Bose gas revisited   Not: λ rapidities

Bare momentum & energy fn

$k_0(\lambda) = \lambda \qquad \varepsilon_0(\lambda) = \lambda^2 - \mu$

$T \to 0$ lim of YY: $\displaystyle \varepsilon(\lambda) - \int_{-\lambda_F}^{\lambda_F} d\lambda' \, \mathcal{K}(\lambda-\lambda')\, \varepsilon(\lambda') = \varepsilon_0(\lambda)$

Define $\mathcal{K}_F(\lambda, \lambda') = \mathcal{K}(\lambda - \lambda')\, \Theta(\lambda_F - |\lambda'|)$

$$(1 - \mathcal{K}_F) * \varepsilon(\lambda) = \varepsilon_0(\lambda)$$

Formal inversion : $(1+\mathcal{L})*(1-\mathcal{C}_F)=1$

$\to (1+\mathcal{L})*\mathcal{C}_F=\mathcal{L}$, $\mathcal{L}*(1-\mathcal{C}_F)=\mathcal{C}_F$

so $\varepsilon(\lambda)=(1+\mathcal{L})*\varepsilon_0$

Lieb eq^n : $\rho_g(\lambda)=\frac{1}{2\pi}(1+\mathcal{L})*1_{|\lambda|<\lambda_F}$

Displacements of TI: 1.97

$$(1 - \mathcal{C}_F) * D(\lambda, \lambda_p) = \frac{-1}{2\pi}\left(\pi + \phi(\lambda - \lambda_p)\right)$$

$$|\lambda| < \lambda_F \,,\ \lambda_p > \lambda_F$$

Trick: RHS

$$= \frac{1}{2\pi}\int_{\lambda_p}^{\infty} d\lambda' \frac{d}{d\lambda'}\phi(\lambda - \lambda')$$

$$= -\int_{\lambda_p}^{\infty} d\lambda' \, \mathcal{C}(\lambda - \lambda')$$

so $$D(\lambda, \lambda_p) = -\int_{\lambda_p}^{\infty} d\lambda' \, \mathcal{L}(\lambda, \lambda')$$

Energy ~~of~~ TI : eq^{n} 1.99

$$\omega_I(\lambda_p) = \mathcal{E}_0(\lambda_p) + \int_{-\lambda_F}^{\lambda_F} d\lambda \, \frac{\partial \mathcal{E}_0(\lambda)}{\partial \lambda} \, D(\lambda, \lambda_p)$$

$$= \mathcal{E}_0(\lambda_p) - \int_{\lambda_p}^{\infty} d\lambda' \int_{-\lambda_F}^{\lambda} d\lambda \, \mathcal{L}(\lambda, \lambda') \frac{\partial \mathcal{E}_0(\lambda)}{\partial \lambda}$$

Use $(1 - \mathcal{L}_F) * \frac{\partial \mathcal{E}(\lambda)}{\partial \lambda} = \frac{\partial \mathcal{E}_0(\lambda)}{\partial \lambda}$

So $\mathcal{L} * \frac{\partial \mathcal{E}_0}{\partial \lambda} = \mathcal{L}_F * \frac{\partial \mathcal{E}}{\partial \lambda}$

$$\rightarrow \omega_I(\lambda_p) = \varepsilon_0(\lambda_p) - \int_{\lambda_p}^{\infty} d\lambda \left(\frac{\partial \varepsilon}{\partial \lambda} - \frac{\partial \varepsilon_0}{\partial \lambda} \right)$$

$$\boxed{\omega_I(\lambda_p) = \varepsilon(\lambda_p)} \quad \left(\text{using } \varepsilon - \varepsilon_0 \Big|_{\lambda \to \infty} = 0 \right)$$

Momentum:
(1.98) $\rightarrow$ $$P_I(\lambda_p) = \lambda_p + \int_{-\lambda_F}^{\lambda_F} d\lambda\, \Theta(\lambda, \lambda_p)$$

c.f. notes ...

$$\longrightarrow P_I(\lambda_p) = \lambda_p - \int_{-\lambda_F}^{\lambda_F} d\lambda\, \phi(\lambda_p - \lambda)\, \rho_g(\lambda) + \lim_{\lambda_p > \lambda_F}$$

Group v of T. I:

$$v_F = \frac{d\omega_I}{dp_I} = \frac{\partial \mathcal{E}_g}{\partial \lambda}\Big|_{\lambda_F} \Big/ \frac{\partial p_I}{\partial \lambda}\Big|_{\lambda_F}$$

We have: $$\frac{\partial p}{\partial \lambda}\Big|_{\lambda_F} = 1 + 2\pi \int_{-\lambda_F}^{\lambda_F} d\lambda \, \mathcal{C}(\lambda_F - \lambda') \rho_g(\lambda')$$

Lieb eq:

$$\Rightarrow \; = 2\pi \rho_g(\lambda_F)$$

So $$v_F = \left(\partial \mathcal{E}_g(\lambda)/\partial \lambda\right)\Big|_{\lambda_F} \Big/ 2\pi \rho_g(\lambda_F)$$

Type II: similar logic

$$\omega_{II}(\lambda_k) = \ldots = -\mathcal{E}_g(\lambda_k)$$

$$p_{II}(\lambda_k) = -p_I(\lambda_k)$$

If you have N_I TI ense^m's, N_{II},

then
$$\omega = \sum_{i=1}^{N_I} \mathcal{E}(\lambda_{p_i}) - \sum_{j=1}^{N_{II}} \mathcal{E}(\lambda_{k_j})$$

$$p = \quad \ldots p \qquad p$$

Finite-size corrections

$$E = \sum_j \varepsilon_0(\lambda_j) \qquad P = \sum_j k_0(\lambda_j)$$

BE for GS: $L k_0(\lambda_j^0) + \sum_l \phi(\lambda_j^0 - \lambda_l^0) =$

$$= 2\pi I_j^0 = 2\pi\left(-\frac{N+1}{2} + j\right)$$

$$\frac{E_g}{L} = \frac{1}{L}\sum \varepsilon_0\left(k_g\left(I_j^0/L\right)\right)$$

Invoke Euler-Maclaurin: (2nd)

$$\frac{E_g}{L} = \int_{-x_F}^{x_F} dx\, \varepsilon_0\left(k_g(x)\right) - \frac{1}{24L^2}\frac{\partial \varepsilon_0(x)}{\partial x}\Bigg|_{-x_F}^{x_F} + \mathcal{O}\left(1/L^4\right)$$

$$x_F \equiv \frac{N}{2L}$$

$$E_g = L\int_{-\lambda_F}^{\lambda_F} d\lambda\, \varepsilon_0(\lambda)\rho_{g,L}(\lambda) - \frac{1}{12L}\frac{1}{\rho_g(\lambda_F)}\frac{\partial \varepsilon_0(\lambda)}{\partial \lambda}\Bigg|_{\lambda_F} + \mathcal{O}(1/\ldots$$

$$\lambda_F \equiv k_g(x_F).$$

where $\rho_{g,L}(\lambda)$ is GS density, correct up to subleading order in $1/L$.

It obeys

$$\rho_{g,L}(\lambda) - \int_{-\lambda_F}^{\lambda_F} d\lambda' \, \mathcal{C}(\lambda-\lambda')\, \rho_{g,L}(\lambda') = \frac{1}{2\pi}\frac{dk_0(\lambda)}{d\lambda}$$

$$+ \frac{1}{24L^2 \rho_g(\lambda_F)} \frac{\partial}{\partial\lambda}\left\{ \mathcal{C}(\lambda,\lambda_F) - \mathcal{C}(\lambda,-\lambda_F) \right\}$$

Can take

$$\rho_{g,L}(\lambda) = \rho_g(\lambda) + \frac{1}{L^2}\rho_g^{(2)}(\lambda)$$

After some steps, get

$$E_g = L e_g + e_g^{(0)} + \frac{1}{L} e_g^{(1)} + O\left(\frac{1}{L^2}\right)$$

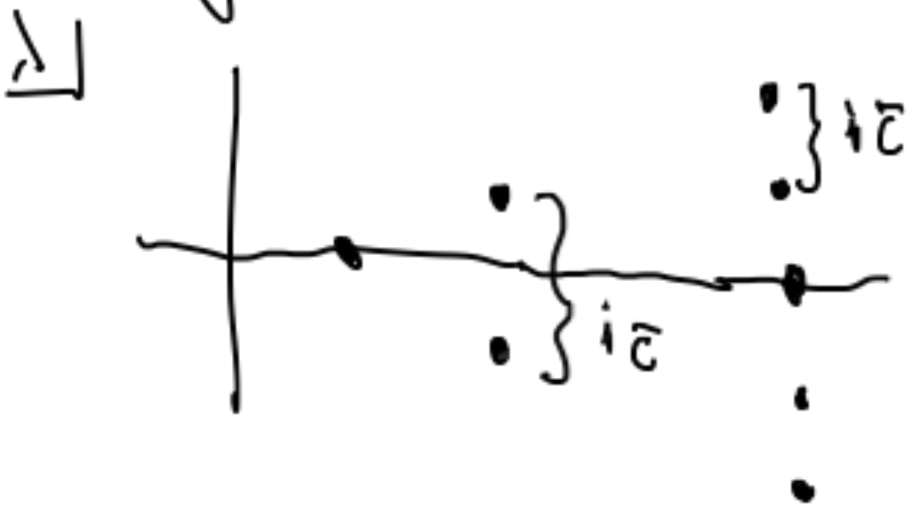
$$e_g = \int_{-\lambda_F}^{\lambda_F} d\lambda \, \varepsilon_0(\lambda) \rho_g(\lambda) \qquad e_g^{(0)} = 0$$

$$e_g^{(1)} = -\frac{\pi}{6} N_F \qquad \text{Casimir term}$$

The $c<0$ case $\quad \tau \equiv -c$

$$e^{i\lambda_\alpha L} = \prod_{\beta\neq\alpha} \frac{\lambda_\alpha - \lambda_\beta - i\tau}{\lambda_\alpha - \lambda_\beta + i\tau}$$

Strings are now admissible:



$$\lambda^{j,a}_{\alpha} = \underset{\in \mathbb{R}}{\lambda^{j}_{\alpha}} + \frac{i\bar{c}}{2}\left(j+1-2a\right) + i\,\delta^{j,a}_{\alpha}$$

$a = 1, \ldots, j$

$\uparrow$

$\sim c^{-\# L}$

j-string

$$E \underset{(j,\alpha)}{=} j\left(\lambda^{j}_{\alpha}\right)^2 - \underbrace{\frac{\bar{c}^{\,2}}{12}\, j\left(j^2-1\right)}_{\text{'binding' energy}} \qquad P = j\,\lambda^{j}_{\alpha}$$