Lecture 1: Majorana fermions in condensed matter

Sunday, February 8, 2015 2:55 PM

Title: Anyonic Defects: A New Paradigm for Non-Abelian Statistics

Plan for lectures:

1. Majorana fermions in condensed matter physics
2. Fractionalized Majoranas and parafermionic algebra on FQH edges
3. Generalized discussion of defects. FQH bilayers. Ising defects

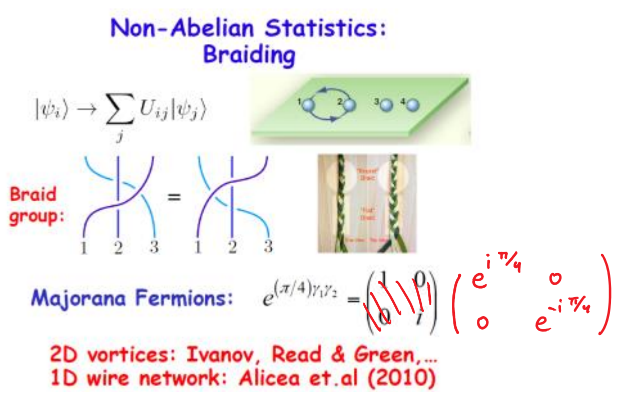
Background

In the preceding lectures, you saw how to describe *topologically ordered states of matter*. These are T=0 phases of interacting particles (in most of these lectures we have focused on two spatial dimensions, although higher dimensional examples are also possible), which are gapped in their bulk. Nevertheless, there is a non-local (topological) order in the ground state wavefunction; the system cannot be adiabatically connected to a different, non-topoligically ordered phase without crossing a phase transition.

Defining properties of topological order:

- Gap in the bulk, not adiabatically continuable to a "trivial" (site-factorizable) phase
- Topological degeneracy on high-genus surfaces: the g.s. wavefunction "feels" the topology of the manifold. This degeneracy is topological, in the sense that it cannot be lifted by any local term: the degenerate states are distinct by a "non-local" (string) operator. Only way to lift the degeneracy is through a phase transition, in which the bulk gap closes.
- The phase supports fractionalized "particle-like" excitations which carry fractional statistics. These are often associated also with fractionalized quantum numbers (as the e/3 charges in the Laughlin FQ state). The statistics of the excitations can be non-Abelian (more on this later). The presence of these excitations is directly related to the degeneracy of the g.s. on high-genus surfaces.
- In some (but not all) cases: the topological order may manifest itself in the presence of gapless edge states in a system with open boundary conditions. These gapless edge states may be described by a CFT. E.g. quantum Hall phases discussed in other lecture. NOT in phases such as Kitaev's toric code (the Z2 gauge theory).

Non-Abelian statistics arises if the topological degeneracy of the ground state *increases exponentially* when excitations are introduced. In this situation, exchanging two excitations ("anyons") corresponds not to a phase factor, but a *matrix* acting on the ground state Hilbert space.



$\begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4}\end{pmatrix}$

There is clearly a connection between the topological degeneracy that appears on a non-zero genus surface and the one that appears when non-Abelian excitations are introduced. In these lectures, I will explore this connection. I will show that introducing defects into these systems mimics the effect of a high-genus surface. As a result, in the presence of these defects, the ground state degeneracy increases. Moreover, one can define a "braiding" operation on defects; they give rise to a new kind of non-Abelian statistics, distinct from that of anyons. Defects in Abelian systems behave as non-Abelian objects; introducing defects in a non-Abelian system enriches its properties, providing additional "quantum gates" that can be implemented in a topologically protected manner.

Majorana zero modes in condesed matter physics

Majorana zero modes are at the heart of the simplest kind of non-Abelian statistics - that of Ising anyons - and also the simplest example of a non-Ablian defect. They are also the kind which is probably closest to experimental realization. I will spend most of this lecture describing them.

Majorana zero modes are closely tied with superconductivity. They can appear at cores of vortices in p-wave superconductors, and at edges of superconducting wires. They also appear as quasi-particles of the Moore-Read $\nu = 5/2$ state; this state turns out to be just a paired (superconducting) state of composite fermions. Let me start by briefly remind you some basic facts about the spectrum of a superconductor.

Bugoliubov - de Gennes (BdG) Formalism

For concreteness, consider a model of spinless fermions on a d-dimensional lattice. $\boxed{\{\psi_i, \psi_j\} = 0, \quad \{\psi_i^\dagger, \psi_j\} = \delta_{ij}}$

If there are attractive interactions of the form (schematically) $-\psi_i^\dagger \psi_j^\dagger \psi_k \psi_\ell$, the system may become unstable to a "condensate" where the fermion pair operator $O_{ij} = \psi_i \psi_j$ becomes long-range ordered: $\langle O_{ij} \, O_{k\ell}^\dagger \rangle \neq 0$ when k, ℓ are far from i, j.

Mean field approximation (BCS)

$$\langle O_{ij} \rangle = \langle \psi_i \psi_j \rangle \neq 0$$

mean-field decoupling $\langle \psi_i^\dagger \psi_j^\dagger \psi_k \psi_\ell \rangle \to \langle \psi_i^\dagger \psi_j^\dagger \rangle \psi_k \psi_\ell + \langle \psi_k \psi_\ell \rangle \psi_k^\dagger \psi_\ell^\dagger$

$\Rightarrow$ "Anomalous" terms that do not consrve the number appear.

BCS Mean-field Hamiltonian:

$$H = \sum_{i,j} \overbrace{-t_{ij}\, \psi_i^\dagger \psi_j}^{\text{hopping}} \; \overbrace{- \mu \sum_i \psi_i^\dagger \psi_i}^{\text{chemical potential}} + \overbrace{\sum_{i,j} \Delta_{ij}\, c_i^\dagger c_j^\dagger}^{\text{pairing potential}} + h.c.$$

$$= (\psi_1^\dagger \dots \psi_N^\dagger \quad \psi_1 \dots \psi_N) \begin{pmatrix} h_{ij} & \Delta_{ij} \\ & {}^* \end{pmatrix} \underbrace{\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \\ \psi^\dagger \end{pmatrix}}_{\Psi}$$

$$= \underbrace{(\psi_1^\dagger \ldots \psi_N^\dagger \quad \psi_1 \ldots \psi_N)}_{\Psi^\dagger} \begin{pmatrix} h_{ij} & \Delta_{ij} \\ \underbrace{(\Delta^\dagger)_{ij} \quad\quad -h^*_{ij}}_{\mathcal{H}:\ 2N\times 2N} \end{pmatrix} \left.\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \\ \psi_1^\dagger \\ \vdots \\ \psi_N^\dagger \end{pmatrix}\right\} \Psi$$

Nambu Spinor (Here $h_{ij} = \frac{1}{2}(-t_{ij} - \mu\,\delta_{ij})$)

Quadratic in ψ's $\rightarrow$ can be solved!

Bugoliubov transformation:

$$\begin{pmatrix} \psi_1 \\ | \\ \psi_N \\ \psi_1^\dagger \\ | \\ \psi_N^\dagger \end{pmatrix} = \overset{\text{Unitary}}{U} \left.\begin{pmatrix} f_1 \\ | \\ f_N \\ f_1^\dagger \\ | \\ f_N^\dagger \end{pmatrix}\right\} F$$

$$\boxed{\text{Check: } \{f_i^\dagger, f_j\} = \delta_{ij}}$$

$$H = \Psi^\dagger \mathcal{H}\, \Psi = F^\dagger(\underbrace{U^\dagger \mathcal{H} U})F$$

Choose U to diagonalize $\mathcal{H}$:

$$U^\dagger \mathcal{H} U = \begin{pmatrix} E_1 & & \\ & \ddots & \\ & & E_{2N} \end{pmatrix}$$

Symmetry of the BdG Hamiltonian:

$$C^{-1}\mathcal{H}\, C = -\mathcal{H}, \quad \text{where} \quad C = \begin{pmatrix} 0 & \mathbb{1}_{N\times N} \\ \mathbb{1}_{N\times N} & 0 \end{pmatrix} \underbrace{K}_{\text{Complex conjugation}}$$

"Particle - hole" symmetry

(You can prove this by direct substitution

$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} K \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^* \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} K = K \begin{pmatrix} -h^* & \Delta^\dagger \\ \Delta & h \end{pmatrix} K$$

$$= \begin{pmatrix} -h & \Delta^T \\ \Delta^* & h^* \end{pmatrix}$$

You can check that $\Delta^T = -\Delta$

$$\Delta^* = (\Delta^T)^\dagger = -\Delta^\dagger$$

since if Δ is symmetric, $\Delta_{ij}\, c_i^\dagger c_j^\dagger = 0$

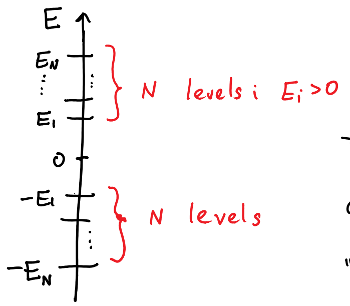
$$\Rightarrow \ldots = -\begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^* \end{pmatrix}.)$$

Note also that $\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \\ \psi_1^+ \\ \vdots \\ \psi_N^+ \end{pmatrix} = C \begin{pmatrix} \psi_1^+ \\ \vdots \\ \psi_N^+ \\ \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} \equiv C\, \Psi^*.$

This implies that the levels of $\mathcal{H}$ come in pairs of opposite sign: if $\mathcal{H}u = Eu$, then

$$\mathcal{H}(Cu) = C(\underbrace{C^{-1}\mathcal{H}C}_{-\mathcal{H}})u$$

$$= -E\, Cu$$



The levels at E_i and $-E_i$ are not independent: they are "two halves of the same state".

After the transformation: $\mathcal{H} = F^+ \begin{pmatrix} E_1 & & & & & \\ & \ddots & & & & \\ & & E_N & & & \\ & & & -E_1 & & \\ & & & & \ddots & \\ & & & & & -E_N \end{pmatrix} F$

$$= \sum_{i=1}^{N} E_i \left(f_i^+ f_i - f_i f_i^+ \right)$$

$$= \sum_{i=1}^{N} E_i \left(2 f_i^+ f_i - 1 \right)$$

Many-body ground state: $f_i^+ f_i |0\rangle = 0 \quad \forall\ i = 1 \dots N$

(Since $E_i > 0$)

Excited states: $\prod_{j=1}^{n} f_{\alpha_1}^+ \dots f_{\alpha_n}^+ |0\rangle$

Energy (rel. to ground state)

$2(E_{\alpha_1} + \dots + E_{\alpha_n})$

Example: p + ip superconductor in d = 2

Take nearest-neighbor hopping: $\sum_{ij} -t_{ij}\, c_i^+ c_j$

$$= -t \sum_{\underbrace{\langle ij \rangle}_{\text{n.n. hopping}}} c_i^+ c_j + \text{h.c.}$$

In momentum space

$$\psi_{\vec{k}} = \frac{1}{\sqrt{N}} \sum_i e^{i \vec{k} \cdot \vec{r}_j}\, \psi_j$$

$$h = \sum_k \Psi_k^\dagger \left(-2t\,(\cos k_x + \cos k_y) - \mu\right) \Psi_k$$

$$\xrightarrow[|\mu+4t| \ll t]{} \quad \sum_k \Psi_k^\dagger \left(\frac{\vec{k}^2}{2m} - \tilde{\mu}\right) \Psi_k + \text{const.}$$

$$\frac{1}{m} = 2t \qquad \tilde{\mu} = \mu - 4t$$

(From now on replace $\tilde{\mu} \to \mu$)

Nearest - neighbor pairing : $-i\Delta \sum_i \left(c_i^\dagger c_{i+\hat{x}}^\dagger + i\, c_i^\dagger c_{i+\hat{y}}^\dagger \right) + h.c.$

$$= \Delta \sum_k \underbrace{(\sin k_x + i \sin k_y)}_{\approx k_x + i k_y \text{ for } |\vec{k}| \ll 2\pi} c_k^\dagger c_{-k}^\dagger + h.c.$$

$$\left(\sum_i c_i^\dagger c_{i+1}^\dagger = \frac{1}{N} \sum_{kk'} \sum_i e^{-i k x_i - i k'(x_i+1)} c_k^\dagger c_{k'}^\dagger \right.$$

$$= \sum_k e^{+ik} c_k^\dagger c_{-k}^\dagger = \frac{1}{2} \sum_k (e^{ik} - e^{-ik})\, c_k^\dagger c_{-k}^\dagger$$

$$\left. = i \sum_k \sin k \; c_k^\dagger c_{-k}^\dagger \right)$$

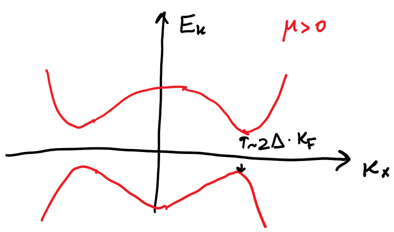
The Hamiltonian in momentum space has the form

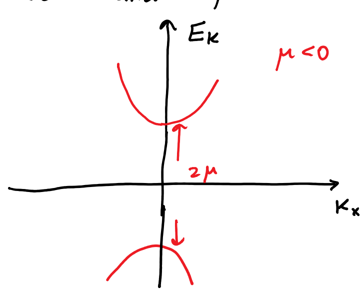
$$H = \frac{1}{2} \sum_k (\Psi_k \;\; \Psi_{-k}^\dagger)\; h_k \begin{pmatrix} \Psi_k \\ \Psi_{-k}^\dagger \end{pmatrix}$$

$$h_k = \begin{pmatrix} \frac{k^2}{2m} - \mu & \Delta(k_x + i k_y) \\ \Delta(k_x - i k_y) & -\left(\frac{k^2}{2m} - \mu\right) \end{pmatrix}$$

Diagonalize : $E_{k,\pm} = \pm \sqrt{\left(\frac{k^2}{2m} - \mu\right)^2 + \Delta^2 |k_x + i k_y|^2}$

Distinguish two cases: $\mu < 0$ and $\mu > 0$





$k_F = \sqrt{2m\mu}$

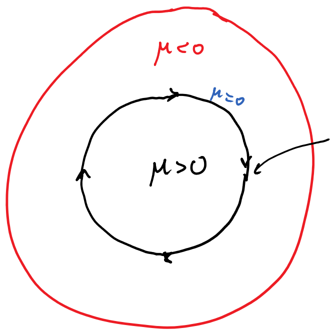
"weak pairing" $\longleftrightarrow$ "strong pairing" ($\Leftrightarrow$ vacuum)

Gap closes
on the way
at $\vec{k} = 0$

These phases are topologically distinct: have a different Chern number.

We will consider the edge of the system.

Edge states consider a system with a disc geometry, where μ changes continuously:



"Chiral" edge states: moving in one direction

$x = r\cos\theta \qquad dx = dr\cos\theta - r\sin\theta\, d\theta$

$y = r\sin\theta \qquad dy = dr\sin\theta + r\cos\theta\, d\theta$

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$

$$\begin{pmatrix}\partial_r \\ \frac{1}{r}\partial_\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix}$$

$$\partial_x = \cos\theta\,\partial_r - \frac{1}{r}\sin\theta\,\partial_\theta$$

$$\partial_y = \sin\theta\,\partial_r + \frac{1}{r}\cos\theta\,\partial_\theta$$

$$\partial_x + i\partial_y = (\cos\theta + i\sin\theta)\,\partial_r + \frac{1}{r}(-\sin\theta + i\cos\theta)\,\partial_\theta$$

$$= e^{i\theta}\partial_r + \frac{ie^{i\theta}}{r}\partial_\theta$$

$$-\partial_x + i\partial_y = (-\cos\theta + i\sin\theta)\,\partial_r + \frac{1}{r}(\overbrace{+\sin\theta + i\cos\theta}^{i(\cos\theta - i\sin\theta)})\,\partial_\theta$$

$$= -e^{-i\theta}\partial_r + \frac{ie^{-i\theta}}{r}\partial_\theta$$

Assume very slow spatial variation, keep only constants and $\partial_{x,y}$ (neglect ∇^2)

BdG equation:

$$\begin{pmatrix} -\mu(r) & \Delta e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta\right) \\ \end{pmatrix}\begin{pmatrix} u(\vec r) \\ \end{pmatrix} = E\begin{pmatrix} u(\vec r) \\ \end{pmatrix}$$

$$\begin{pmatrix} -\mu(r) & \Delta e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta\right) \\ \Delta e^{-i\theta}\left(-\partial_r + \frac{i}{r}\partial_\theta\right) & +\mu(r) \end{pmatrix} \begin{pmatrix} u(\vec r) \\ v(\vec r) \end{pmatrix} = E \begin{pmatrix} u(\vec r) \\ v(\vec r) \end{pmatrix}$$

$$\left.\begin{aligned} u &= e^{i(m+1)\theta} \frac{f(r)}{\sqrt{r}} \\ v &= e^{im\theta} \frac{g(r)}{\sqrt{r}} \end{aligned}\right\} \text{Ansatz} \qquad \partial_r\left(r^{-1/2} f\right) = -\frac{1}{2} r^{-3/2} f + r^{-1/2}\partial_r f = r^{-1/2}\left(\partial_r - \frac{1}{2r}\right) f$$

$$\begin{pmatrix} -\mu(r) & \Delta\left(\partial_r - \frac{m+\frac{1}{2}}{r}\right) \\ \Delta\left(-\partial_r - \frac{m+\frac{1}{2}}{r}\right) & +\mu(r) \end{pmatrix} \begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = E \begin{pmatrix} f(r) \\ g(r) \end{pmatrix}$$

assume $R \gg \frac{\Delta}{\mu}$ $\Rightarrow$ replace $r \to R$

For simplicity take $\mu(r) = \begin{cases} +\mu_0 & r < R \\ -\mu_0 & r > R \end{cases}$

$(\mu_0 > 0)$

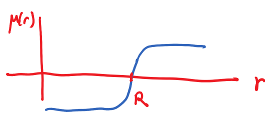
We will first find solutions by setting $\Delta \frac{m+\frac{1}{2}}{R} \to 0$,

and then compute its effects perturbatively.

zeroth order in $\frac{1}{R}$:

$$\left(-\mu(r)\,\sigma^z + \Delta\, i\partial_r\, \sigma^y\right) \begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = E \begin{pmatrix} f(r) \\ g(r) \end{pmatrix}$$

Dirac equation with "kink" in mass term $\mu(r)$



Jackiw - Rebbi (70's): zero energy solution localized near $r = R$!

To find the zero energy solution, write

$$\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\text{const.}} e^{-\alpha_\pm (r-R)}$$ with $\alpha_\pm$ for $r > R$, $r < R$ respectively.

$\Downarrow$ (ignore $\frac{m+1/2}{r}$ to 1st approx.)

$$r < R: \quad \begin{pmatrix} -\mu_0 & -\alpha\Delta \\ \alpha\Delta & +\mu_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

solution: $\alpha_\pm \Delta = \pm \mu_0$

for $\Delta > 0$, pick $\alpha_- < 0$ solution so $f(r \ll R) \to 0$:

$$\begin{pmatrix} -\mu_0 & \mu_0 \\ -\mu_0 & +\mu_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$a = +b$

solution: $$\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = \begin{pmatrix} 1 \\ +1 \end{pmatrix} e^{+\frac{\mu_0}{\Delta}(r-R)} \equiv |\psi_0\rangle$$

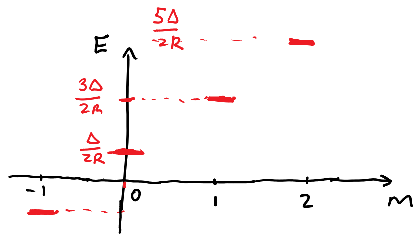
Similarly for $r > R$, pick $\alpha_+ = + \frac{\mu_0}{R}$

$$\begin{pmatrix} +\mu_0 & -\mu_0 \\ \mu_0 & -\mu_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$a = b$

Now do 1st order perturbation theory in $\Delta \frac{m+\frac{1}{2}}{R}$:

$$E_n \approx \frac{\langle \psi_0 | \Delta \frac{m+\frac{1}{2}}{R} \sigma^x | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} = \Delta \frac{m+\frac{1}{2}}{R} \qquad m \in \mathbb{Z}$$



No $E = 0$ solution!

However, is a vortex in the order parameter, all the levels shift by $\frac{1}{2} \frac{\Delta}{R}$ and there is a zero energy solution.

(A "vortex" means that the phase of the order parameter twists slowly in space, and changes by 2π when we go around a certain point, the vortex core).

in H_{BdG}, change

$$\int dr \; \psi^\dagger(\vec{r}) \, \Delta (\partial_x + i\partial_y) \, \psi^\dagger(\vec{r}) \to \int dr \, \psi^\dagger(r) \, e^{i\frac{\varphi(r)}{2}} \Delta (\partial_x + i\partial_y) \, e^{i\frac{\varphi(r)}{2}} \psi^\dagger(r)$$

$+ \text{h.c.} \qquad\qquad + \text{h.c.}$

$$\boxed{\varphi(r,\theta) = -\theta \quad \text{"anti-vortex"}}$$

Subtlety: when we write the pairing term

$$\int dr \, dr' \; \psi^\dagger(r) \underbrace{e^{i\frac{\varphi(r)}{2}} \Delta \, \partial_{x'} \delta(r'-r) \, e^{i\varphi(r')}}_{} \psi(r')$$

$\Delta(r,r')$ has to be anti-symmetric!

$$\boxed{\varphi(r,\theta) = -\theta \quad \text{"anti-vortex"}}$$

$\Delta(r,r')$ has to be anti-symmetric !

Inserting this into the BdG equation:

$$\begin{pmatrix} \mu(r) & e^{i\frac{\theta}{2}}\left(\partial_r + \frac{i}{r}\partial_\theta\right)e^{-i\frac{\theta}{2}} \\ \Delta e^{-i\frac{\theta}{2}}\left(-\partial_r + \frac{i}{r}\partial_\theta\right)e^{i\frac{\theta}{2}} & -\mu(r) \end{pmatrix}\begin{pmatrix} u(\vec{r}) \\ v(\vec{r}) \end{pmatrix} = E\begin{pmatrix} u(\vec{r}) \\ v(\vec{r}) \end{pmatrix}$$

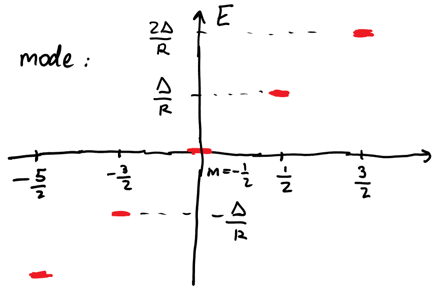
To get rid of the phase factors, write $u = e^{-i\theta/2}\tilde{u}$ $\quad v = e^{i\theta/2}\tilde{v}$. Then the Hamiltonian looks just as before, but the boundary conditions changed!

$\tilde{u}(r, \theta + 2\pi) = -\tilde{u}(r,\theta)$ (and same for $\tilde{v}$)

Spectrum

$$\boxed{E_m = \Delta\,\frac{m+\frac{1}{2}}{R}, \quad m \in \mathbb{Z} + \frac{1}{2}}$$

zero mode:



Solutions to BdG have to come in pairs

$\Rightarrow$ Another zero mode at the vortex core!

Notes:

1. If we solved the model on the lattice, the number of solutions has to be even, since the dimension of the Hamiltonian is even. Therefore, there cannot be an isolated zero mode.
2. The edge modes are chiral, i.e., they move in one direction (similar to quantum Hall edges).
3. The E = 0 state (the *zero mode*) is special: the corresponding *f* operator is Hermitian, i.e. it is equal to its own conjugate:

Recall that the eigenmodes f_i of $\mathcal{H}$ are

$f_i = u^+ \Psi, \quad f_i^+ = \Psi^+ u$

where $u = \begin{pmatrix} | \\ 2N \\ | \end{pmatrix}$ is an eigenstate of $\mathcal{H}$.

A spatially isolated eigenstate of $\mathcal{H}$, $f_i = \gamma$, satisfies is an eigenstate of $\mathcal{H}$.

A spatially isolated eigenstate of $\mathcal{H}$, $\epsilon_i = 0$, satisfies

$$u = \mathcal{C}\,u = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} u^*$$

$$\Downarrow$$

$$\gamma = u^\dagger \Psi = u^\dagger \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \\ \psi_1^\dagger \\ \vdots \\ \psi_N^\dagger \end{pmatrix}$$

$$\gamma^\dagger = (\psi_1^\dagger - \psi_N^\dagger \;\; \psi_1 - \psi_N)\, u = \underbrace{u^T \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}} \begin{pmatrix} \psi_1 \\ | \\ \psi_N \\ \psi_1^\dagger \\ | \\ \psi_N^\dagger \end{pmatrix} = \gamma$$

$$\left[\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} u \right]^T = (\mathcal{C}\, u^*)^T = (u^*)^T = u^\dagger$$

γ is a so called "Majorana zero mode."

Note
The weak pairing phase of the p+ip SC is *topologically distinct* from the trivial phase. This can be seen from the fact that it has chiral edge states, or alternatively, from the fact that it has a zero mode at a vortex core ("defect"). Such an isolated mode at E=0 cannot disappear without a phase transition in which the bulk gap closes.

Extra note on Jackiw- Rebbi:

The $E=0$ solution can be found for arbitrary $\mu(r)$ as follows:

$$\left(-\mu(r)\,\sigma^z + \Delta\, i\,\partial_r\, \sigma^y\right) \begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = 0$$

$$\Downarrow$$

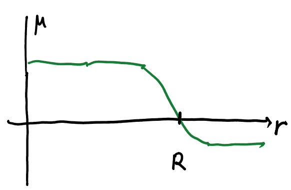
$$i\,\partial_r \begin{pmatrix} f \\ g \end{pmatrix} - i\,\sigma^x\, \frac{\mu(r)}{\Delta} \begin{pmatrix} f \\ g \end{pmatrix} = 0$$

choose $\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = \begin{pmatrix} w(r) \\ w(r) \end{pmatrix}$

$$\partial_r w = \frac{\mu(r)}{\Delta}\, w$$

$$w(r) = w(0)\, e^{+\int_0^r \frac{\mu(r)}{\Delta} dr}$$

I.e. for



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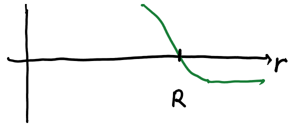
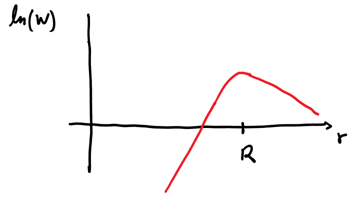


we get



Lecture 2: Majorana zero modes (cont.)

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More on the properties of Majoranas (e.g. braiding) soon. We will now change gears, and discuss Majoranas in *one dimensional* systems.
Here, we are mostly following Kitaev (2003).

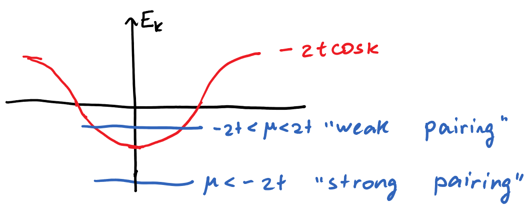
Consider the following lattice model for a 1D s.c.:

$$H = \sum_j \left(-t\,\psi_j^\dagger \psi_{j+1} - \Delta\, \psi_j^\dagger \psi_{j+1}^\dagger \right) + h.c. - \mu \sum_j \psi_j^\dagger \psi_j$$

This can be solved for a system with periodic boundary conditions, as before, by a Fourier transform.

The spectrum is given by

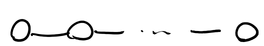
$$E_k = \pm\sqrt{(-2t\cos k - \mu)^2 + (2\Delta \sin k)^2}$$



Again, there is a gap closing (phase transition) between the weak and strong pairing phases.

What happens <u>at the edge</u> of the weak pairing phase?

Consider a finite system with N sites, at the special case <u>$\Delta = t,\ \mu = 0$</u>.



Write every fermion ψ_j as two Majorana operators, α_j, β_j:

$$\psi_j = \frac{1}{2}(\alpha_j + i\beta_j) \qquad \alpha_j = (\psi_j + \psi_j^\dagger)$$

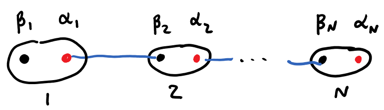
$$\psi_j^\dagger = \frac{1}{2}(\alpha_j - i\beta_j) \qquad \beta_j = \frac{1}{i}(\psi_j - \psi_j^\dagger)$$

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}$$

Insert into H:

$$H = \sum_{j=1}^{N-1} -t\,\frac{1}{4}(\alpha_j + i\beta_j)(\alpha_{j+1} - i\beta_{j+1}) - \frac{t}{4}(\alpha_j - i\beta_j)(\alpha_{j+1} - i\beta_{j+1}) + h.c.$$

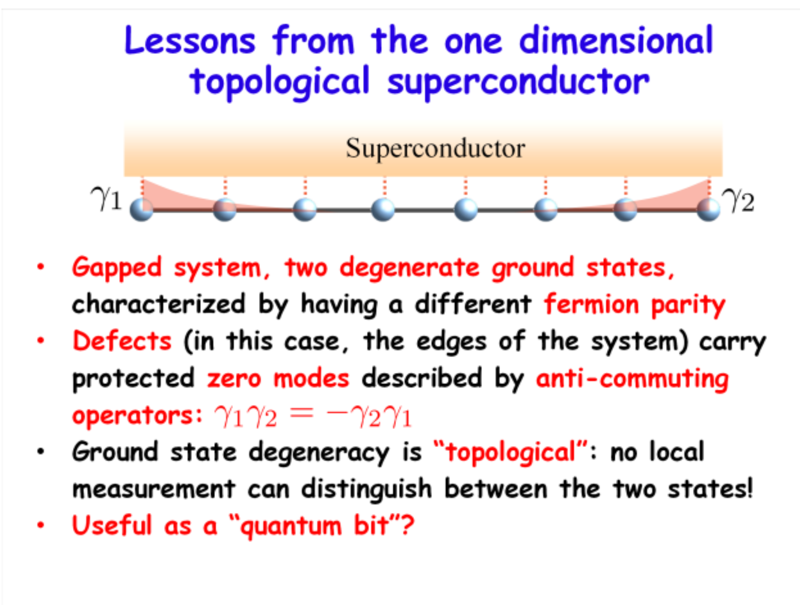
$$= \sum_{j=1}^{N} t\; i\alpha_j \beta_{j+1}$$



$[\beta_1, H] = [\alpha_N, H] = 0 \Rightarrow$ These are

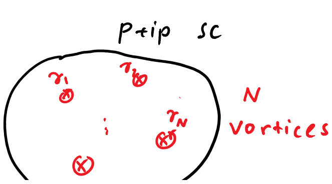
zero modes.

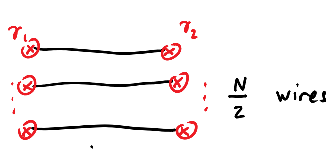
As before, the zero modes are robust

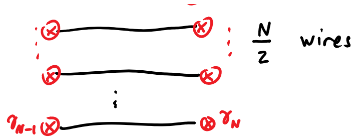


- Braiding of vortices / zero modes.
- Robustness to interactions, many-body interpretation in 1D.
- Fractionalized Majoranas.

In the presence of N Majorana zero modes, the g.s. manifold is $2^{N/2}$ fold degenerate.







Every pair of Majoranas form a 2-level system.

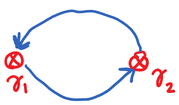
Basis : $\quad i\gamma_{2j-1}\gamma_{2j} = \pm 1$

$\Longleftrightarrow$ form a fermion:

$$f_j = \frac{1}{2}\left(\gamma_{2j-1} + i\gamma_{2j}\right)$$

and then $\quad f_j^\dagger f_j = 0, 1.)$

What happens when we exchange ("braid") two Majoranas?



(See Kareljan's talk)

How do the operators $\gamma_{1,2}$ transform under braiding?

$\gamma_1 \to \gamma_1' = U_{12}^\dagger \gamma_1 U_{12}$ $\quad U_{12}$: Unitary adiabatic evolution

$\gamma_2 \to \gamma_2' = U_{12}^\dagger \gamma_2 U_{12}$ $\quad$ operator.

We expect

$$\gamma_1' \propto \gamma_2 \quad \text{(up to a phase)}$$

$$\gamma_2' \propto \gamma_1$$

$(\gamma_1')^2 = U_{12}^\dagger \gamma_1 U_{12} U_{12}^\dagger \gamma_1 U_{12} = 1 \Rightarrow$ phases are ± 1

suppose $\quad \gamma_1' = \gamma_2$

transformation has to conserve $i\gamma_1\gamma_2$ (fermion parity of 1,2)

$\to \quad \gamma_2' = -\gamma_1$!

One can check that the transformation that does this is

$$\boxed{U_{12} = \underbrace{e^{i\phi}}_{\text{phase we can't determine from the present considerations.}} e^{\frac{\pi}{4}\gamma_1\gamma_2}}$$

check: $\quad e^{\frac{\pi}{4}\gamma_1\gamma_2} = \frac{1}{\sqrt{2}}\left(1 + \gamma_1\gamma_2\right)$

$$e^{-\frac{\pi}{4}\gamma_1\gamma_2}\,\gamma_1\,e^{\frac{\pi}{4}\gamma_1\gamma_2} = \frac{1}{2}(1-\gamma_1\gamma_2)\,\gamma_1\,(1+\gamma_1\gamma_2)$$

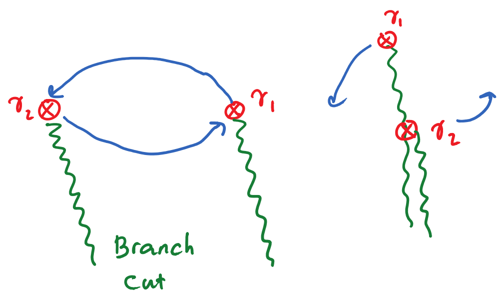
$$= \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_2 - \gamma_1)$$

$$= \gamma_2$$

$$e^{-\frac{\pi}{4}\gamma_1\gamma_2}\,\gamma_2\,e^{\frac{\pi}{4}\gamma_1\gamma_2} = \frac{1}{2}(1-\gamma_1\gamma_2)\,\gamma_2\,(1+\gamma_1\gamma_2)$$

$$= \frac{1}{2}(\gamma_2 - \gamma_1 - \gamma_1 - \gamma_2) = -\gamma_2$$

Ivanov (2001) argument: When a fermion goes around a vortex in a S.C., it gets a (−1) sign.



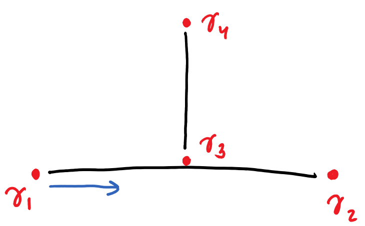
γ_2 collides with the branch cut of γ_2 once.

How to "braid" Majoranas on edges of wires?

(Alicea, Oreg, von Oppen, Refael, Fisher 10')

"exchange" γ_1, γ_2?

T - junction geometry: (Two "auxiliary" Majoranas γ_3, γ_4:)



Change position of γ_i (e.g., by applying gate potentials):

When γ_3 is close to γ_4, $H_{eff} = i\,J_{34}(r_{34})\,\gamma_3\gamma_4$.

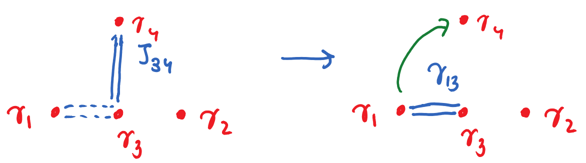
..

More generally $H_{eff} = \sum_{i,j} i J_{ij}(t)\, \gamma_i \gamma_j$

Make the J_{ij}'s time dependent!

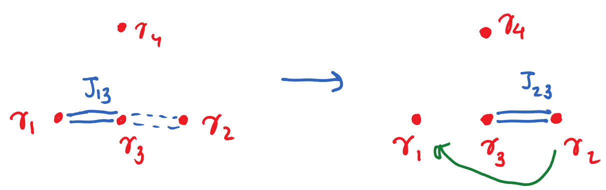
"Discrete Braiding protocol":

Step I :

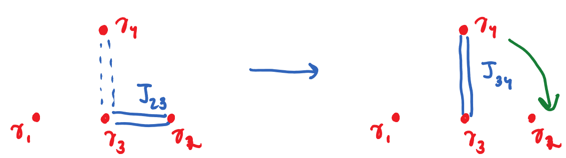


(During this step, the g.s. degeneracy does not change!)

Step II:



Step III :



H returned to itself. However, the wavefunction does not.

E.g., in step I : $H_\lambda = -J(1-\lambda)\, i\gamma_3\gamma_4 - J\lambda\, i\gamma_1\gamma_3$

(γ_2 never participates)

λ goes from 0 to 1 adiabatically

Find zero mode of H_λ : $\gamma(\lambda) = \sum_{j=1,3,4} a_j \gamma_j$

require $[\gamma(\lambda), H_\lambda] = 0$

$[\gamma_1, \gamma_1\gamma_3] = 2\gamma_3$ etc.

$\Downarrow$

$$[\gamma(\lambda), H] = a_1\left(-2\lambda J\, i\, \gamma_3\right) + a_3\left(-2(1-\lambda) J\, \gamma_4 + 2\lambda J\, \gamma_1\right)$$
$$+ a_4\left(+2(1-\lambda) J\, i\, \gamma_3\right) = 0$$

$$\begin{cases} -\lambda a_2 + (1-\lambda) a_4 = 0 \\ a_3 = 0 \end{cases}$$

$$\Downarrow$$

$\lambda = 0: \quad a_4 = 0, \quad a_2 = 1$

$\lambda = 1: \quad a_2 = 0, \quad a_4 = +1$ (you can find the sign by requiring that $a_4(\lambda)$ is continuous.)

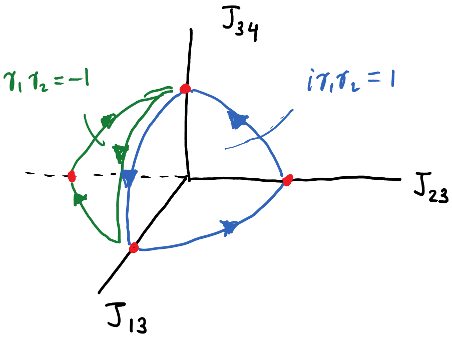
One can follow the procedure through steps II, III, and find that at the end $\begin{cases} \gamma_1 \to \pm \gamma_2 \\ \gamma_2 \to \mp \gamma_1. \end{cases}$ as anticipated.

What we have done here is drawn a closed loop in Hamiltonian space, rather than real space.

The braiding protocol can be visualized as follows.

The Hamiltonian has 3 parameters, J_{13}, J_{23}, and J_{34}.

Draw them in a 3D space and fix $J_{13}^2 + J_{23}^2 + J_{34}^2 = J^2 =$ const.



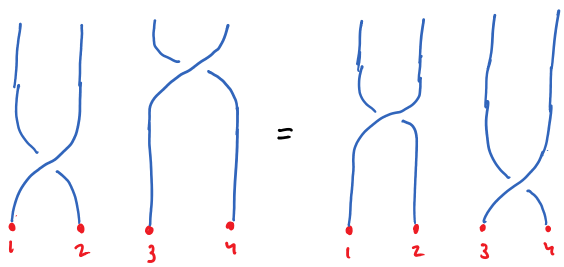
At any given intermediate time, there are only two of the three parameters which are non-zero. The trajectory covers $\frac{1}{8}$ of the area of the sphere; For a spin $\frac{1}{2}$ (2-level system), that gives a Berry phase of $\phi = \frac{1}{2} \cdot 4\pi \cdot \frac{1}{8} = \frac{\pi}{4}$. For the other eigenstate of $i\gamma_1\gamma_2$, we get the opposite phase.

Hence

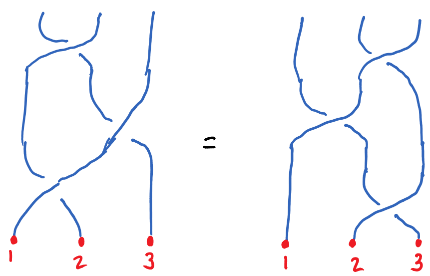
$$U_{12} = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} = e^{-\frac{\pi}{4}\gamma_1\gamma_2}$$

Notes:

- The result of the braiding protocol is topological; it does not depend on deviations from the path in Hamiltonian space, $\alpha J_{ij}(\lambda)$, as long as γ_2 remains decoupled in step 1, etc.
- The resulting braiding matrices U_{ij} form a projective representation of the braid group. By "projective" here we mean that the group relations are obeyed up to a phase.



$$U_{12}\, U_{34} = U_{34}\, U_{12}$$



$$U_{12}\, U_{23}\, U_{12} = U_{23}\, U_{12}\, U_{23} \quad (\text{Yang - Baxter equation})$$

- As mentioned earlier in this school, the Majorana (Ising anyon) braiding rules are non-universal for quantum computation purposes. This can be seen, e.g., from taking a single "q-bit" made out of four zero modes, and consider all the possible braiding operations.

To see this take e.g. $N=4$ zero modes, $\gamma_1 \dots \gamma_4$.

γ_1 • • γ_2 fix the total fermion parity, $P = \gamma_1\gamma_2\gamma_3\gamma_4$.

γ_3 • • γ_4 The g.s. is now two fold degenerate.

e.g. it is spanned by $i\gamma_1\gamma_2 = \pm 1$;
you can check that on this basis
the following representation is possible

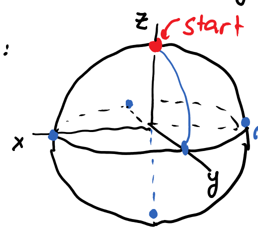
$$i\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^z$$

$$i\gamma_1\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^x$$

$$i\gamma_1\gamma_4 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma^y$$

I.e. these form a "Pauli group".
The group is preserved by the braiding
(meaning that every element is mapped to another element, but never a sum of two).
The braiding can be thought of as rotations by $\frac{\pi}{2}$
around x, y, z:

start here



you can reach these points, but nothing else.

This result generalizes to an arbitrary number of qbits.
One can show that if the braiding preserves the
a Pauli group defined on an n-qbit Hilbert space,
then it is not universal.

Can we go beyond Majorana zero modes in one spatial dimension?

Suppose we allow for arbitrary interactions (beyond mean field theory). Could we get a phase with zero modes that extend the behavior of Majorana zero modes, and obtain the missing gates for universal TQC?

The answer to this question is believed to be no. To shed light on it, it is useful to understand the topological phase in the wire and the nature of Majoranas in one dimension in a more general way.

The Majorana chain can be mapped to the transverse field Ising model:

$$\psi_j = \Big(\prod_{k<j} \sigma_k^z\Big)\sigma_j^-$$

$$\psi_j^\dagger \psi_j = \sigma_j^+ \sigma_j^- = \frac{1+\sigma_j^z}{2}$$

In terms of these variables

$$H = -t \sum_j \psi_j^\dagger \psi_{j+1} - \Delta \sum_j \psi_j^\dagger \psi_{j+1}^\dagger + h.c. - \mu \sum_j \psi_j^\dagger \psi_j$$

$$= -t \sum_j \sigma_j^+ \sigma_j^z \sigma_{j+1}^- - \Delta \sum_j \sigma_j^+ \sigma_j^z \sigma_{j+1}^+ + h.c. - \mu \sum_j \frac{1+\sigma^z}{2}$$

$$= -t \sum_j \sigma_j^+ \sigma_{j+1}^- - \Delta \sum_j \sigma_j^+ \sigma_{j+1}^+ + h.c. - \frac{\mu}{2}\sum_j \sigma_j^z + \text{const.}$$

$$= -t \sum_j \frac{\sigma_j^x + i\sigma_j^y}{2}\,\frac{\sigma_{j+1}^x - i\sigma_{j+1}^y}{2} + h.c. - \Delta \sum_j \frac{\sigma_j^x + i\sigma_j^y}{2}\,\frac{\sigma_{j+1}^x + i\sigma_{j+1}^y}{2} + h.c.$$

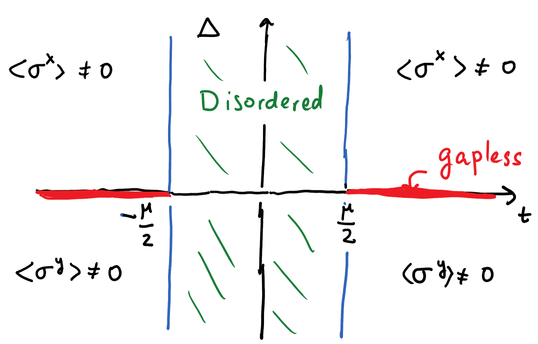
$$- \frac{\mu}{2}\sum_j \sigma_j^z$$

$$= -\frac{t}{2}\sum_j \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y\right) - \frac{\Delta}{2}\sum_j \sigma_j^x \sigma_{j+1}^x - \sigma_j^y \sigma_{j+1}^y - \frac{\mu}{2}\sum_j \sigma_j^z$$

$$= -\sum_j \left(\frac{t+\Delta}{2}\sigma_j^x \sigma_{j+1}^x + \frac{t-\Delta}{2}\sigma_j^y \sigma_{j+1}^y\right) - \frac{\mu}{2}\sum_j \sigma_j^z$$

This is the well-known XYZ model, which can have three gapped phases: disordered, ordered with $\langle\sigma^x\rangle \neq 0$, and ordered with $\langle\sigma^y\rangle \neq 0$.

Schematic phase diagram:



This can be read from the spectrum:

$$E_k = \pm\sqrt{(-2t\cos k - \mu)^2 + (2\Delta \sin k)^2}$$

by locating the regions where the gap closes.

(For $t = \pm\Delta$, this is the usual transverse field Ising model.)

Note the symmetry that protects the $\langle\sigma^x\rangle \neq 0$ and $\langle\sigma^y\rangle \neq 0$ phases:

This is

$[P, H] = 0$ with $P = \prod_j \sigma_j^z = \prod_j e^{i\pi(1-\psi_j^\dagger \psi_j)}$ = fermion parity.

A field in the x or y direction is a non- local operator in terms of the fermions.

This would still be true in the presence of arbitrary interactions between fermions.

There are general arguments (Turner, Pollmann, EB (2010); Fidkowski, Kitaev (2010); Schuch et al. (2011); Chen, Gu, Wen (2011)) that this is the end of the story in 1D; in an arbitrary interacting system of fermions, there could be only two stable gapped phases, the topological phase with Majoranas at the end and a trivial phase. If extra symmetries are present, more phases - symmetry protected topological (SPT) phases - are possible. However, as long as our system is made out of fermions, this won't give any new non-Abelian properties beyond those of Majoranas. To find something richer, our basic degrees of freedom need to be anyons, as we will see next.