Thermo limit

Def$^n$: $\frac{1}{2\pi}\frac{d\phi(k)}{dk} = \frac{1}{\pi}\frac{c}{k^2+c^2} =: \varrho(k)$

$$0 \leq \varrho(k) \leq \frac{2}{c} \qquad k \in \mathbb{R}$$

$$\phi(k_1)-\phi(k_2) = \int_{k_2}^{k_1} dk\, \varrho(k) \leq \frac{2}{c}(k_1-k_2)$$

$$\frac{2\pi}{L}|I_j - I_1| \geq |k_j - k_1| \geq \frac{2\pi}{L}\frac{|I_j - I_1|}{(1+2m/c)}$$

Let's view $\frac{I_j}{L} = x_j \in \mathbb{R}$

BE: $k(x_j) + \frac{1}{L}\sum_l \phi(k(x_j) - k(x_l)) = 2\pi x_j$

$k(x_j) \equiv k_j$ solving BE

Introduce a density $\rho(x) = \frac{1}{L}\sum_j \delta\left(x - \frac{I_j}{L}\right)$

Extend def$^{\underline{n}}$ of $k(x)$ to $\mathbb{R}$ by requiring

$$k(x) + \int_{-\infty}^{\infty} dy\, \phi(k(x) - k(y))\,\rho(y) = 2\pi x$$

Define other densities:

$$\rho_h(x) = \frac{1}{L} \sum_{n \in \{\tilde{I}\}} \delta\left(x - \frac{n}{L}\right)$$

$\{I\}$: occupied # 

$\{\tilde{I}\}$: unoccup.

$$\rho_t(x) = \frac{1}{L} \sum_{n \in \{I\}+\{\tilde{I}\}} \delta\left(x - \frac{n}{L}\right)$$

$$\rho(x) + \rho_h(x) = \rho_t(x)$$

$$\lim_{Ah} \rho_t(x) \longrightarrow 1$$

$k(x)$ is a mapping $x \longleftrightarrow k$

equiv, $x(k)$

Move to k space, defining densities

$$\rho(k) = \rho(x(k))\frac{dx(k)}{dk} \qquad \rho_h(k) = \rho_h(x(k))\frac{dx}{dk}$$

$$\rho_t(k) = \frac{dx(k)}{dk} \qquad \rho(k) + \rho_h(k) = \rho_t(k)$$

BE:
$$k + \int_{-\infty}^{\infty} dk' \, \phi(k-k')\rho(k') = 2\pi x(k)$$

Take $\frac{d}{dk}$ " $\Rightarrow$ $\boxed{\rho(k) + \rho_h(k) = \frac{1}{2\pi} + \mathcal{C} * \rho(k)}$

where $f * g(k) = \int_{-\infty}^{\infty} dk' \, f(k-k') g(k')$

Given a $\rho(k)$, eigenvalues of simple charges are

$$n = \frac{N}{L} = \int_{-\infty}^{\infty} dk\, \rho(k) \qquad p = \frac{P}{L} = \int_{-\infty}^{\infty} dk\, k\, \rho(k)$$

$$e = \frac{E}{L} = \int_{-\infty}^{\infty} dk\, k^2\, \rho(k)$$

The ground state: $\rho(k) = \begin{cases} \rho_t(k) & |k| < k_F \\ 0 & \text{oth} \end{cases}$

BE for GS:

$$\rho(k) - \int_{-k_F}^{k_F} dk'\, \mathcal{C}(k-k')\rho(k') = \frac{1}{2\pi} \qquad |k| < k_F$$

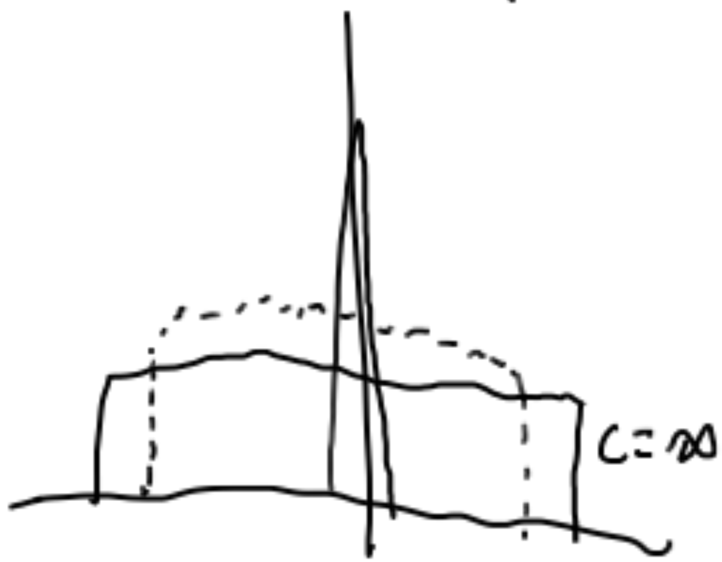
$$n = \int_{-k_F}^{k_F} dk\, \rho(k)$$

Only easy case: $c \to \infty$, $\varepsilon \to 0$

so Lieb eq$^{n}$ solved by $\rho(k) = \frac{1}{2\pi}\Theta(k_F^2 -$

$k_F = \pi n$

Sketch:



Excitations around the GS.

Particle-like exc$\overset{n}{=}$s (type 1)

Q#: $\{I_j\} = \left\{-\frac{N}{2}, -\frac{N}{2}+1, \ldots, \frac{N}{2}-1, \frac{N}{2}+m\right\}_{N+1}$

GS Q#: $\{I_j^0\} = \left\{-\frac{(N-1)}{2}, \ldots, \frac{N-1}{2}\right\}_N$

Exc state: momentum $P = \frac{2\pi}{L} m$, rap. $\{\{k_j\}, q\}$

GS: $\{k_j^0\}_{j=1,\ldots,N}$

Expect: $k_j - k_j^0 \equiv \frac{d_j}{L}$ small

Subtract BE for $k_j^0$ from that for $k_j$:

get eq$^{\underline{n}}$ for $d_j$:

$$d_j = -\pi - \phi(k_j - q) - \sum_{l=1}^{N}\left[\phi(k_j - k_l) - \phi(k_j^0 - k_l^0)\right]$$

Expanding in $1/L$:

$$d_j = -\pi + \phi(k_j - q) - \frac{1}{L}\sum_{l=1}^{N} \frac{2c}{(k_j - k_l)^2 + c^2} \cdot (d_j - d_l) + O\left(\frac{1}{L^2}\right)$$

$\Rightarrow$

$$d_j\left\{1 + \frac{1}{L}\sum \frac{2c}{(k_j - k_l)^2 + c^2}\right\} = -\pi - \phi(k_j - q) + \frac{1}{L}\sum \frac{2c}{(k_j - k_l)^2 + c^2} d_l$$

$d_j \to d(k, q)$

Introduce a $\int^{N}$ $D(k,q) = d(k,q)\,\rho(k)$

get:

$$D(k,q) - \int_{-k_F}^{k_F} dk'\, \ell(k-k')\, D(k',q) = -\frac{1}{2\pi}\left\{\pi + \phi(k-q)\right\}$$

$$q > k_F$$

Change in momentum:

$$\Delta P = q + \sum_{l=1}^{N} k_j - k_j^0 = q + \int_{-k_F}^{k_F} dk\, D(k,q)$$

$$\Delta E = q^2 - \mu + 2\int_{-k_F}^{k_F} dk\, k\, D(k,q)$$

Simple limits:

$c = 0^+$: $D = 0 \to \varepsilon(p) = p^2$

$c \to \infty$: $D = -1/2 \to \varepsilon(p) = p^2 + 2\pi m p$

Hole-like exc$^{ns}$ (type II)

Equilibrium thermodynamics

$$Z = \mathrm{Tr}\, e^{-(E-\mu N)/T} = \sum_{N=0}^{\infty} \sum_{\{I\}_N} e^{-(E_{\{I\}}-\mu N)/T}$$

$$Z \underset{TL}{\to} \int \mathcal{D}[\rho(k), \rho_h(k)]\Big|_{BE} e^{S[\rho]} e^{-(E-\mu N)/K}$$

In an interval $[k, k+\Delta k]$, $\frac{1}{L} \ll \Delta k \ll 1$

there are $L\rho(k)\Delta k$ particles

$L\rho_h(k)\Delta k$ holes

in $L\rho_t(k)\Delta k$ 'slots'

Nr of states described by $\rho(k)$ in th. lim
differing by perm$^n$s in $[k, k+\Delta k]$

$$= [L(\rho+\rho_h)\Delta k]! \Big/ [L\rho\Delta k]!\,[L\rho_h\Delta k]!$$

$\rightarrow$ get the (Yang-Yang) entropy

$$S[\rho,\rho_h] = L\int_{-\infty}^{\infty} dk \left\{(\rho+\rho_h)\ln(\rho+\rho_h) - \rho\ln\rho - \rho_h\ln\rho_h\right\}$$

so $$Z = \int \mathcal{D}[\rho, \rho_h]\, e^{-G[\rho,\rho_h]/T} \Big|_{BE}$$

$$G[\rho,\rho_h] = L\int_{-\infty}^{\infty} dk \Big[ \rho(k)\Big\{ k^2 - \mu - T\ln[\rho+\rho_h] + T\ln\rho \Big\} + \rho_h(k)\Big\{ -T\ln[\rho+\rho_h] + T\ln\rho_h \Big\} \Big]$$

Saddle-pt evaluation: consider variations

$$\rho(k) \to \rho(k) + \delta\rho(k)$$

$$\rho_h(k) \to \rho(k) + \delta\rho_h(k)$$

get $\dfrac{\delta G}{L} = \int_{-\infty}^{\infty} dk \Big\{ \delta\rho(k) \big\{ k^2 - \mu - T \ln[1 + \rho_h/\rho] \big\}$

$+ \delta\rho_h(k) \big\{ -T \ln[1 + \rho/\rho_h] \big\} \Big\}$

$\delta\rho$ & $\delta\rho_h$ are related by BE:

$$\delta\rho(k) - \mathcal{C} * \delta\rho(k) = -\delta\rho_h(k)$$

$$\Rightarrow \frac{\delta G}{L} = \int_{-\infty}^{\infty} dk \, \delta\rho(k) \Big\{ k^2 - \mu - T \ln\frac{\rho_h}{\rho} - \mathcal{C} * T \ln[1 + \rho/\rho_h] \Big\}$$

Saddle-pt: $\{\} = 0 \ \forall k$ Define $\varepsilon(k) = T \ln \dfrac{\rho_h}{\rho}$

SP cond$^{n}$: $\boxed{\varepsilon(k) = k^2 - \mu - \varphi * T\ln\left[1+e^{-\varepsilon(k)/T}\right]}$

The free energy itself becomes:

$$\left[ g(T,\mu) = \frac{G}{L} = -T\int_{-\infty}^{\infty} \frac{dk}{2\pi} \ln\left[1+e^{-\varepsilon(k)/T}\right] \right]$$

Simple limits:

$c \to \infty$: $\varphi \to 0$, $\varepsilon(k) = k^2 - \mu$

$c \to 0$: $\varphi \to \delta$ $\varepsilon(k) = T\ln\left[e^{\frac{k^2-\mu}{T}} - 1\right]$

$\to$ free bosons