\times e^{\frac{i}{2}\operatorname{sgn}(x_2-x_1)\phi(k_{P_2}-k_{P_1})}$$

Property: $\sim$ Pauli!

$$\Psi_2^{(a)}(x_1,x_2|k_2,k_1) = -\Psi_2^{(a)}(x_1,x_2|k_1,k_2)$$

$$\Psi_2^{(s)}(x_1,x_2|k_1,k_2) = \mathrm{sgn}(k_2-k_1)\,\Psi^{(a)}$$

Many particles: solving SE equations

Define D_N: $x_1 < x_2 < \ldots < x_N$

$$(H^{(N)} - E_N)\Psi_N = 0 \quad \text{in } D_N$$

$$(\partial_{x_{j+1}} - \partial_{x_j} - c)\Psi_N\Big|_{x_{j+1}-x_j=0^+} = 0$$

Sol$^{\underline{n}}$: Bethe Ansatz

$$\Psi_N(x_1,\dots,x_N|k_1,\dots,k_N)\Big|_{\mathbb{R}^N} = \prod_{N\geq j_1>j_2\geq 1} \operatorname{sgn}(x_{j_1}-x_{j_2})\operatorname{sgn}(k_{j_1}-k_{j_2})$$

$$\times \sum_{P\in\Pi_N} (-1)^P e^{i\sum_{j=1}^N k_{P_j}x_j + \frac{i}{2}\sum_{j_1>j_2}^N \operatorname{sgn}(x_{j_1}-x_{j_2})\,\phi(k_{P_{j_1}}-k_{P_{j_2}})}$$

$$E_N = \sum_{j=1}^N k_j^2 \qquad P_N = \sum_{j=1}^N k_j$$

Quantization & the Bethe eq$^{\underline{n}}$s.

Put system on ring of length L

Impose PBCs

$$\Psi_2(x_1+L, x_2) = \Psi_2(x_1, x_2+L)$$
$$= \Psi_2(x_1, x_2)$$

This gives

$$e^{ik_1 L} = -e^{-i\phi(k_1-k_2)}$$

$$e^{ik_2 L} = -e^{i\phi(k_1-k_2)}$$

Equiv: $e^{ik_1L} = \frac{k_1 - k_2 + ic}{k_1 - k_2 - ic}$ $e^{ik_2L} = \frac{k_2 - k_1 + ic}{k_2 - k_1 - ic}$

Take logs:

$$k_1 + \frac{1}{L}\phi(k_1 - k_2) = \frac{2\pi}{L} I_1 \qquad I_{1,2} \in \mathbb{Z} + \frac{1}{2}$$

$$k_2 + \frac{1}{L}\phi(k_2 - k_1) = \frac{2\pi}{L} I_2$$

Comments: if $I_1 = I_2$, then $k_1 = k_2$.

Proof: easy ex.

Many particles:

$$\Psi_N(x_1,\dots,x_j+L,\dots)=\Psi_N(x_1,\dots,x_j,\dots)$$

Bethe eq$\underset{\sim}{n}$s:

$$e^{ik_jL}=\prod_{l\neq j}\frac{k_j-k_l+ic}{k_j-k_l-ic}=(-1)^{N-1}e^{-i\sum_{l=1}^{N}\phi(k_j-k_l)}$$

or $$k_j+\frac{1}{L}\sum_l\phi(k_j-k_l)=\frac{2\pi}{L}I_j\qquad j=1,\dots,N$$

$$I_j\in\begin{cases}\mathbb{Z}+1/2 & N \text{ even}\\ \mathbb{Z} & N \text{ odd}\end{cases}$$

Properties of sol$^{\underline{n}}$s to Bethe eq$^{\underline{n}}$s

Theorem $\underline{1}$: for $c > 0$ (repulsive gas), all sol$^{\underline{n}}$s are real.

Proof: consider e^{ikL}.

Note: if $\mathrm{Im}(k) \gtrless 0$, $|e^{ikL}| \lessgtr 1$

Also, $\left|\dfrac{k + ic}{k - ic}\right| \geq 1$ for $\mathrm{Im}(k) \geq 0$

Consider a set $\{k_j\}$ solving BE.

Pick k with largest im part: k_{max}

$$\left|e^{i k_{max} L}\right| = \left| \prod_j \frac{k_{max} - k_j + ic}{k_{max} - k_j - ic} \right| \geq 1$$

so $\mathrm{Im}\, k_{max} \leq 0$

Conversely, $\mathrm{Im}\, k_{min} \geq 0$. $\square$

(for $c < 0$, $\exists$ complex k, bound states)

Theorem 2 (Yang & Yang 1969):

given a proper set of quantum nos, the sol^n of the BE exists and is unique

Proof: define an action

$$S(\{k\}) = \frac{L}{2}\sum_j k_j^2 + \sum_{j<l} \Phi(k_j - k_l) - 2\pi I_j k_j$$

$$\Phi(k) = \int_0^k dk' \, \phi(k') = 2k \arctan k/c - c \ln(1+k^2/c^2)$$

Statement: BE $\Leftrightarrow$ extremum of $\tilde{S}$

$$\partial_{k_j} S = 0 = k_j L + \sum_l \phi(k_j - k_l) - 2\pi I_j$$

Consider curvature of S:

$$S_{jl} = \frac{\partial^2 S}{\partial k_j \partial k_l} = \delta_{jl}\left\{ L + \sum_{m=1}^{N} \frac{2c}{(k_j - k_m)^2 + c^2} \right\} - \frac{2c}{k_{jl}^2 + c^2}$$

$\forall$ real vector $\underset{\sim}{v}_j$ can show

$$\sum_{jl} v_j S_{jl} v_l = \ldots > 0$$