Firenze 2015

Markovian Open Quantum Systems
A Basic Introduction
Lecture I

Introduction: Open Quantum Systems

About the lectures

Some paradigmatic systems

Back to basics: kinematics, dynamics, measurements

The fundamental equation of Markovian open quantum systems in discrete time

Examples

Introduction : Open Quantum Systems

Elementary quantum mechanics deals with simple quantum systems : an harmonic oscillator, an hydrogen atom, etc. A charcteristic property of these systems, in fact an assumption, is that they are isolated. The rest of the universe has no relevance for their time evolution, which by the postulates of the theory is given by an unitary operator $U(t) := e^{-iHt}$ $(\hbar=1)$ acting on the Hilbert space of the system. Yet measurement, which in general affects the system, is a central instance when the outside world comes into play. And even without measurements, isolation is never truely exact, it is only an abstraction, or an excellent approximation. Even in outer space, every system is surrounded by the relic photons at 3 K.

In fact one of the deep messages of open quantum systems, and of these lectures in particular, is in the close relationship between measurement and interaction with an environment. This is usually advocated for macroscopic objects, which are never found in quantum superposition (a coin which is on a table in a superposition of head and tail) if only because collisions with the surrounding atmosphere are more than enough to turn them into classical objects, with either a deterministic or probabilistic description.

The theory of open quantum systems is a, largely successful, attempt to describe non isolated systems: systems which can exchange energy, particles, etc with the rest of the universe, systems which are open. Of course, if we knew the Hamiltonian of the Universe and could deal with it, open quantum systems would be easy, or even useless. This is not the case, and the need for a good description of an open system, taking into account its interaction with the environment while involving only its internal degrees of freedom, is motivated by theoretical and practical considerations. There is a whole range of situation, from those where neglecting the outside Universe is enough to those where taking it into account requires to incorporate at least some of its degrees of freedom into the description. Even when a good description of the system in terms of only its internal degrees of freedom is possible, the price to pay for that can be quite high, at least technically. For instance the state of the system at $t+dt$ can be computed only from the knowledge of its states in the whole interval $]-\infty, t]$, i.e there are strong memory effects. But in these lectures, we shall concentrate on situation which in many respect are the easiest to deal with just after isolated systems, the so called Markovian open quantum systems, when the knowledge of the state at time t is enough to determine the state at time $t+dt$. This is of course the case with the Schrödinger equation for isolated systems, and the big difference is that

Markovian open quantum systems may have a non-unitary and/or non deterministic time evolution

As with many scientific theories, there are two complementary aspects:

- To establish the general equations / formalism of the theory
- To extract physical information from the above, ie to solve analitically or numerically for some interesting physical situation and to gain intuition

About the lectures

The lectures are about Markovian open quantum systems.
The basic notions are explained within a discrete time approach, though some contact points with continuous time are provided, and sometimes used to get information.
We focus on simple quantum systems, with a finite dimensional Hilbert space, mostly of dimension 2 or 3 in concrete examples.
This is not only for pedagogical reasons. In fact, the last decades have seen spectacular progresses in the detailed manipulation of simple systems. This was made possible by parallel progresses in fast electronics and low temperatures. Part of the motivation comes from quantum metrology, but the most fashionable excuse is quantum computing.

Main points / take home messages

- Close connection between open systems and indirect measurements
- Use non demolition (to be defined) measurement to observe open quantum systems

Quantum jumps and all that!

Engages only the lecturer: this approach is an ideal starting point to begin thinking what quantum mechanics really is. There are many connections with the interpretation of quantum mechanics. These notes do not propose one, or even hint at one. But it seems that certain thing which are obvious within one interpretation require some computation / thinking within other. Alas this criteria does not always select the same interpretation as the intuitive one.

Some paridigmatic systems

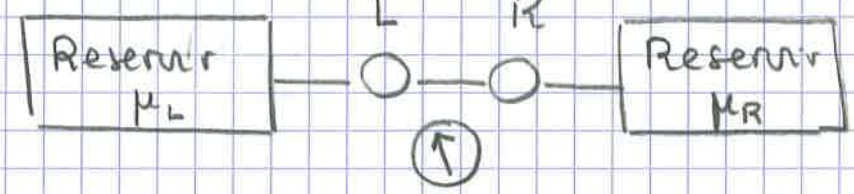
- Equilibrium situation:
  system coupled to a heat bath

- Non equilibrium situation: Show Quantum Dots
  system coupled to two heat bath at different temperatures, or to two reservoirs of particles at different chemical potentials

--- and then measurement comes into play.

- Cavity QED: Show cavity, show experiment
  - non demolition measurements
  - life and death of photons: heat bath + non demolition measurements
- Circuit QED: Show Toffoly Gate

- Quantum Dots: Show Quantum Dots again



Does not commute

→ Hamiltonian couples L and R → Rabi oscillations
Reservoirs inject or extract electrons with different rates
→ Indirect measurements give hints of the presence and position of an electron in the system

Interesting even without reservoirs!

Back to basics.

Kinematics:

Physical system → Hilbert space $\mathcal{H}$ (over $\mathbb{C}$)

- Different shades of states

* First course in quantum mechanics:

A state is a vector of norm 1 in $\mathcal{H}$ $|\varphi\rangle$

but phase is irrelevant $\qquad \| |\varphi\rangle \|^2 = \langle \varphi | \varphi \rangle = 1$

* A state is a 1d orthogonal projector of dimension 1 on $\mathcal{H}$; $P = |\varphi\rangle\langle\varphi|$

$P^2 = P \quad \mathrm{Tr}\, P = 1 \quad \mathrm{Ker}\, P \perp \mathrm{Im}\, P$

* A state is a density matrix, ie a positive opérateur of trace 1 on $\mathcal{H}$; ρ

$\mathrm{Tr}\, \rho = 1 \qquad \langle \varphi | \rho | \varphi \rangle \geq 0 \quad \forall\ |\varphi\rangle$

Automatically self adjoint if $\mathcal{H}$ over $\mathbb{C}$

Exercice: prove it if $\dim \mathcal{H} < +\infty$

- Obserables: self-adjoint operators on $\mathcal{H}$: O

If O is measured on many systems in state φ, the average of O is $\langle O \rangle = \langle \varphi | O | \varphi \rangle = \mathrm{Tr}\, PO$ if $P = |\varphi\rangle\langle\varphi|$

in state ρ

$\langle O \rangle = \mathrm{Tr}\, \rho O$

This formula (will be explained / is a consequence) of the rules for measurement

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Convention $\{|\alpha\rangle\}$ orthonormal basis of $\mathcal{H}_s$ (greek)

$\{|i\rangle\}$ orthonormal basis of $\mathcal{H}_e$ (latin)

$\mathcal{O}$ sur $\mathcal{H}$ $\quad \mathcal{O} = \sum_{\alpha,i} \sum_{\beta,j} |\alpha i\rangle \langle \alpha i | \mathcal{O} | \beta j\rangle \langle \beta j|$

$\langle \alpha i | \mathcal{O} | \beta j\rangle$ has several notations/interpretations

$\mathcal{O}_{\alpha i, \beta j}$: matrix on $\mathcal{H}$

$\mathcal{O}_{\alpha\beta, ij}$: collection, indexed by α, β, of matrices on $\mathcal{H}_e$, or matrix on $\mathcal{H}_s$ whose coefficients are matrices on $\mathcal{H}_e$

$\mathcal{O}_{ij, \alpha\beta}$: collection, indexed by ij, of matrices on $\mathcal{H}_s$, or matrix on $\mathcal{H}_e$ whose coefficients are matrices on $\mathcal{H}_s$

$\mathcal{H}om_{\mathbb{C}}(\mathcal{H}, \mathcal{H}) \cong \mathcal{H}om_{\mathcal{H}om(\mathcal{H}_e, \mathcal{H}_e)}(\mathcal{H}, \mathcal{H})$

$\cong \mathcal{H}om_{\mathcal{H}om(\mathcal{H}_s, \mathcal{H}_s)}(\mathcal{H}, \mathcal{H})$ need for a better notation

for intrinsic stuff

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Compound systems → The system ΣL is the union of two systems Σ and E with respective Hilbert spaces $\mathcal{H}_s$ and $\mathcal{H}_e$. Then $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$

2 In these notes we think of Σ as the system we are interested in, E as the (relevant part of the) environment and ΣL as the (relevant part of the rest of the) Universe.

An observable is said to apply only to Σ (resp. to E) if it is of the form $O_s \otimes \mathbb{1}_e$ (resp. $\mathbb{1}_s \otimes O_e$).

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Why density matrices?

Path 1 : effective description

Suppose ρ is the density matrix of a state in ΣL, is there an effective description of averages $\langle O \rangle$ for $O = O_s \otimes \mathbb{1}_e$ only in terms of $\Sigma, \mathcal{H}_s$?

Answer: there is a single operator ρ_s on $\mathcal{H}_s$ such that $\mathrm{Tr}_{\mathcal{H}}\, \rho(O_s \otimes \mathbb{1}_e) = \mathrm{Tr}_{\mathcal{H}_s}\, \rho_s O_s$ for every observable O_s on Σ

Remember traces are basis independent, so use basis

A simple but useful property of the partial trace: it acts only on $\mathcal{H}_e$

$$\mathrm{Tr}_{\mathcal{H}_e} : \mathrm{End}(\mathcal{H}_e) \to \mathbb{C} \quad \text{extended to}$$

$$\mathrm{Tr}_{\mathcal{H}_e} : \mathrm{End}(\mathcal{H}_s \otimes \mathcal{H}_e) \cong \mathrm{End}\,\mathcal{H}_s \otimes \mathrm{End}\,\mathcal{H}_e \to \mathrm{End}\,\mathcal{H}_s$$

Then if $\Phi : \mathrm{End}\,\mathcal{H}_s \to \mathcal{K}$ extend

$$\Phi : \mathrm{End}(\mathcal{H}_s \times \mathcal{H}_e) \to \mathcal{K} \otimes \mathrm{End}\,\mathcal{H}_e$$

By definition Φ and $\mathrm{Tr}_{\mathcal{H}_e}$ commute.

This is used freely in the following in different disguises

$$\mathrm{Tr}_{\mathcal{H}}\, \rho\, (O_s \otimes \mathbb{1}_e) = \sum_{\alpha, i} \sum_{\beta j} \rho_{\alpha\beta\, ij}\, O_{\beta\alpha}\, \delta_{ji}$$

$$= \sum_{\alpha,\beta} \sum_{i} \rho_{\alpha\beta, ii}\, O_{\beta\alpha}$$

$$= \mathrm{Tr}_{\mathcal{H}_s} \underbrace{(\mathrm{Tr}_{\mathcal{H}_e}\, \rho)}_{\text{partial trace}}\, O$$

$$(\mathrm{Tr}_{\mathcal{H}_e}\, \rho)_{\alpha\beta} := \sum_{i} \rho_{\alpha\beta, ii}$$

$\rho_s = \mathrm{Tr}_{\mathcal{H}_e}\, \rho$: trace out the environment,

keep only the relevant degrees of freedom.

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Exercice: if ρ_s is a density matrix on $\mathcal{H}_s$,

there is an environment E and a pure state $|v\rangle$ on $\mathcal{H}$ (resp. 1d orthogonal projector P on $\mathcal{H}$) such that $\rho_s = \mathrm{Tr}_{\mathcal{H}_e} |v\rangle\langle v|$ (resp. $= \mathrm{Tr}_{\mathcal{H}_e} P$)

Sol: choose $\dim \mathcal{H}_e = \dim \mathcal{H}_s$, diagonalize

$$\rho_s = \sum p_\alpha |\alpha\rangle\langle\alpha| \qquad p_\alpha \geq 0 \quad \sum p_\alpha = 1$$

Pair $|\alpha\rangle$ in $\mathcal{H}_s$ with $|i_\alpha\rangle$ in $\mathcal{H}_e$ ⊥N bases

$$|v\rangle := \sum_{\alpha} \sqrt{p_\alpha}\, |\alpha\rangle \otimes |i_\alpha\rangle$$

$$P = |v\rangle\langle v| = \sum_{\alpha\beta} \sqrt{p_\alpha p_\beta}\, |\alpha\rangle \otimes |i_\alpha\rangle\, \langle\beta| \otimes \langle i_\beta|$$

$$= \sum_{\alpha\beta} \sqrt{p_\alpha p_\beta}\, |\alpha\rangle\langle\beta| \otimes |i_\alpha\rangle\langle i_\beta|$$

$$(P)_{\alpha\beta\, jk} = \sqrt{p_\alpha p_\beta}\, \delta_{j i_\alpha}\, \delta_{k i_\beta}$$

$$(\mathrm{Tr}_{\mathcal{H}_e} P)_{\alpha\beta} = \sum_{j} \sqrt{p_\alpha p_\beta}\, \delta_{j i_\alpha}\, \delta_{j i_\beta} = \sqrt{p_\alpha p_\beta}\, \delta_{i_\alpha, i_\beta} = \sqrt{p_\alpha p_\beta}\, \delta_{\alpha\beta}$$

2 This result is archetypal: "one can do everything on $\mathcal{H}_s$ using a pure state on an appropriate $\mathcal{H}_s \otimes \mathcal{H}_e$!"

Even if you don't like density matrices, they pop out. Think of $\mathcal{H}$ large and $\mathcal{H}_s$ small. Taking the partial trace may be hard/impossible, but a phenomenological parameterization by a density matrix on $\mathcal{H}_s$ can be very useful

Path 2: statistical ensembles

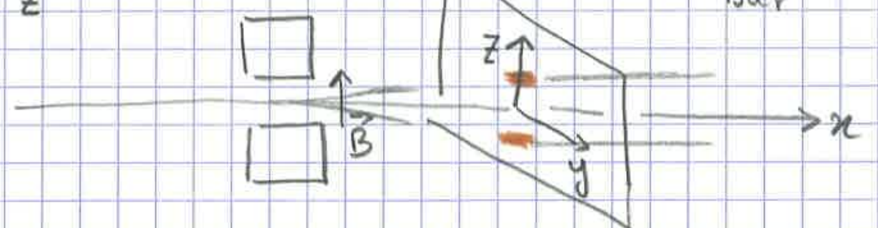
Suppose you have a large collection (N) of copies of a Σ, with N_α copies in state $|\alpha\rangle$, but you do not know who is who. If you select uniformly M of them at random, M large, then a fraction $f_\alpha \sim \frac{N_\alpha}{N}$ will be in state $|\alpha\rangle$, and measuring O on them will give

$$\langle O \rangle \sim \mathrm{Tr}\, \rho\, O \quad \text{with} \quad \rho = \sum_\alpha f_\alpha |\alpha\rangle\langle\alpha| .$$

This is true whether or not the $|\alpha\rangle$'s are orthogonal!

Example: Take a beam of electrons with spin $+1/2$ along the y axis and perform a Stern Gerlach experiment along z

but



Before $|+y\rangle \propto \frac{1}{\sqrt{2}}\left(|+z\rangle + i|-z\rangle\right) \rightarrow \rho = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$

After $\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Does it apply to a single electron?

Elementary geometry of density matrices

* Density matrices on $\mathcal{H}$ are a convex set
... but not a simplex in général

* $\dim \mathcal{H} = 2$ $\quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- $\rho = \frac{1}{2}\left(\mathbb{1} + \vec{r}\cdot\vec{\sigma}\right) \quad \vec{r}^2 \leq 1$ The Bloch ball

$= \frac{1}{2}\left(\mathbb{1} + x\,\sigma_x + y\,\sigma_y + z\,\sigma_z\right) \quad x^2+y^2+z^2 \leq 1$

Exercice: check!

- pure states: $\det \rho = 0$ ie $x^2+y^2+z^2 = 1$
The Bloch sphere

Exercice: what is orthogonality on the Bloch sphere
→ diametrically opposed

Exercice: is there uniqueness of the decomposition of a density matrix as a convex combination of pure states
Picture on the Bloch disk and circle of real density matrices. Consequences on the physical interpretation

* $\dim \mathcal{H} \geq 3$ The pure state are only a small piece of the boundary of density matrices ($\det \rho = 0$).
The geometry is more complicated

Exercice: draw the space of density matrices of the form

$$\begin{pmatrix} 1/3 & a & b \\ a & 1/3 & c \\ b & c & 1/3 \end{pmatrix} \qquad \begin{array}{l} 1/9 - a^2 \geq 0 \quad 1/9 - b^2 \geq 0 \quad 1/9 - c^2 \geq 0 \\ \frac{1}{27} - \frac{a^2+b^2+c^2}{3} + 2abc \geq 0 \end{array}$$

Back to basics

Dynamics :

$U^{+} = U^{-1}$

- Time evolution on $\mathcal{H}$ given by an unitary operator U

continuous time $\quad |t\rangle = U(t)\,|0\rangle \qquad U(t) = e^{-iHt}$

discrete time $\quad |n\rangle = U_n\,|0\rangle \qquad U_n = U_1^{n}$

$$\rho(t) = e^{-itH}\,\rho(0)\,e^{itH}$$
$$\rho_n = U_1^{n}\,\rho_0\,U_1^{-n}$$

if time is homogeneous

$$\rho(t) = U(t)\,\rho(0)\,U(t)^{+}$$
$$\rho_n = U_n\,\rho_0\,U_n^{+}$$

Back to basics

Measurement

It is an understatement that measurement is one of the most debated issue in quantum physics. At the same time, there is a whole theory of measurement from a purely technical viewpoint, because most real-life measurements are quite far from what is presented in elementary courses. Here, we shall content with slight variants of the elementary theory of von Neumann measurement:

* Measure O (self adjoint) on $\mathcal{H}$. If $|v\rangle$ is the state just before measurement and $O = \sum \lambda_a P_a$ is the spectral decomposition of O then the state just after measurement is $\dfrac{P_a|v\rangle}{\| P_a|v\rangle \|}$ if the outcome λ_a is recorded, and this occurs with probability $\| P_a|v\rangle \|^2 = \langle v|P_a|v\rangle$

? This assumes instantaneous measurement which **must** be an idealization. In general there is a piece of Unitary evolution that "superimposes" to those formulae. We shall put this to good use later.

* It is straighforward to generalize to density matrices:

  if ρ just before measurement, then $\dfrac{P_a \rho P_a}{\mathrm{Tr}\, P_a \rho}$ just after, and this occurs with probability $\mathrm{Tr}\, P_a \rho$

Exercice: Recover the formulae for $\langle O \rangle$ from these expressions

2 It is not O that is important, but the P_a's

$$O = \sum \lambda_a P_a \quad \text{and} \quad \tilde{O} = \sum \tilde{\lambda}_a P_a \qquad \begin{cases} \lambda_a \text{ distinct} \\ \tilde{\lambda}_a \text{ distinct} \end{cases}$$

yield the same effect on $|v\rangle$, ρ, ...

This is just like changing the scale on a meter.

* One last refinement: suppose $a \in I$ and that after the measurement by the apparatus only a partial reading is made, i.e. one checks only if the result $a \in I_r$ where $I = \bigsqcup_r I_r$ is a partition. For instance build a measurement apparatus that determines an eigenvalue of O with a certain precision (say a position up to a milimeter uncertainty) but read the result rounded to the centimeter.

Then
$$\rho \to \frac{\sum_{a \in S_r} P_a \rho P_a}{\operatorname{Tr} \sum_{a \in S_r} P_a \rho} \quad \text{with proba } \operatorname{Tr} \sum_{a \in S_r} P_a \rho$$

This is the last generalization we shall need

Exercice: explain this equation within the statistical ensemble framework; what about the effective description framework?

The fundamental equations of Markovian open quantum systems in discrete time

* The basic computation (no measurement case)

$\mathcal{H} = \mathcal{H}_s \times \mathcal{H}_e$ $\qquad \rho$ density matrix at time 0

$\qquad\qquad\qquad\qquad \rho'$ ——————————— 1

$\mathbb{U}$ the one timestep evolution operator

$$\rho' = \mathbb{U} \rho \mathbb{U}^+ \qquad \rho_s = \mathrm{Tr}_{\mathcal{H}_e} \rho \qquad \rho'_s = \mathrm{Tr}_{\mathcal{H}_e} \rho'$$

There is no simple relationship between ρ_s and ρ'_s !

Exercice : find $\mathcal{H}_s, \mathcal{H}_e, \mathbb{U}, \rho, \tilde{\rho}$ such that

$$\rho_s = \tilde{\rho}_s \quad \text{but} \quad \rho'_s \neq \tilde{\rho}'_s$$

? This must be possible but I have no example, I guess factorized ρ and $\tilde{\rho}$ should be enough

However, suppose $\rho = \rho_s \otimes \rho_e$ and write

$$\rho_e = \sum_i p_i |i\rangle\langle i| \quad \text{in some orthonormal basis}$$

(note $\quad \mathrm{Tr}_{\mathcal{H}_e} \rho = \rho_s$ so notations are coherent!)

Take $|\alpha\rangle$ orthonormal basis of $\mathcal{H}_s$ and set

$$(A_{ij})_{\alpha\beta} := \sqrt{p_j} \langle \alpha i | \mathbb{U} | \beta j \rangle . \text{ Then}$$

$$\rho'_s = \mathrm{Tr}_{\mathcal{H}_e} \mathbb{U}(\rho_s \otimes \rho_e)\mathbb{U}^+ = \sum_{i,j} A_{ij} \rho_s A_{ij}^+$$

Exercice : check this formula

Note: $A_{ij}^{+} := (A_{ij})^{+}$ the adjoint of A_{ij} as an operator on $\mathcal{H}_s$, so that

$$(A_{ij}^{+})_{\alpha\beta} = \overline{(A_{ij})_{\beta\alpha}} = \sqrt{p_j}\ \overline{\langle \beta i | U | \alpha j \rangle}$$
$$= \sqrt{p_j}\ \langle \alpha j | U^{+} | \beta i \rangle$$

Note: $\sum_{ij} A_{ij}^{+} A_{ij} = Id_{\mathcal{H}_s}$, and $\sum_{ij} A_{ij} \rho_s A_{ij}^{+}$ is a sum of positive operators so ρ'_s is indeed a density matrix

Exercise: check this

Our next aim is to show that: if A_a on $\mathcal{H}_s$ are such that $\sum_a A_a^{+} A_a = Id_{\mathcal{H}_s}$ it is "reasonnable" to consider the rule $\rho_s \to \sum A_a \rho_s A_a^{+}$ as a dynamical system, ie iteration makes sense.

2 It is not true in general that

$U(\rho_s \otimes \rho_e) U^{+} = \rho'_s \otimes \rho_e$ so iteration does not always work. Sometimes one argues that this equation holds to a good approximation under the physical assumption that the evolution does not perturb the environment (because it is large, weakly coupled, etc). Such physical arguments are the heart of the understanding of open quantum systems.

What we are after is a mathematical argument of consistency, and happily this argument points towards some physical picture!

* Theorem: Suppose $\dim \mathcal{H}_s < +\infty$ and A_a is a collection of operators on $\mathcal{H}_s$ such that $\sum_a A_a^\dagger A_a = \mathrm{Id}_{\mathcal{H}_s}$. Then there is a Hilbert space $\mathcal{H}_e$ an unitary operator U on $\mathcal{H}_s \otimes \mathcal{H}_e := \mathcal{H}$ and a density matrix $\rho(0)$ on $\mathcal{H}$ such that iteration of $\rho(n+1) = U\rho(n)U^\dagger$ leads via partial trace to iteration of $\rho_s(n+1) = \sum_a A_a \rho_s(n) A_a^\dagger$

Step 1 : one time-step with arbitrary A_a's :

Take $\mathcal{K}$ a Hilbert space with an orthonormal basis indexed by a, $|a\rangle$, and take $|\psi\rangle \in \mathcal{K}$ of norm 1. Define $F = \{ |\varphi\rangle \otimes |\psi\rangle, |\varphi\rangle \in \mathcal{H}_s \} \subset \mathcal{H}_s \otimes \mathcal{K}$, a linear subspace, and for $|\varphi\rangle \otimes |\psi\rangle \in F$ set

$$U(|\varphi\rangle \otimes |\psi\rangle) := \sum_a A_a |\varphi\rangle \otimes |a\rangle .$$

Then $\sum_a A_a^\dagger A_a = \mathrm{Id}\,\mathcal{H}_s$ implies that U maps F unitarily to $U(F)$. Then $F^\perp$ and $U(F)^\perp$ have the same dimension and U can be extended (in many ways) to all of $\mathcal{H}_s \otimes \mathcal{K}$ by choosing an unitary map from $F^\perp$ to $U(F)^\perp$.

Then $U(|\varphi\rangle\langle\varphi| \otimes |\psi\rangle\langle\psi|)U^\dagger = \sum_{a,b} A_a |\varphi\rangle\langle\varphi| A_b^\dagger \otimes |a\rangle\langle b|$ and $\mathrm{Tr}_{\mathcal{K}}\, U(|\varphi\rangle\langle\varphi| \otimes |\psi\rangle\langle\psi|)U^\dagger = \sum_a A_a |\varphi\rangle\langle\varphi| A_a^\dagger$

By linearity this holds for all density matrices on $\mathcal{H}_s$

- Illustration : take $\{\alpha\} = \{a\}$ ie $\mathcal{K} \simeq \mathcal{H}_s$ and take $A_\alpha := |\alpha\rangle\langle\alpha|$ which is ok. $\sum_\alpha A_\alpha^\dagger A_\alpha = \mathbb{1}_{\mathcal{H}_s}$

Then $\sum_\alpha A_\alpha \rho_s A_\alpha^\dagger$ = the diagonal part of ρ_s

In general, unitary dynamics on $\mathcal{H}$ leads to non unitary dynamics on $\mathcal{H}_s$!

Step 2: finitely many time steps.

Take $\mathcal{K}$, $|\psi\rangle$, U as before, take $\mathcal{H}_e = \mathcal{K}^{\otimes n}$ and define U_m on $\mathcal{H}_s \otimes \mathcal{H}_e$ by:

if $U(|\varphi\rangle \otimes |a\rangle) = \sum_b U_{ba}(|\varphi\rangle) \otimes |b\rangle$ then

$U_m(|\varphi\rangle \otimes |a_1\rangle \cdots \otimes |a_m\rangle \otimes \cdots |a_n\rangle) :=$

$\sum_b U_{b a_m}(|\varphi\rangle) \otimes |a_1\rangle \cdots \otimes |b\rangle \otimes \cdots |a_n\rangle)$ and then by linearity.

Each U_m is unitary and acting on

$\rho_s(0) \otimes |\psi\rangle\langle\psi| \otimes \cdots \otimes |\psi\rangle\langle\psi|$ with U_1, then U_2, ..., then U_n gives an effective dynamics on $\rho_s(m)$ which is iteration of $\rho_s \to \sum_a A_a \rho_s A_a^\dagger$

Step 3: (skip: infinitely many time steps)

$\mathcal{H}_e = \mathcal{K}_{|\psi\rangle}^{\otimes \mathbb{N}^*}$ restricted tensor product and copy step 2

Step 4: the evolution operator looks time dependent, but take $\mathcal{H}_e = \mathcal{K}_{|\psi\rangle}^{\otimes \mathbb{Z}}$ and include the shift from by one in the sequence of copies of $\mathcal{K}$ into the unitary evolution.

2 Physical picture

At each time step, a "new" probe in state $|\psi\rangle\langle\psi|$ interacts with the system and then goes away.

Any (Markovian) environment can be simulated that way!

This is one of the important messages of this lecture.

Study iterates of $\rho_s \to \sum_a A_a^\dagger \rho_s A_a$ dynamical map

* The basic computations (indirect measurement)

What about measuring a probe observable on each probe (for simplicity, always the same observable)

Let O be a (non-degenerate for simplicity) observable on $\mathcal{H}$, $O = \sum_a \alpha_a |a\rangle\langle a|$ in diagonal form

Protocol: Let Σ interact with a probe (described by $\mathcal{H}$) with evolution U and then make a standard measurement on the probe.

$$1_{d_{\mathcal{H}_S}} \otimes O = \sum \alpha_a P_a \qquad P_a = 1_{d_{\mathcal{H}_S}} \otimes |a\rangle\langle a| \qquad \rho = \rho_s \otimes |\psi\rangle\langle\psi|$$

So $\rho \to \dfrac{P_a U \rho U^\dagger P_a}{\mathrm{Tr}\, P_a U \rho U^\dagger}$ with proba $\mathrm{Tr}\, P_a U \rho U^\dagger$

$$U \rho U^\dagger = \sum_{bc} A_b \rho_s A_c^\dagger \otimes |b\rangle\langle c|$$

$$P_a U \rho U^\dagger P_a = A_a \rho_s A_a^\dagger \otimes |a\rangle\langle a|$$

So $\quad \rho_S \to \dfrac{A_a \rho_S A_a^\dagger}{\mathrm{Tr}\, A_a \rho_S A_a^\dagger}$ with proba $\mathrm{Tr}\, A_a \rho_S A_a^\dagger$

This is if measurement outcome is read accurately.

If not, partition $\{a\} = I = \bigsqcup_r I_r$

$$\rho \to \frac{\sum_{a \in S_r} P_a U \rho U^\dagger P_a}{\mathrm{Tr} \sum_{a \in S_r} P_a U \rho U^\dagger} \quad \text{with proba } \mathrm{Tr} \sum_{a \in I_r} P_a U \rho U^\dagger$$

$\left(\mathrm{Tr}_{\mathcal{H}} \; A_a \rho_S A_a^\dagger \otimes |a\rangle\langle a| = A_a \rho_S A_a^\dagger \text{ of course!} \right)$

$$\boxed{\rho_S \to \frac{\sum_{a \in I_r} A_a \rho_S A_a^\dagger}{\mathrm{Tr} \sum_{a \in I_r} A_a \rho_S A_a^\dagger} \quad \text{with proba } \mathrm{Tr} \sum_{a \in I_r} A_a \rho_S A_a^\dagger \quad (*)}$$

A slight problem with iteration within the statistical ensemble approach: Upon iteration we find outcomes $a_1, a_2, \ldots$ (or $r_1, r_2, \ldots$ if fuzzy reading), so to keep a history, the number of representative system decreases exponentially. Of course, this issue does not occur if we view ρ_S as a characteristic of a single Σ, or if we have a Bayesian/subjective view of ρ

A modest miracle: Taking the (probabilistic) average over measurement outcomes is the same as taking the partial trace:

$$\mathbb{E}(\rho_S') = \sum_r \frac{\sum_{a \in I_r} A_a \rho_S A_a^\dagger}{\mathrm{Tr} \sum_{a \in I_r} A_a \rho_S A_a^\dagger} \mathrm{Tr} \sum_{a \in I_r} A_a \rho_S A_a^\dagger$$

$$= \sum_a A_a \rho_S A_a^\dagger$$

$=$ no reading at all!

Trivial but with deep consequences!

(*) together with $\sum A_a^\dagger A_a = Id_{sys}$ is all one needs to work with Markovian open quantum systems in discrete time

However, given a physical problem, finding all the right choice of A_a's is highly non trivial. But this is not proper to quantum mechanics. Given a classical problem in statistical mechanics or out of equilibrium physics, finding an appropriate Markovian dynamics is ambiguous but at the same time a difficult art.

Exercice : Try to interpret the close connection between putting a system in contact with a reservoir and doing reapeted indirect measurements without reading on it.

Remarks:

* We have already observed that the linear map $\rho \to \sum_a A_a \rho A_a^\dagger$ transforms positive matrices into positive matrices: such a map is called positive

If $\mathcal{L}$ is any (super)operator on operators on $\mathcal{H}$, and O is an operator on $\mathcal{H} \otimes \mathcal{K}$ where $\mathcal{H}$ and $\mathcal{K}$ are (finite dimensional) Hilbert space we may define

$$\tilde{\mathcal{L}}(O) := \sum_{ij} \mathcal{L}(O_{ij}) \otimes |i\rangle\langle j| \quad \text{if}$$

$$O = \sum_{ij} O_{ij} \otimes |i\rangle\langle j|$$

Exercice: write this with all indices

Question: if $\mathcal{L}$ is positive, is it the case that $\tilde{\mathcal{L}}$ is automatically positive? The answer is no in general. (examples on the web)

An operator $\mathcal{L}$ such that $\tilde{\mathcal{L}}$ is positive for any $\mathcal{K}$ is called completely positive

Exercice: check that $O \to \sum_a A_a O A_a^\dagger$ is completely positive: This is simply because $\tilde{\mathcal{L}}$ is given by

$$O \to \sum_a (A_a \otimes Id_{\mathcal{K}}) \, O \, (A_a^\dagger \otimes Id_{\mathcal{K}})$$

Choi's theorem states that $\rho \to \sum_a A_a \rho A_a^\dagger$ is the most general completely positive map. This is true also in infinite dimension if $\{a\}$ is allowed to be countable (separable Hilbert spaces) but it is harder!

$*$: Set $\mathcal{L}(O) = \sum_a A_a O A_a^+$ and compute $\mathcal{L}^+$ for the trace Hilbert space norm: $\langle \tilde{O}, O \rangle := \mathrm{Tr}\, O \tilde{O}^+$

$$\langle \tilde{O}, A O A^+ \rangle = \mathrm{Tr}(A O A^+ \tilde{O}^+) = \mathrm{Tr}(O\, A^+ \tilde{O}^+ A)$$
$$= \mathrm{Tr}(O (A^+ \tilde{O} A)^+) = \langle A^+ \tilde{O} A, O \rangle$$

Thus $\mathcal{L}^+(O) = \sum_a A_a^+ O A_a$. If $\sum_a A_a^+ A_a = \mathrm{Id}_{\mathcal{H}_S}$ we get $\mathcal{L}^+(\mathrm{Id}_{\mathcal{H}_S}) = \mathrm{Id}_{\mathcal{H}_S}$ so 1 is an eigenvalue of $\mathcal{L}^+$, and then $\mathcal{L}$. On can show that all other eigenvalues have modulus ≤ 1, i.e. $\mathcal{L}$ is contracting. To be a bit more precise, any O is such that OO^+ has a unique positive square root and $\|O\|_{tr}$, the trace norm, is defined by

$\|O\|_{tr} = \mathrm{tr}\sqrt{OO^+}$. ($OO^+ = \sum d_i^2 |i\rangle\langle i|$ in some orthonormal basis because it is self-adjoint and positive, with $d_i \geq 0$, then $\sqrt{OO^+} := \sum d_i |i\rangle\langle i|$), and

$\|\mathcal{L}(O)\|_{tr} \leq \|O\|_{tr}$.

Examples

* Random unitary evolutions

U_a unitary on $\mathcal{H}_s$ $\quad \sum_a p_a = 1 \quad p_a \geq 0$

$A_a = \sqrt{p_a}\, U_a$ measurement is just dice throwing to decide which unitary comes next

The probability of outcome a is p_a (does not depend of ρ_s)

Exercice : simulate some simple cases, with (possibly fuzzy) measurements

Examples

* Rabi oscillations $\qquad \dim \mathcal{H}_S = 2$

hopping of an electron

indirect measure of where the electron is:

$\{a\} = \{+,-\}$
$$A_+ = \begin{pmatrix} \cos\alpha \cos\beta & \sin\alpha \cos\beta \\ -\sin\alpha \sin\beta & \cos\alpha \sin\beta \end{pmatrix}$$
$$A_- = \begin{pmatrix} \cos\alpha \sin\beta & \sin\alpha \sin\beta \\ -\sin\alpha \cos\beta & \cos\alpha \cos\beta \end{pmatrix}$$

Exercice: check that $A_+^\dagger A_+ + A_-^\dagger A_- = \mathrm{Id}$

Exercice: simulate on real density matrices, with measurement or in average, vary the parameters,

At $\beta = \frac{\pi}{4}$ $\quad A_+ = A_-$ ie measuring does nothing (proportional to unitaries)

At $\alpha = 0$ $\quad |L\rangle$ and $|R\rangle$ are invariant.

Is there a continuous time limit?

Exercice: simulate the $\alpha = 0$ case

Examples

* Thermal fluctuations in a two-level system $\dim \mathcal{H}_S = 2$

plus an indirect measurement of the energy

Proposal: three possible measurement outcomes ie $\dim \mathcal{H} = 3$

$$A_+ = \begin{pmatrix} 0 & \cos\alpha \\ 0 & 0 \end{pmatrix} \qquad A_0 = \begin{pmatrix} \sin\gamma & 0 \\ 0 & \sin\beta \end{pmatrix} \qquad A_- = \begin{pmatrix} 0 & 0 \\ \cos\gamma & 0 \end{pmatrix}$$

Exercice: check that $A_+^\dagger A_+ + A_0^\dagger A_0 + A_-^\dagger A_- = \mathrm{Id}$

Exercice: simulate on density matrices, with measurement, possibly with fuzzy reading (for instance do not distinguish between outcomes + and −)

Is there a continuous time limit?

Exercice: split $A_0 = \underbrace{\sqrt{p_+}\, A_0}_{A_{0+}} + \underbrace{\sqrt{p_-}\, A_0}_{A_{0-}}$ $\qquad p_+ + p_- = 1$, $p_+, p_- \geq 0$

and group $A_\pm$ with $A_{0\pm}$

Is there a continuum limit

How do the outcomes of the exercices fit with the initial goal of describing the observation of thermal fluctuations

Could you imagine better A_a's

Firenze 2015

Markovian Open Quantum Systems
A Basic Introduction
Lecture II

Quantum non-demolition experiments

Quantum non-demolition measurements

Some mathematical considerations

Progressive wave function collapse in non-degenerate quantum non-demolition measurements

The continuous time limit (average evolution)

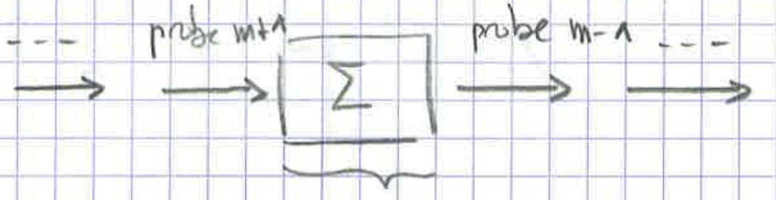
The stochastic continuous time limit

Quantum non-demolition experiments

We are now ready to embark on the analysis of (*) in a special case. Before explaining what it is, let us list some of its virtues

- it is simple
- it can be analyzed thoroughly by different methods
- it leads to some remarkable conclusions
- it is a basic building block to understand certain more complicated instances of (*)
- last but not least it is used daily in real experiments

Recall our setting for generic experiments



U on $\mathcal{H}_S \otimes \mathcal{K}$

contact with probe m $\rightarrow$ U_m which acts as U on $\mathcal{H}_S \otimes \mathcal{K}^{(m)}$ where $\mathcal{K}^{(m)}$ is the Hilbert space of the m^{th} probe

The experiment is called a non-demolition experiment if the U_m s commute among each other

In such a situation, a man in the middle could permute the probe after they have passed through Σ and nobody could discover it.

Lemma: the experiment is non-demolition if and only if there is an orthonormal basis $\{|\alpha\rangle\}$ of $\mathcal{H}_S$ and a collection $U^{(\alpha)}$ (same index set) of unitary operation on $\mathcal{K}$ such that $U = \sum_\alpha |\alpha\rangle\langle\alpha| \otimes U^{(\alpha)}$

One direction is simple. First, $U = \sum_\alpha |\alpha\rangle\langle\alpha| \otimes U^{(\alpha)}$ is unitary if and only if the $U^{(\alpha)}$'s are unitary

Exercice : prove it!

Compute $\langle \beta j | U U^\dagger | \gamma k\rangle$

Then
$$U_1 = \sum_\alpha |\alpha\rangle\langle\alpha| \otimes U^{(\alpha)} \otimes 1d_{\mathcal{K}}$$
$$U_2 = \sum_\alpha |\alpha\rangle\langle\alpha| \otimes Id_{\mathcal{H}} \otimes U^{(\alpha)}$$
$$U_1 U_2 = U_2 U_1 = \sum_\alpha |\alpha\rangle\langle\alpha| \otimes U^{(\alpha)} \otimes U^{(\alpha)} \quad \text{etc} \dots$$

For the other direction, take an orthonormal basis of $\mathcal{K}$, say $\{|a\rangle\}$ and write

$U = \sum_{a,b} U_{ab} \otimes |a\rangle\langle b|$ where each U_{ab} is an operator on $\mathcal{H}_S$, $(U_{ab})_{\alpha\beta} = \langle \alpha . a | U | \beta b\rangle$, and

check that $U_1 U_2 = U_2 U_1 \iff U_{ab} U_{cd} = U_{cd} U_{ab}$ whatever a, b, c, d, i.e. the U_{ab}'s form a family of commuting operators

$$U_{ab} U_{cd} - U_{cd} U_{ab} = 0$$
$$\sum_{ab} U^\dagger_{aa'} (U_{ab} U_{cd} - U_{cd} U_{ab}) U^\dagger_{b'b} = 0$$

where $U^\dagger = \sum_{a,b} U^\dagger_{ab} \otimes |b\rangle\langle a|$ i.e.
$$(U^\dagger)_{ab} = U^\dagger_{ba}$$

From $U^\dagger U = Id_{\mathcal{H}_S} \otimes Id_{\mathcal{K}} = U U^\dagger$ infer

$$\sum_a U^\dagger_{aa'} U_{ab} = \delta_{a'b}\, Id_{\mathcal{H}_S} \qquad \sum_b U_{ab} U^\dagger_{b'b} = \delta_{ab'}\, Id_{\mathcal{H}_S}$$

So $\quad U_{cd} U^{\dagger}_{b'a'} - U^{\dagger}_{b'a'} U_{cd} = 0 \quad$ ie the

U_{ab} form a family of commuting normal operators.

Thus there is an orthonormal basis $|\alpha\rangle$ of $\mathcal{H}_s$ such

that $\quad U_{ab} |\alpha\rangle = c_{\alpha ab} |\alpha\rangle \quad \forall \alpha, a, b \quad$ ie

$$U_{ab} = \sum_{\alpha} c_{\alpha ab} |\alpha\rangle\langle\alpha| \quad \text{and then}$$

$$U = \sum_{\alpha} |\alpha\rangle\langle\alpha| \otimes U^{(\alpha)} \quad \text{where } U^{(\alpha)} := \sum c_{\alpha ab} |a\rangle\langle b|$$

The states $|\alpha\rangle\langle\alpha|$ are called pointer states for

the experiment because if $\quad \rho = |\alpha\rangle\langle\alpha| \otimes \rho_e$

$$U \rho U^{\dagger} = \sum_{\beta,\gamma} \left(|\beta\rangle\langle\beta| \otimes U^{(\beta)}\right)\left(|\alpha\rangle\langle\alpha| \otimes \rho_e\right)\left(|\gamma\rangle\langle\gamma| \otimes U^{(\gamma)\dagger}\right)$$

$$= |\alpha\rangle\langle\alpha| \otimes U^{(\alpha)} \rho_e U^{(\alpha)}$$

→ The states $|\alpha\rangle\langle\alpha|$ are preserved by the dynamics

Quantum non demolition measurements

What if one measures an observable O on the probes in a non-demolition experiment:

Remember
$$U(|\varphi\rangle \otimes |\psi\rangle) := \sum_a A_a |\varphi\rangle \otimes |a\rangle$$

the initial state of probes

so that
$$(\langle\alpha| \otimes \langle a|) U (|\varphi\rangle \otimes |\psi\rangle) = \langle\alpha| A_a |\varphi\rangle$$

ie
$$\langle\alpha| A_a |\beta\rangle = \delta_{\alpha\beta} \langle a| U^{(\alpha)} |\psi\rangle := \delta_{\alpha\beta} \underbrace{C(a|\alpha)}_{\text{notation}}$$

In the orthonormal basis $|\alpha\rangle$, all the operators A_a are diagonal. Thus the operators A_a form a set of commuting normal operators
$$\left.\begin{array}{l} [A_a, A_b^\dagger] = 0 \\ [A_a, A_b] = 0 \end{array}\right\} \forall a, b$$

Thus, we could also define directly non demolition measurements by the condition above on the A_as: they have to commute among themselves and with their adjoints.

<u>Exercice</u>: check the equivalence of the two definitions, i.e construct $|\alpha\rangle$ and $U^{(\alpha)}$ from the A_as.

From now on, we write all quantities in the pointer state basis, ie write $\rho_S = \sum_{\alpha\beta} \rho_{\alpha\beta} |\alpha\rangle\langle\beta|$. Then non-demolition measurements lead to a specialization of (*):

$$\rho_{\alpha\beta} \rightarrow \rho_{\alpha\beta} \frac{C(a|\alpha)\,\overline{C(a|\beta)}}{\sum_\gamma \rho_{\gamma\gamma} |C(a|\gamma)|^2} \quad \text{with} \quad (*)_{ND}$$

probability $\sum_\gamma \rho_{\gamma\gamma} |c(a|\gamma)|^2$, where

$\sum_a A_a^\dagger A_a = Id_{\mathcal{H}_S}$ specializes to $\sum_a |c(a|\alpha)|^2 = 1$ for each α.

We set $p(a|\alpha) := |c(a|\alpha)|^2$ so that for each α

$p(\cdot|\alpha)$ is a probability measure on the set of a's.

Definition: we say that a quantum non-demolition measurement is non-degenerate if the probability measures $p(\cdot|\alpha)$ are different for different αs.

Illustration: the cavity QED experiment:

The basis α is the basis of photon number n

The probes are Rydberg atoms, which behave as two-level systems (very sensitive to electromagnetic fields)

- $U = e^{i\Theta N \otimes \sigma_z}$ N is the photon number operator in the cavity, σ_z acts on the probes
- the observable is a Pauli matrix along an axis perpendicular to the z axis (in probe space)
- the initial state of Rydberg atoms is $+x$ (in probe space)
- the parameter Θ depends on the time spent by probes in the cavity

In the experiment, the value of Θ is such that $p(\cdot|n)$ is periodic with period 8 in n. Thus the measurement is degenerate. It is formally non-degenerate if $N \pmod 8$ is considered instead of N.

Show pictures!

Exercice : do a numerical simulation of the experiment!

Exercice : do numerical simulations of degenerate or non-degenerate quantum non-demolition measurements for modest sizes of the probe and system Hilbert spaces

Exercice : what is gained with fuzzy reading, ie

$$\rho_{\alpha\beta} \to \rho_{\alpha\beta} \frac{\sum_{a\in I_r} c(a|\alpha)\overline{c(a|\beta)}}{\sum_\gamma \sum_{a\in I_r} \rho_{\gamma\gamma} |c(a|\gamma)|^2}$$

Look in particular at diagonal elements

Some mathematical considerations

* Explicit solution of the random recursion (*)$_{ND}$

Lemma: If $\rho(0)$ is given and the outcomes of experiments are $a_1, \ldots a_n$ then for $m \le n$

$$\rho_{\alpha\beta}(m) = \rho_{\alpha\beta}(0) \frac{\prod_a \left[c(a|\alpha)\,\overline{c(a|\beta)}\right]^{N_m(a)}}{\sum_\gamma \rho_{\gamma\gamma}(0) \prod_a |c(a|\gamma)|^{2N_m(a)}}$$

where $N_m(a) := \#\{\ell \le m,\ a_\ell = a\}$ = number of times a was the outcome up to the m^{th} measurement.

The probability of $a_1, \ldots, a_m$ is

$$\sum_\gamma \rho_{\gamma\gamma}(0) \prod_a |c(a|\gamma)|^{2N_m(a)}$$

Exercice: Prove the lemma, eg by recursion. Note the pretty compensations / cancellations

Then formulae show clearly that any permutation $a_{\sigma(1)} \ldots a_{\sigma(n)}$ has the same probability and leads to the same $\rho(n)$ as $a_1, \ldots a_n$. This is because $N_n(a)$ has the same property. In probabilistic language, the sequence $a_1, \ldots, a_n, \ldots$ is called "exchangeable", a fact related to the commutation of the W_m s or of the A_a, A_a^+ s. This exchangeability property has deep consequences in general. In our case, these can be checked straightforwardly!

Some mathematical considerations

* Probability measures on sequences :

We imagine that the experiment goes on forever, ie that an infinite sequence $a_1, \ldots a_n, \ldots$ can be observed. So an experiment outcome is a point in $I^{\mathbb{N}^*}$ where I is the set of a's "$I = \{a\}$". We set $\Omega := I^{\mathbb{N}^*}$, and use the letter ω to denote a point in Ω, i.e $\omega = (a_1, \ldots a_n, \ldots)$ for some sequence of outcomes.

For each α, we consider a probability measure

$\mathbb{P}_\alpha$ on Ω : it just makes the outcomes $(a_1, \ldots, a_n, \ldots)$ a sequence of independent random variables with values in I, with law $p(\cdot|\alpha)$ for each. This is characterized by $\mathbb{P}_\alpha$ (the n first outcomes are $a_1, \ldots a_n$)

$$:= \prod_{m=1}^{n} p(a_m|\alpha) = \prod_a p(a|\alpha)^{N_n(a,\omega)}$$

Note : $N_n(a,\omega)$ is a more precise notation for $N_n(a)$

$N_n(a,\omega)$:= number of times a appears among the first n components of $\omega \in \Omega$.

Note : for each α $\mathbb{P}_\alpha$ is very simple, much simpler that we get via the formula $\circledast_{ND}$. Well, not that much simpler !

As seen on the previous page, the probability of $a_1 \ldots a_n$ in the GND measurement is

$$\sum_\gamma \rho_{\gamma\gamma} \prod_a p(a|\gamma)^{N_m(a,\omega)}$$

So the probability measure for the GND is simply

$\sum_\alpha \varrho_{\alpha\alpha}(0)\, \mathbb{P}_\alpha$, and to get (fair) samples of a GND experiment outcomes, just pick an α, with probability measure $\varrho_{\alpha\alpha}(0)$ on the αs, and then independent $a_1, a_2, \dots$, with proba $p(a_1|\alpha)$, $p(a_2|\alpha)$.

Look, this is amazing! (*)(*1) looks much more complicated than it really is. Contrary to what it seems to indicate, one can simulate the GND measurement process with ever computing the $\varrho(n)$s!

To get the full powers of these remarks, we shall need the non-degeneracy assumption that the $p(\cdot|\alpha)$ are different for different αs (as probability measures on I)

Some mathematical considerations

* Martingales and super-martingales

Question: suppose that $\rho(0), \dots \rho(n)$ are known, what is $\rho(n+1)$ in average? We use the standard probabilistic notation $\mathbb{E}$ for such an average.

$\mathbb{E}(\rho_{\alpha\beta}(n+1)$ knowing the past up to time $n)$ abbreviated by $\mathbb{E}(\rho_{\alpha\beta}(n+1) \mid \mathcal{F}_n)$. By the Markov property (embodied in $(*_{ND})$) this depends only on $\rho(n)$ and one finds

$$\mathbb{E}(\rho_{\alpha\beta}(n+1) \mid \mathcal{F}_n) = \rho_{\alpha\beta}(n) \sum_a c(a|\alpha) \overline{c(a,\beta)}$$

(notice again the nice cancellation between denominators and probabilities) and

$$\mathbb{E}(|\rho_{\alpha\beta}(n+1)| \mid \mathcal{F}_n) = |\rho_{\alpha\beta}(n)| \sum_a |c(a|\alpha) \overline{c(a|\beta)}|$$

In particular, for $\alpha = \beta$, and setting $\rho_{\alpha\alpha} := Q_\alpha$

$$\mathbb{E}(Q_\alpha(n+1) \mid \mathcal{F}_n) = Q_\alpha(n) \sum_a p(a|\alpha) = Q_\alpha(n)$$

"Knowing the past, the best prediction for the future is the present!". This is called a martingale in probability theory

For off-diagonal element we use Cauchy-Schwarz

$|c(a|\alpha)\, c(a|\beta)| = p(a|\alpha)^{1/2}\, p(a|\beta)^{1/2}$ so

$$\sum_a |c(a|\alpha)\, c(a|\beta)| \leq \left(\sum_a p(a|\alpha)\right)^{1/2} \left(\sum_a p(a|\beta)\right)^{1/2} = 1$$

with equality if and only if the GND measurement is degenerate between α and β

Thus $\mathbb{E}(\,|\rho_{\alpha\beta}(n+1)|\,|\,\mathcal{F}_n) \leq |\rho_{\alpha\beta}(n)|$

" Knowing the past, the best prediction for the future is lower than the present!" This is called a supermartingale in probability theory.

Note: the confusing term "super" for $\leq$ is to be consistent with superharmonic, historically due to the fact that it is $-\Delta$ which is a non negative operator. There are deep relations (involving Brownian motion) between martingales and harmonic functions.

The importance of martingales and supermartingales in probability stems from several facts.

- The obvious one: by iteration $\mathbb{E}(Q_\alpha(n)) = Q_\alpha(0)$

- The deep one: martingales and submartingales are ruled by convergence theorems. A special case that is enough for our needs is: a bounded (super) martingale converges almost surely and in $\mathbb{L}^1$. As $Q_\alpha(n) \in [0,1]$ for n and α and $|\rho_{\alpha\beta}|^2 \leq Q_\alpha Q_\beta$ by positivity of ρ we infer: there is a random variable $\rho(\infty)$ such that $\rho(n) \to \infty$ a.s and in $\mathbb{L}^1$: $\mathbb{E}(\,|\rho(n)-\rho(\infty)|\,) \to 0$ as $n \to \infty$ and for (almost) every GND measurement process $\rho(n) \to \rho(\infty)$

For diagonal elements we have $\mathbb{E}(\rho_\alpha(\infty)) = \rho_\alpha(0)$

For non diagonal elements we have $\mathbb{E}(|\rho_{\alpha\beta}(\infty)|) \leq |\rho_{\alpha\beta}(0)|$

To make progress, it turns out that the decomposition $\sum \rho_\alpha(0)\, \mathbb{P}_\alpha$ is the most efficient tool. Non-degeneracy is a crucial assumption. In the martingale approach, this is clear: if $p(\cdot|\alpha) \neq p(\cdot|\beta)$ we get

$$\mathbb{E}(|\rho_{\alpha\beta}(n)|) \leq |\rho_{\alpha\beta}(0)| \Big(\underbrace{\sum_a |C(a|\alpha)\, C(a|\beta)|}_{<1} \Big)^n$$

leading to $\mathbb{E}(|\rho_{\alpha\beta}(\infty)|) = 0$ ie $|\rho_{\alpha\beta}(\infty)| = 0$ a.s

whereas if $p(\cdot|\alpha) = p(\cdot|\beta)$ we get that $|\rho_{\alpha\beta}(n)|$ is a martingale and $\mathbb{E}(|\rho_{\alpha\beta}(\infty)|) = |\rho_{\alpha\beta}(0)|$

At least in the non-degenerate case, off-diagonal element of ρ in the pointer state basis go to 0 a.s. We can even do a bit better: if $\rho(0)$ is a pure state, then so is each $\rho(n)$ and by continuity $\rho(\infty)$. But the diagonal pure states are precisely the pointer states, so $\rho(\infty)$ must be one of the pointer states if $\rho(0)$ is pure and the QND is non-degenerate. It turns out that this is also true for general $\rho(0)$, but a proof via martingales is not so easy!

The deep martingale convergence theorem will be replaced by another deep theorem, the strong law of large numbers:

Under $\mathbb{P}_\alpha$ the (vector) random variable $N_n(a,\omega)$ is such that $\frac{1}{n} N_n(a,\omega) \to p(a|\alpha)$ $\mathbb{P}_\alpha$ a.s ie

there is an $\Omega_\alpha \subset \Omega$ $\mathbb{P}_\alpha(\Omega_\alpha)=1$ and

$$\frac{1}{n} N_n(a,\omega) \to p(a|\alpha) \quad \text{a.s.}$$

Under the assumption of non degeneracy, we get that $\Omega_\alpha \cap \Omega_\beta = \phi$, meaning that the $\mathbb{P}_\alpha$s are mutually singular: as n gets larger and larger we get better and better indications of which measure is sampled by looking at the $N_n(a)$

Fix γ_0 and look at $\rho_{\alpha\beta}(n)$ under $\mathbb{P}_\sigma$:

$$\rho_{\alpha\beta}(n) = \rho_{\alpha\beta}(0) \frac{\prod_a \left(c(a|\alpha)\overline{c(a|\beta)}\right)^{N_n(a)}}{\sum_\gamma \rho_{\gamma\gamma}(0) \prod_a p(a|\gamma)^{N_n(a)}} \quad \text{with probability}$$

$$\prod_a p(a|\sigma)^{N_n(a)} = p(a_1|\sigma)\cdots p(a_n|\sigma)$$

We might be in trouble with this formula (a vanishing denominator with non-zero probability) but if $\rho_{\sigma\sigma}(0)>0$ this cannot occur. Thus we assume $\rho_{\sigma\sigma}(0)>0$.

Then $\rho_{\sigma\sigma}(n)>0$ for all n's with $\mathbb{P}_\sigma$-probability 1.

Thus $$\frac{\rho_{\alpha\beta}(n)}{\rho_{\sigma\sigma}(n)} = \frac{\rho_{\alpha\beta}(0)}{\rho_{\sigma\sigma}(0)} \prod_a \left(\frac{c(a|\alpha)\overline{c(a|\beta)}}{p(a|\sigma)}\right)^{N_n(a)}$$

Progressive collapse in ...

We assume in this discussion that if $\alpha \neq \beta$ there is some a such that $p(a|\alpha) \neq p(a|\beta)$

We prove the following lemma: (add some a.s. where needed)

Lemma: $\rho(n)$ converges to a random density matrix $\rho(\infty)$ as $n \to \infty$. The possible values for $\rho(\infty)$ are the pointer states, and the probability to end in state $|\alpha\rangle\langle\alpha|$ is $\rho_{\alpha\alpha}(0)$

The meaning of this is that an infinite sequence of non-degenerate quantum non-demolition measurements is exactly equivalent to a standard Von Neumann measurement of an observable have the $|\alpha\rangle$s as (non degenerate) eigenvectors.

So a sequence of ND QND measurements leads to a progressive wave function collapse (or density matrix collapse) and the number of probes serves as a parameter to observe the transition from a quantum (= small) to a classical (= large) measurement device.

We sketch the proof:
If we take the martingale results into account, the only thing that remains to be proved is that the limit must be a pointer state. However we reprove everything using the "extremal" decomposition
$\sum G_\alpha(0) \mathbb{P}_\alpha$ and its properties.

Start with $\alpha = \beta$. We infer,

$$\left(\frac{\rho_\alpha(n)}{\rho_\sigma(n)}\right)^{1/n} = \left(\frac{\rho_\alpha(0)}{\rho_\sigma(0)}\right)^{1/n} \prod_a \left(\frac{p(a|\alpha)}{p(a|\sigma)}\right)^{\frac{N_n(a)}{n}}$$

so with $\mathbb{P}_\alpha$-probability 1, we have

$$\lim_{n\to\infty} \left(\frac{\rho_\alpha(n)}{\rho_\sigma(n)}\right)^{1/n} = \mathbb{1}_{\rho_\alpha(0)\neq 0}\ e^{-S_{rel}(\sigma|\alpha)}$$

where $S_{rel}(\sigma|\alpha) := \sum_a p(a|\sigma) \log \frac{p(a|\sigma)}{p(a|\alpha)}$ is nothing but the relative entropy of two distributions $p(\cdot|\sigma)$ and $p(\cdot|\alpha)$. This relative entropy is always in $[0,+\infty]$, and is in $]0,+\infty]$ in our case because $p(\cdot|\alpha) \neq p(\cdot|\sigma)$.

<u>Note</u>: from $\log x \leq x-1$ $x \in [0,+\infty]$ get

$\log \frac{p(a|\alpha)}{p(a|\sigma)} \leq \frac{p(a|\alpha)}{p(a|\sigma)} - 1$, multiply both sides by $p(a|\sigma)$ and sum over a to get $-S_{rel}(\sigma|\alpha) \leq 0$

For general α, β we get

$$\lim_{n\to\infty} \left|\frac{\rho_{\alpha\beta}(n)}{\rho_{\sigma\sigma}(n)}\right|^{1/n} = \mathbb{1}_{\rho_{\alpha\beta}(0)\neq 0}\ e^{-\frac{1}{2}\left(S_{rel}(\sigma|\alpha) + S_{rel}(\sigma|\beta)\right)}$$

So under $\mathbb{P}_\sigma$ and if $\rho_{\sigma\sigma}(0) \neq 0$ $\frac{\rho_{\alpha\beta}(n)}{\rho_{\sigma\sigma}(n)}$ converges exponentially fast towards 0 with probability 1 (unless $\alpha = \beta = \sigma$) with an explicit rate related to a cornerstone of information theory. Because $\operatorname{Tr} \rho(n) = 1$ for all n, this is only possible if under $\mathbb{P}_\sigma$ $\rho_{\alpha\beta}(n)$ converges to $\delta_{\alpha\sigma}\delta_{\beta\sigma}$

(unless $\rho_{\sigma\sigma}(0)=0$). Thus under $\mathbb{P}_\sigma$ ρ converges (exponentially fast, but with fluctuations) towards $|\sigma\rangle\langle\sigma|$. If $\mathcal{Q}_\alpha$ is any probability measure on the α's, then under $\sum_\sigma \mathcal{Q}_\sigma \mathbb{P}_\sigma$ ρ converges to the pointer state $|\sigma\rangle\langle\sigma|$ with probability $\mathcal{Q}_\sigma$ (unless $\rho_{\sigma\sigma}(0)=0$).

Let's twist the notation a little bit: suppose $\rho(0)$ is the "real" density matrix of the system (ie we have an ensemble described by $\rho(0)$) but we do not know $\rho(0)$ and work with a guess $\tilde{\rho}(0)$ which we update by the rule $\circledast_{ND}$ for $\tilde{\rho}$ but with probabilities as imposed by $\sum \mathcal{Q}_\sigma(0)\,\mathbb{P}_\sigma$ i.e. the real probabilities. Then as soon as our guess was not too bad, ie as soon as $\rho_{\alpha\alpha}(0)=0$ if $\tilde{\rho}_{\alpha\alpha}(0)=0$ ie in fancy language $\mathcal{Q}_\alpha$ has a density with respect to $\tilde{\mathcal{Q}}_\alpha$, $\rho(n)$ and $\tilde{\rho}(n)$ have the same limit.

Thus we have two ways (at least) to identify the large n limit:

- If the frequencies $\frac{N_n(\cdot)}{n}$ approach $p(\cdot|\alpha)$ at large n (to such an extent that it cannot be mistaken for another $p(\cdot|\beta)$) then the limit must be $|\alpha\rangle\langle\alpha|$
- Follow a trial density matrix, for instance

$\tilde{\rho}(0) \propto \mathrm{Id}_{\mathcal{H}_S}$ and update it. For large n $\tilde{\rho}(n)$ will be close to the limiting $|\alpha\rangle\langle\alpha|$ with large probability.

Application: the cavity QED experiment

Show pictures of wave function update

Note: The lemma does not prove the wave function collapse ex-nihilo: we use the usual wave function collapse axiom on each probe measurement!

Note: However, this result is fascinating for at least two reasons.

- The decomposition of the probability measure as $\sum Q_\alpha(0)\,\mathbb{P}_\alpha$ means that the experiment outcome can be decided in advance: first choose $\mathbb{P}_\alpha$ with probability $Q_\alpha(0)$ and then run the experiment knowing that the finite state will be $|\alpha\rangle\langle\alpha|$. Another way to phrase this is that starting from a ρ-statistical ensemble, a man in the middle performing secretely a measurement a la Von Neumann projecting to the $|\alpha\rangle$s before we start our experiment would be totally undetectable. True, the a_n's are not independent. However an infinite sample $(a_1, a_2, \cdots)$ has all statistical properties of a sequence of independent random variables with probability distribution $\lim_{n\to\infty} \frac{1}{n} N_n(a)$.

- The update rule for diagonal elements of ρ is

$$\rho'(\alpha) = \rho_\alpha \frac{p(a|\alpha)}{\sum_\beta \rho_\beta \, p(a|\beta)} \quad \text{with proba } \sum_\beta \rho_\beta \, p(a|\beta)$$

Compare with Bayesian / subjective inference:

Two random variables X and Y, Bayes rule

$$p(X=\alpha \mid Y=a)\, p(Y=a) = p(Y=a \mid X=\alpha)\, p(X=\alpha) \quad *$$

because both equal $p(X=\alpha \text{ and } Y=a)$

$p(Y=a \mid X=\alpha)$ assumed to be known

$p(X=\alpha)$ is estimated a priori as $p_{prior}(\alpha)$

Suppose $Y=a$ is observed, (probability $p(Y=a)$),

then one gains information and gets $p_{post}(\alpha)$

$p_{post}(\alpha)$ is a guess for $p(X=\alpha \mid Y=a)$ so by $*$

$$p_{post}(\alpha)\, p(Y=a) = p(Y=a \mid X=\alpha)\, p_{prior}(\alpha)$$

$$p_{post}(\alpha) = p_{prior}(\alpha) \frac{p(Y=a \mid X=\alpha)}{p(Y=a)}$$

and $p(Y=a) = \sum_\beta p_{prior}(\beta)\, p(Y=a \mid X=\beta)$ to

ensure $\sum_\alpha p_{post}(\alpha) = 1$ thus

$$p_{post}(\alpha) = p_{prior}(\alpha) \frac{p(Y=a \mid X=\alpha)}{\sum_\beta p_{prior}(\beta)\, p(Y=a \mid X=\alpha)}$$

with probability $\sum_\beta p_{prior}(\beta)\, p(Y=a \mid X=\alpha)$

The quantum update rule is nothing but the Bayesian subjective probability update rule!

Exercice: comment on the possible interpretations of this fact

Conclusions

Quantum non demolition measurements provide a model to study quantum measurement in general. This model can be analysed in great detail, and has several nice features:

- the possibility to study the transition from the classical to the quantum regime: when many probes have passed by one has the equivalent of a classical measurement apparatus, with (almost) perfect wave function collapse, while for a small number of probes fluctuation play a dominant effect. Remember that if $Q_\alpha(n)$ is known, it gives the probability to end in state $|\alpha\rangle\langle\alpha|$ in the end, so that even if $Q_\alpha(n) \sim 1$, there is still some tiny probability $1-Q_\alpha(n)$ to end up in another state.

Show simulations

- the fact that a real measurement apparatus has a non uniform sensitivity, ie the precision depends on the outcome is nicely embodied in relative entropies. If $S_{rel}(\sigma|\alpha)$ is "large" for every $\alpha \neq \sigma$ then σ is easily identified by the measurement apparatus whereas if some $S_{rel}(\sigma|\alpha)$ are small it has a hard time to separate σ from those αs.
- one can use different protocols (like changing the initial state of the probes or the probe observable) to accelerate convergence

Done in real experiments

The continuous time limit (average evolution)

Our aim is to explore the possible continuous time limits of the average / partial trace Markovian evolution

$\rho_{n+1} = \sum_a A_a \rho_n A_a^+$ where $\sum_a A_a^+ A_a = Id_{\mathcal{H}_S}$

[As ρ_S is the only density matrix that appears here, we omit the "S" subscript.

If the time step is ε, we thus want to start with ε-dependant operateurs $A_a(\varepsilon)$ such that

$\sum_a A_a(\varepsilon) \rho A_a^+(\varepsilon) = \rho + \varepsilon \mathcal{L}(\rho) + o(\varepsilon)$, those operators having to fulfil $\sum_a A_a^+(\varepsilon) A_a(\varepsilon) = Id_{\mathcal{H}_S}$.

The starting point is to take $\varepsilon = 0$ so that

$\sum_a A_a(\varepsilon) \rho A_a^+(\varepsilon) = \rho$ for every (positive, then

self-adjoint then arbitrary) ρ.

<u>Lemma</u>: $\sum_a A_a \rho A_a^+ = \rho \quad \forall \rho \Leftrightarrow A_a = s_a Id_{\mathcal{H}_S}$ for complex numbers s_a such that $\sum_a |s_a|^2 = 1$

$\forall \rho \quad \sum_a A_a \rho A_a^+ = \rho \quad \Leftrightarrow \quad \sum_a (A_a)_{\alpha\mu} (A_a^+)_{\nu\beta} = \delta_{\alpha\mu} \delta_{\nu\beta}$

and $\beta = \alpha \quad \nu = \mu$ gives $\sum_a |(A_a)_{\alpha\mu}|^2 = \delta_{\alpha\mu}$ ie the

A_a's are diagonal. This must be true in any basis so they must be scalar

To make the expansion of $A_a(\varepsilon)$ at small ε we make an educated guess, that we shall "justify" or better "postmotivate" later.

We postulate a form $A_a(\varepsilon) = s_a \, Id_{\mathcal{H}_s} + \varepsilon^{1/2} B_a + \varepsilon C_a + o(\varepsilon)$

This lead to $\sum_a |s_a|^2 = 1$ i) $\sum_a \bar{s}_a B_a \rho + \rho s_a B_a^\dagger = 0$ ii)

$\sum_a |s_a|^2 = 1$ $\quad \sum_a s_a B_a^\dagger + \bar{s}_a B_a = 0$ iii) $\quad \sum_a s_a C_a^\dagger + \bar{s}_a C_a + B_a^\dagger B_a = 0$ iv)

and if they hold $\mathcal{L}(\rho) = \sum_a \bar{s}_a C_a \rho + \rho s_a C_a^\dagger + B_a \rho B_a^\dagger$

From ii) we get $\sum \bar{s}_a B_a = ib \, Id_{\mathcal{H}_s}$ for some real b

This solves iii) as well. Set $C = \sum \bar{s}_a C_a$. Then

iv) says that $C + \frac{1}{2} \sum_a B_a^\dagger B_a = -iH$ for some

selfadjoint H and all the constraints are solved

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_a \mathcal{L}_{B_a}(\rho)$$

$$\mathcal{L}_B(\rho) := B\rho B^\dagger - \frac{1}{2}\left(B^\dagger B \rho + \rho B^\dagger B\right)$$

linear in ρ but quadratic in B

Thus the continuous time limit of the discrete evolution is

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) \qquad \mathcal{L}(\rho) := -i[H, \rho] + \sum_a \mathcal{L}_{B_a}(\rho)$$

This equation is called the Lindblad equation,

and a (super)operator $\mathcal{L}$ of the above type is called a Lindbladian.

The term $-i[H, \cdot]$ represents a unitary evolution

The terms $\mathcal{L}_B(\cdot)$ represent a dissipative evolution in general and are sometimes called dissipators.

It is wrong to think of H as the intrinsic initial hamiltonian evolution and the $\mathcal{L}_{B_a}$ s as the effect of the environment because

$B_a \to B_a + i\, b_a\, Id_{\mathcal{H}_S}$ $\quad H \to H - \frac{1}{2}\sum_a (\bar{b}_a B_a + b_a B_a^\dagger)$

leaves $\mathcal{L}$ invariant: interaction with the environment may lead to an effective internal Hamiltonian evolution on top of dissipation!

Remark: the constraint $\sum \bar{s}_a B_a = i b\, Id_{\mathcal{H}_S}$ is empty because we can get the most general $\mathcal{L}$ by taking $s_a = \delta_{a,a_0}$ for some a_0 and $B_{a_0} = 0$

Remark: why not $\varepsilon^{1/3}$ or $\varepsilon^{1/4}$ or ? in the initial ansatz. The reason is that $\mathcal{L}$ is trivial $\mathcal{L} \equiv 0$ iff H and the B_a are all scalar matrices (a variant of the lemma). Suppose ε^α $\alpha > 0$ is the next to lowest order term in the expansion ie $A_a(\varepsilon) = s_a Id_{\mathcal{H}_S} + \varepsilon^\alpha X_a + \cdots$. If $2\alpha < 1$, call Y_a the $\varepsilon^{2\alpha}$ term

then the above computation carry over to yield that the X_a and $\sum \bar{s}_a Y_a = Y$ must be scalars (no associated Lindbladian because $2\alpha < 1$ so no term of order $\varepsilon^{2\alpha}$ in the expansion). On can use this easily to show that adding $\varepsilon^{1/3}$ and $\varepsilon^{2/3}$ to $\varepsilon^0, \varepsilon^{1/2}, \varepsilon$ does not lead to anything more general...

Remark: why $\varepsilon^{1/2}$ is in fact a more interesting question: $A_a = \langle a | U | \Psi \rangle$. Why should this matrix element contain non integer powers of ε for $U(\varepsilon)$ a unitary matrix which we would expect to be something like $e^{-i\varepsilon H}$. The answer is maybe that such an operator has smooth matrix elements only in finite dimension. This $\varepsilon^{1/2}$ is also related to the zeno effect, see later

The Lindblad equation is the equation for the deterministic evolution of $\mathrm{Tr}_{\mathcal{H}_e}\, \rho(t)$ when the interaction between the system and the environment lead to no memory effect. There are some "moral" theorems to justify this, but this is still the subject of active research:

A version of the moral theorem:

$$H = \underbrace{H_s \otimes Id_{\mathcal{H}_e} + Id_{\mathcal{H}_s} \otimes H_e}_{H_0} + \lambda V$$

$$\tilde{\rho}_s(t) := \lim_{\lambda \to 0} \mathrm{Tr}_{\mathcal{H}_e}\, e^{it/\lambda^2 H_0}\, e^{-it/\lambda^2 H}\, \rho(0)\, e^{it/\lambda^2 H}\, e^{-it/\lambda^2 H_0}$$

(small interaction acting for a long time) exists and satisfies a Lindblad equation whose form is computable from H_s, H_e, V. To get non trivial dissipation, $\mathcal{H}_e$ would better be infinite dimensional

Note the appearance of λ^2 scaling.

The stochastic continuous time limit.

We shall only sketch this more intricate discussion

For orientation, go back to non demolition, with the formulae

$$\rho_{\alpha\beta}(m) = \rho_{\alpha\beta}(0) \frac{\prod_a \left(c(a|\alpha)\, \overline{c(a|\beta)} \right)^{N_m(a)}}{\sum_\gamma \rho_{\gamma\gamma}(0) \prod_a |c(a|\gamma)|^{2N_m(a)}}$$

The probability measure on the counting processes $N_m(a)$ is $\sum_\alpha \rho_{\alpha\alpha}(0)\, \mathbb{P}_\alpha$ where under $\mathbb{P}_\sigma$

$N_m(a) = \sum_{\ell=1}^{m} \varepsilon_\ell(a)$ the $\varepsilon_\ell(\cdot)$ being random unit vectors, iid with law

$\text{prob}(\varepsilon_\ell(a) = \delta_{a,a_0}) = p(a_0|\sigma)$

Thus all the results of standard random walk theory. $\sqrt{\frac{\varepsilon}{p(a|\sigma)}} \left(N_{\lfloor \frac{t}{\varepsilon} \rfloor}(a) - \lfloor \frac{t}{\varepsilon} \rfloor p(a|\sigma) \right)$ converges, in the sense of the central limit theorem, towards a process $W_t(a)$ which is Gaussian with mean 0 and covariance

$$\mathbb{E}(W_t(a)\, W_s(b)) = \min(t,s) \left(\delta_{ab} - \sqrt{p(a|\sigma)\, p(b|\sigma)} \right)$$

$\mathbb{E}(\dot{W}_t(a)\, \dot{W}_s(b)) = \delta(t-s) \left(\text{- - -} \right)$ is just a vector white noise with an appropriate sum rule reflecting

$\sum_a N_m(a) = m \quad \forall m$

Now what we want to do is to let also let the $(A_a)_{\alpha\beta} = C(a|\alpha)\,\delta_{\alpha\beta}$ depend on ε, remember $p(a|\alpha) = |C(a|\alpha)|^2$ will depend on ε as well.

It turns out that the scaling introduced in the non-random case is just what it takes to have also a limiting value for $\rho_{\alpha\beta}\left(\lfloor \frac{t}{\varepsilon} \rfloor\right)$ other scalings → Poisson noise

$$= \frac{\rho_{\alpha\beta}(0) \prod_a \left(c_\varepsilon(a|\alpha)\, \overline{c_\varepsilon(a,\beta)} \right)^{N_{\lfloor \frac{t}{\varepsilon} \rfloor}(a)}}{\sum_\gamma \rho_{\gamma\gamma}(0) \prod_a |c_\varepsilon(a|\gamma)|^{2\, N_{\lfloor \frac{t}{\varepsilon} \rfloor}(a)}}$$

and this is expressed naturally in terms of the $W_t(a)$

Exercice: Try to do it for yourself!

It is not difficult to transfer this result for the "real" case $\mathbb{P} = \sum_\gamma \rho_{\gamma\gamma}(0)\, \mathbb{P}_\gamma$ to get an explicit formula for the continuous quantum non demolition measurement process. From this explicit formula, one gets also an "equation of motion" which is stochastic with white noise randomness.

In the general case, no such explicit formula is available due to non-commutativity issues, but a stochastic equation of motion can be obtained. To get it, the counting processes again play a crucial role

We just quote the result:

The continuous time limit of

$$\rho_{n+1} = \frac{\sum_{a\in I_r} A_a \rho_n A_a^+}{\mathrm{Tr} \sum_{a\in I_r} A_a \rho_n A_a^+} \qquad N_{n+1}^r(q) = N_n(q) + \delta_{q,r}$$

with probability $\mathrm{Tr} \sum_{a\in I_r} A_a \rho_n A_a^+$

when $A_a(\varepsilon) = s_a \mathbb{1}d_{\mathcal{H}_s} + \varepsilon^{1/2}(\dots) + \varepsilon(\dots) + o(\varepsilon)$

(and under the assumption that $p_r := \sum_{a\in I_r} |s_a|^2 > 0$ for each r) is the following:

$$\begin{cases} \rho_{\lfloor \frac{t}{\varepsilon} \rfloor} \xrightarrow{w} \rho(t) \\ \left(\frac{\varepsilon}{p_r}\right)^{1/2} (N_n(r) - n p_r) \rightarrow X_t^{(r)} \end{cases}$$

and $$d\rho_t = \sum_r \left(\mathcal{G}_{D_r}(\rho_t)\, dW_t^{(r)} + \mathcal{L}_{D_r}(\rho_t)\, dt \right) + \left(\sum_a{}' \mathcal{L}_{B_a}(\rho_t) - i[H, \rho_t] \right) dt$$

$$d X_t^{(r)} = dW_t^{(r)} + \left(\mathrm{Tr}_{\mathcal{H}_s} (D_r - p_r^{1/2} D)\rho_t + \rho_t (D_r^+ - p_r^{1/2} D^+) \right) dt$$

where D_r, B_a can be expressed in terms of $(\dots)$ but are wholly arbitrary and $D := \sum p_r^{1/2} D_r$

$$\mathbb{E}\left(\dot{W}_t^{(r)} \dot{W}_s^{(q)} \right) = \delta(t-s) \left(\delta^{qr} - p_r^{1/2} p_q^{1/2} \right)$$

$$\mathcal{L}_B(\rho) := B\rho B^+ - \frac{1}{2}(B^+B\rho + \rho B^+B)$$

$$\mathcal{G}_B(\rho) := B\rho + \rho B^+ - \rho\, \mathrm{Tr}_{\mathcal{H}_s}(B\rho + \rho B^+)$$ non linear, comes from denominator

These are the Belavkin equations. Taking the average and setting $\mathbb{E}(\rho_t) = \bar{\rho}_t$ we get

$$d\bar{\rho}_t = \left(\sum_r \mathcal{L}_{D_r}(\bar{\rho}_t) + \sum_a{}' \mathcal{L}_{B_a}(\bar{\rho}_t) - i[H, \bar{\rho}_t] \right) dt$$

which is, as expected, an equation of the Lindblad type, with $\sum_r \mathcal{L}_{D_r} + \sum_a{}' \mathcal{L}_{B_a}$ reconstructing the dissipator. (2 Itô convention dW_t independant of past!)

2 The D_r's and B_a's here are linear combinations of the original B_a's (that appear in the Lindblad equation if $A_a(\varepsilon) = S_a \, Id_{\mathcal{H}_S} + \varepsilon^{1/2} B_a + O(\varepsilon)$)

The continuous time equations are mathematically more tractable the discrete time ones. But for numerical simulations discrete time is a good option because the numerical simulation of several Brownian motions, more precisely of stochastic equations involving several Brownian motions, is difficult.

Conclusion: At this point, we have the general equations describing open quantum systems in the Markovian setting taking into account fluctuations due to indirect repeated measurements. It is unclear wether other environment fluctuations can also be described that way.

Firenze 2015

Markovian Open Quantum Systems
A Basic Introduction

Lecture III

Quantum trajectories

Quantum trajectories for GND measurements

Some phenomenology of quantum trajectories for GND in the strong measurement regime.

Quantum noises (on a post stamp).

Open quantum counting processes as an example.

Quantum trajectories

This part is meant to be non-technical and descriptive.
The random iteration system (*) or its continuous time limit describes a time evolution of ρ_s.
Such a time evolution is called a <u>quantum trajectory</u>. Thus, from any solution of $\sum_a A_a^\dagger A_a = Id$ on a Hilbert space $\mathcal{H}_s$ and any partition $I = \bigsqcup I_r$ of the set of as, or with any set of Bs and Ds in the Belavkin equations we can get quantum trajectories.

This is easy but the physical interpretation is unclear in general! Even so, we encourage the readers to play that game because quantum trajectories are still full of mysteries, and we shall only scratch upon the surface.

Quantum trajectories for GND measurements

This case is well understood! So well that we want to use it as a tool.

The basic idea is the following: suppose that the system Σ interacts with an environment E (possibly with measurement or other types of randomness) but the time scale for fluctuations is too small to keep a record, so that the evolution (which we write in discrete time here for simplicity) is

$$\rho_{n+1} = \sum_a A_a \rho A_a^\dagger \qquad \text{(average evolution)}$$

What if is is observed with a non demolition measurement process?

To model this, we use that if $\sum_{a\in I} A_a^\dagger A_a = \mathrm{Id}_{\mathcal{H}_s}$ and $\sum_{b\in J} B_b^\dagger B_b = \mathrm{Id}_{\mathcal{H}_s}$ then $\sum_{a,b} (B_b A_a)^\dagger B_b A_a = \mathrm{Id}_{\mathcal{H}_s}$

ie $\quad K = I \times J$ and $C_c = B_b A_a$ if $c = (a,b)$

gives $\sum_c C_c^\dagger C_c = \mathrm{Id}$ so that

ρ_{n+1} = any version of $(*)$ with the C_c's is ok.

This just means that $(*)$ can be iterated in basically any way.

Here we content with a modest one:

$$\rho_{n+1} = \frac{B_b\left(\sum_a A_a \rho_n A_a^\dagger\right)B_b^\dagger}{\mathrm{Tr}\, B_b\left(\sum_a A_a \rho_n A_a^\dagger\right)B_b^\dagger} \quad \text{with probability}$$

$$\mathrm{Tr}\; B_b\left(\sum_a A_a \rho_n A_a^\dagger\right)B_b^\dagger$$

where the B_bs are QND operators (ie they all commute among each other and the adjoints)

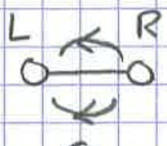
That means: we let the system evolve due to contact with E for $(\Delta t)_e$, then make a QND measurement on it taking $(\Delta t)_m \ll (\Delta t)_e$ and go on

Example: QND to observe an unitary evolution:

$\rho_{n+1} = U \rho_n U^\dagger$ $\quad$ U unitary on $\mathcal{H}_S$, in general non diagonal in the pointer state basis of the B_b's

$$\rightarrow \rho_{n+1} = \frac{B_b U \rho_n U^\dagger B_b^\dagger}{\mathrm{Tr}\, B_b U \rho_n U^\dagger B_b^\dagger} \quad \text{proba } \mathrm{Tr}\, B_b U \rho_n U^\dagger B_b^\dagger$$

Exercice: Take the exercice on rabi oscillations from Lecture I



hopping of an electron

⊕ indirect measure of where the electron is

Write $A_+ = B_+ U$ $\quad A_- = B_- U$ where U is a rotation

$$\left(A_+ = \begin{pmatrix} \cos\alpha\cos\beta & \sin\alpha\cos\beta \\ -\sin\alpha\sin\beta & \cos\alpha\sin\beta \end{pmatrix} \right.$$

$$\left. A_- = \begin{pmatrix} \cos\alpha\sin\beta & \sin\alpha\sin\beta \\ -\sin\alpha\cos\beta & \cos\alpha\cos\beta \end{pmatrix} \right)$$

Example: GND to observe thermalization:

Take H Hamiltonian on $\mathcal{H}_S$ and suppose that

$$\rho_{n+1} = \sum_a A_a \rho_n A_a^\dagger \quad \text{yields} \quad \rho_\infty = \frac{e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}}$$

Note: cooking up the A_as for that may be non trivial, and deciding of a physically relevant set of A_as is even more non-trivial

Then make GND measurements to observe the "approach to equilibrium" in two cases:

- The pointer state basis diagonalizes H
- The pointer state basis does not diagonalize H.

(expect some surprises)

Exercice: suppose $M_{\alpha\gamma}$ is a "good" classical Markov matrix to go to equilibrium for the classical system with energies $\mathrm{spec}(H) = \{E_\alpha\}$. That means

$$M_{\alpha,\gamma} \geq 0 \ \forall \alpha \quad , \quad \sum_\alpha M_{\alpha\gamma} = 1 \quad \forall \gamma \quad \text{and}$$

$$\sum_\gamma M_{\alpha\gamma} e^{-\beta E_\gamma} = e^{-\beta E_\alpha}$$ (plus some conditions ensuring there is no other fixed point).

Take $I = \{\alpha\}^2$ and for $a = (i,j)$

$$(A_a)_{\mu\nu} = \delta^i_\mu \, \delta^j_\nu \, M_{\mu\nu}^{1/2}$$

Check that $\dfrac{e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}}$ is a fixed point of

Phenomenology of quantum trajectories in the strong measurement regime.

Example: Thermal case (two states)

Simulations with $g \to g\delta$ $n = \left\lfloor \frac{t}{\delta} \right\rfloor$ which keeps the equilibration time finite.

Take $\cos^2\varphi = \frac{1}{2} + h\,\delta^{1/2} + \cdots$ to get close to the continuous time limit if δ is small.

Strong measurement: h^2 large g fixed

Exercice: let Q be the diagonal of ρ ie

$Q_\alpha = \rho_{\alpha\alpha}$ and write a thermal equilibration step as

$Q_\alpha(n+1) = \sum_\beta M_{\alpha\beta} Q_\beta(n)$. Assume that

$M_{\alpha\beta} = M_{\alpha\beta}(\delta) = (e^{\delta L})_{\alpha\beta}$ when L is a continuous time Markov generator. Compute the overlap between $Q(n)$ and $Q(n+1)$ to lowest non trivial order in δ

(The overlap is $\mathrm{Tr}\,\rho\rho'$ when $\rho = \mathrm{diag}\,Q(n)$
$\rho' = \mathrm{diag}\,Q(n+1)$ so we look for

$$\mathrm{Tr}\,\rho\rho' = \sum_{\alpha\beta} Q_\alpha (e^{\delta L})_{\alpha\beta} Q_\beta = \sum_\alpha Q_\alpha^2 + \delta \sum_{\alpha\beta} Q_\alpha L_{\alpha\beta} Q_\beta + \cdots)$$

What if $Q = Q(n)$ is a pure state?

$$\mathrm{Tr}\,\rho\rho' = 1 + \delta L_{\alpha_0\alpha_0} \qquad \text{if } Q_\alpha = \delta_{\alpha,\alpha_0}$$

$\rho_{n+1} = \sum_a A_a \rho_n A_a^\dagger$. Can you check the large n convergence? Is this physical? Interpret in terms of decoherence!

For a two level system with gap $E_2 - E_1 = \Delta E$ check that the most general Markov matrix is

$$\begin{pmatrix} 1-g e^{-\beta\Delta E} & g \\ g e^{-\beta\Delta E} & 1-g \end{pmatrix} \quad \text{for } g \in]0, \min(1, e^{\beta\Delta E})]$$

Exercise: Check this. Use it to simulate the observation of thermal equilibration in a two-level system by GND measurements in the H eigenstates basis. What happens for GND measurements in other bases? Play with parameters

Due to strong decoherence, one can work in the diagonal basis all along. Setting $\mathcal{C} = \rho_{11}$ we get

if $B_+ = \begin{pmatrix} \cos\varphi & 0 \\ 0 & \sin\varphi \end{pmatrix}$ $\quad B_- = \begin{pmatrix} \sin\varphi & 0 \\ 0 & \cos\varphi \end{pmatrix}$

$$\mathcal{C}_{n+1} = \frac{\cos^2\varphi\left[(1-g e^{-\beta\Delta E})\mathcal{C} + g(1-\mathcal{C})\right]}{\cos^2\varphi\left[(1-g e^{-\beta\Delta E})\mathcal{C} + g(1-\mathcal{C})\right] + \sin^2\varphi\left(g e^{-\beta\Delta E}\mathcal{C} + (1-g)(1-\mathcal{C})\right)}$$

with proba: the denominator

$$= \frac{\sin^2\varphi\left[(1-g e^{-\beta\Delta E})\mathcal{C} + g(1-\mathcal{C})\right]}{\sin^2\varphi\left[(1-g e^{-\beta\Delta E})\mathcal{C} + g(1-\mathcal{C})\right] + \cos^2\varphi\left(g e^{-\beta\Delta E}\mathcal{C} + (1-g)(1-\mathcal{C})\right)}$$

with proba the denominator.

$L_{\alpha_0\alpha_0} = -\sum_{\alpha} L_{\alpha\alpha_0} < 0$ if α_0 is not absorbing!

Thus after a standard Von Neumann measurement of the pointer state basis, in one time step $1 - \mathrm{Tr}\, \rho\rho'$ scales as δ.

Show images Show the experiment

Example: Hamiltonian evolution (two states)

$$U = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \quad B_+ = \begin{pmatrix} \cos\varphi & 0 \\ 0 & \sin\varphi \end{pmatrix} \quad B_- = \begin{pmatrix} \sin\varphi & 0 \\ 0 & \cos\varphi \end{pmatrix}$$

Again take $\cos^2\varphi = \frac{1}{2} + h\,\delta^{1/2}$ and take $\alpha = \omega\,\delta$

δ small to get the continuous time limit

The naive strong measurement regime with fixed ω and h large exhibit zeno freezing.

Talk about Zeno

Show images of Zeno freezing

Exercice: "Show" that Hamiltonian evolution and measurement with accurate reading preserve purity

$$|\psi\rangle\langle\psi| \to e^{-itH}\,|\psi\rangle\langle\psi|\,e^{itH}$$

$$|\psi\rangle\langle\psi| \to \propto\ A_a|\psi\rangle\langle\psi|A_a^\dagger$$

Exercice on the zeno effect: assume $\rho(n)$ is pure state and evolution is unitary $U = e^{-i\delta H}$

compute the overlap with $\rho(n+1)$ to lowest order in δ

$$\mathrm{Tr}\,\rho\rho' = \mathrm{Tr}\,|\psi\rangle\langle\psi| e^{-i\delta H}|\psi\rangle\langle\psi| e^{i\delta H}$$
$$= |\langle\psi|e^{-i\delta H}|\psi\rangle|^2$$

$$\langle\psi|e^{-i\delta H}|\psi\rangle = 1 - i\delta \underbrace{\langle\psi|H\psi\rangle}_{\text{real}} - \frac{\delta^2}{2}\underbrace{\langle\psi|H^2|\psi\rangle}_{\text{real}}$$

$$|\langle\psi|e^{-i\delta H}|\psi\rangle|^2 = 1 - \delta^2\left(\langle\psi|H^2|\psi\rangle - \langle\psi|H|\psi\rangle^2\right) + \cdots$$

Order δ^2 ! Until $H \sim \delta^{-1/2}$

This is the Zeno effect for standard Von-Neumann measurements: such measurement in a basis where the Hamiltonian is non diagonal freeze the evolution if they are repeated to often: the probability to measure at time δ in another state than the one prepared at time 0 is not of order δ but δ^2!

In our problem, the time scale for measurements is $\sim h^{-2}$ $\quad H = \omega\sigma_y = \omega\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ so to fight the Zeno effect we should take $\omega = h\gamma$ $\quad h$ large and γ^{-2} is the time scale that remains

Show images

Zeno's paradoxes

The arrow and the target

Achille and the tortoise

Once properly normalized, the two problem (thermal equilibration, Rabi oscillation) exhibit common features (that are present also in more complicated systems)

Here is what is "really" understood:

- In the strong measurement regime of QND, there is a limiting random evolution of the diagonal of the density matrix which is described by a continuous time finite state Markov process:
  * the system jumps from one pointer state to another
  * the corresponding Markov generator can be computed explicitely
  * in the particular case when the system is described by a Markov chain $M = e^{tL}$ the generator is exactly L (thus QND measurements allow to interpolate between individual trajectories and the master equation) and in particular the

trajectories are ergodic

- There are some decoration (or spikes) on top of the Markov process. These spikes are mostly understood. They have universal features and exhibit scale invariance. They are described by a simple Poisson point process:

  We throw points in the simplex in a appropriate way (in fact along edges) and connect them to the appropriate vertex.

Conclusions: QND measurement are a very versatile tool, used daily in labs. They allow to challenge the most fundamental and intriguing features of measurement in quantum mechanics. They also allow to observe quantum jumps as anticipated 100 years ago by Bohr (though in reality these quantum jumps are measurement apparatus dependent). They even allow to make real some famous gedanken experiments of the father of quantum mechanics

Quantum noises

* Up to now, we have always measured the environment, i.e we have always concentrated on $\mathcal{H}_s$. But we could keep track of $\mathcal{H}_e$.

The study of quantum noises is, in very short the study of an unitary map U on a Hilbert space $\mathcal{H}_s \otimes \mathcal{H}_e$ with a density matrix ρ. In fact U depends on a (discrete or continuous) time parameter (ie $U(n)$ or $U(t)$) and one studies how $\mathcal{H}_s$ "dissolves" into $\mathcal{H}_e$ as time goes by so that partial traces lead to interesting phenomena: decoherence, dissipation, ...

* In the framework developped in these lectures, a typical setting is the following.

$\mathcal{H}_e = \mathcal{K} \otimes \mathcal{K} \otimes \cdots \otimes \mathcal{K} \, (\otimes \cdots)$ and

$U(n) = U_n \cdots U_2 U_1$ where each U_m acts on $\mathcal{H}_s \otimes$ the m^{th} copy of $\mathcal{K}$, acting on the other copy as the identity map. The most natural setting is when time is homogeneous, which means that each U_m is a clone of a given $U : \mathcal{H}_s \otimes \mathcal{K} \circlearrowleft$, acting on the appropriate copy.

* Why quantum noises? The relationship between noises and quantum noises is the same as the relationship between classical probability and probability in quantum mechanics: $U(n)\rho U(n)^{+}$ allows to compute expectations and probabilities using traces on Hilbert spaces instead of integrals on classical spaces (so to say).

* A simple analogy:

Let $a(t)$ $t \in [0,+\infty[$ be a family of bosonic independent annihilation operators

$$[a(s), a^+(t)] = \delta(t-s) \qquad [a(s), a(t)] = [a^+(s), a^+(t)] = 0$$

Consider $A_t := \int_0^t a(s)\, ds$ and $W_t := A_t + A_t^+$.

Let $|\Omega\rangle$ be the vacuum for the $a(s)$. One checks that $\langle \Omega | W_{t_1} \cdots W_{t_n} | \Omega \rangle = \mathbb{E}(B_{t_1} \cdots B_{t_n})$ (*)

where B_t is a standard Brownian motion.

2 Let us be a bit careful: because of the singular nature of the commutator, we should write

$A_t = \int_{[0,t[} a(s)\, ds$ so that for

$t_0 = 0 < t_1 < t_2 \cdots < t_n$ the increments $A(t_i) - A(t_{i-1})$ commute with the increments $A^+(t_j) - A^+(t_{j-1})$ $j \neq i$

The reason for (*) is that Gaussian integrals and oscillator expectation values are both regulated by the same Wick theorem for contractions.

Note that $a(t)$ plays for A the same role as white noise for Brownian motion, an object much subtler to define correctly. In fact the A_t can be constructed directly exactly like Brownian motion:

Exercise: let $(f_n)_{n\geq 1}$ be an orthogonal basis of $\mathbb{L}^2([0,+\infty[)$ and $(a_n)_{n\geq 1}$ a sequence of bosonic anihilation operators, acting on a Fock space $\mathcal{F}$ with vacuum $|\Omega\rangle$. Finally, let $(\varphi_n)_{n\geq 1}$ be a family of independent standard Gaussians (mean 0, variance 1)

Show that $\widetilde{A}_t := \int_0^t ds \sum_n a_n f_n(s)$ (and the corresponding $\widetilde{A}_t^+$) have the same commutation relations as the $\int_{[0,t[} a(s) ds$ and the corresponding adjoints.

Show that if f_n made of real functions $\widetilde{B}_t := \int_0^t ds \sum_n \varphi_n f_n(s)$ is a standard Brownian motion.

Remark: with the operator, we could replace the trivial density matrix $|\Omega\rangle\langle\Omega|$ with other ones, like thermal for instance, to get a theory whose relationship with classical Brownian motion is less clear.

Exercice: show how to give a meaning to thermal averages of $\langle W_{t_1} \cdots W_{t_n} \rangle_\beta$ at temperature β.

Can you relate this to Brownian motion?

(use $\omega \sum_n a_n^+ a_n$ as energy operator).

The lesson to be learned from this simple example is that many things like random processes, stochastic calculus, etc can be defined in the quantum setting. There is probably still a lot to work out in this domain.

Open quantum counting processes as an example

* A (naive) idea

Markov chain with transition matrix $M_{\alpha\beta}$

$\alpha \in I$ which is a stochastic matrix

Take $\mathcal{H}_c$ Hilbert space with basis

$|r, \alpha\rangle$ $\quad r \in \mathbb{Z}^I$ and define

$$T\, |r, \alpha\rangle = \sum_\beta M_{\alpha\beta}\, |r + \underset{\text{canonical basis}}{e_\beta}, \beta\rangle$$

This is just a trick to keep track of occupation times

A naive idea to quantize is to replace the stochastic matrix $M_{\alpha\beta}$ by a unitary matrix U to go from classical randomness to quantum randomness

Thus

$$T\, |r, \alpha\rangle = \sum_\beta U_{\alpha\mu}\, |r + e_\beta, \beta\rangle$$

T is unitary, for instance

$$\left(T|r,\alpha\rangle \,|\, T|s,\beta\rangle\right) = \sum_{\mu\nu} \overline{U_{\alpha\mu}}\, U_{\beta\nu} \underbrace{\langle r+e_\mu, \mu \,|\, s+e_\nu, \nu\rangle}_{\delta_{\mu\nu}\,\delta_{rs}}$$

$$= \delta_{rs}\,\delta_{\alpha\beta}$$

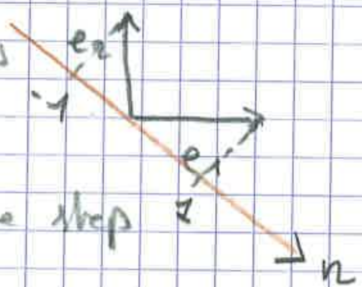
donc $T^\dagger T = \mathbb{1}_{\mathcal{H}}$

However the properties of T in the quantum case a very different than in the classical case: there is no diffusion.

Example : $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ $\quad \alpha = \pm$

compared to $M = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

As the process "conserves" the sum of the components of $r \in \mathbb{Z}^2$, we can change basis



"conserves": grows by 1 at each time step

$$T\,|n,+\rangle = \frac{1}{\sqrt{2}}\left(|n+1,+\rangle + |n-1,-\rangle\right)$$

$$T\,|n,-\rangle = \frac{1}{\sqrt{2}}\left(-|n+1,+\rangle + |n-1,-\rangle\right)$$

Take as initial condition

$$\cos\theta\,|0,+\rangle + \sin\theta\,|0,-\rangle = |\theta\rangle$$

Exercice : compute (numerically ?)

$|\langle m,+|\,T^n\,|\theta\rangle|^2 + |\langle m,-|\,T^n\,|\theta\rangle|^2$ ie the "spatial structure of the wave function" at time n for $|m| \leq n$ for various theta's, or average over θ. Show picture

Compare with the Markov chain case with initial condition $\frac{1}{2}\left(|0+\rangle + |0-\rangle\right)$

* Open quantum counting processes

Using notation from the first lecture, suppose U on $\mathcal{H}_S \otimes \mathcal{K}$ and take state $|\psi\rangle$ in $\mathcal{K}$, so that only the A_a matter:

$U|\varphi\rangle \otimes |\psi\rangle = \sum_a A_a|\varphi\rangle \otimes |a\rangle$ where unitarity imposes $\sum_a A_a^\dagger A_a = \mathbb{1}_{\mathcal{H}_S}$ and define $U_1, U_2, \ldots$ by acting on the m^{th} copy of $\mathcal{K}$ in $\mathcal{H}_S \otimes \mathcal{K} \otimes \mathcal{K} \otimes \cdots$

$U(n) = U_n \cdots U_1$.

As we have seen for measurements, it is important to keep track of the counting of results so we enlarge $\mathcal{H}_S$: we call $\mathcal{H}_g$ the old $\mathcal{H}_S$ and we take $\mathcal{H}_S := \mathcal{H}_g \otimes \mathcal{H}_c$

we view $\mathcal{H}_g$ as an internal state and $\mathcal{H}_c$ as a position, a counting process. We use a little trick:

Suppose A_a on $\mathcal{H}$ satisfies $\sum A_a^\dagger A_a = \mathbb{1}_{\mathcal{H}}$ and take U_a unitary on $\tilde{\mathcal{H}}$ then $\tilde{A}_a := A_a \otimes U_a$ on $\mathcal{H} \otimes \tilde{\mathcal{H}}$ satisfies $\sum \tilde{A}_a^\dagger \tilde{A}_a = \mathbb{1}_{\mathcal{H} \otimes \tilde{\mathcal{H}}}$.

So from the A_a's we may construct a unitary map $U : \mathcal{H}_g \otimes \mathcal{H}_c \otimes \mathcal{K}$ as in lecture I; $\mathcal{H}_c$ has a basis $|n\rangle$ $n \in \mathbb{Z}^I$ ($I = \{a\}$, the set of possible outcomes) and

$$U \, |\varphi\rangle \otimes |n\rangle \otimes |\psi\rangle = \sum_a A_a|\varphi\rangle \otimes |n + e_a\rangle \otimes |a\rangle$$

So if $|\varphi\rangle \otimes |0\rangle$ is the initial state in $\mathcal{H}_s$

so $|\Phi\rangle |\varphi\rangle \otimes |0\rangle \otimes |\psi\rangle \otimes |\psi\rangle \otimes \cdots$ is the initial state in $\mathcal{H}$ we get

$$U(1)\,|\Phi\rangle = \sum_{a_1} A_{a_1}|\varphi\rangle \otimes |e_{a_1}\rangle \otimes |a_1, \psi, \psi \cdots\rangle$$

$$U(2)\,|\Phi\rangle = \sum_{a_1, a_2} A_{a_2}A_{a_1}|\varphi\rangle \otimes |\underbrace{e_{a_1} + e_{a_2}}_{N_2(a)}\rangle \otimes |a_1, a_2, \psi, \psi \cdots\rangle$$

and so on.

Then environment keep track of all precise outcomes, while $\mathcal{H}_c$ just keep a collective variable, so it is fair to see it as part of the system.

This could be called the "open quantum counting process". We could make measurements projecting on the a's and recover our previous discussion. But we have the freedom to make other measurements, i.e. to project on a basis of states rotated with respect to the basis $|a\rangle$.

In the model, there is diffusion for the counting process. This is an appropriate point to close the lectures!