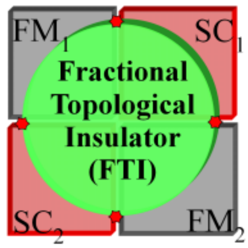
L3a

Thursday, February 12, 2015 10:09 PM

Beyond Majorana fermions

Consider the effectively 1D boundaries
of 2D a topological phase
which supports (abelian) anyons.

"Fractional topological insulator":
Laughlin Quantum Hall state with:
$\nu=1/m$ for spin up
$\nu=-1/m$ for spin down (m odd)

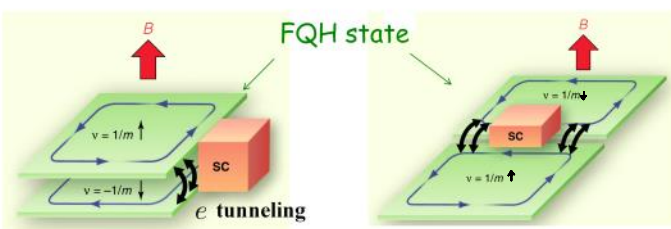


Stable phase: Levin and Stern (2010)

Majorana fermions at SC/FM interfaces: Fu and Kane (2009)

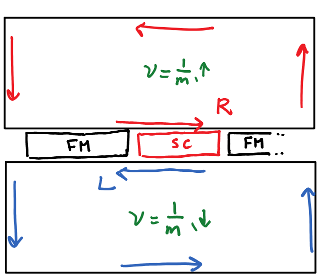
Beyond Majorana fermions

Fractional quantum Hall "realizations"
of a Fractional Topological Insulator



Lindner, EB, Stern, Refael (2013);
Clarke, Alicea, Shtengel (2013);
Cheng (2013)

To analyze this system, write down an effective low E effective model for the edge states:



(m = odd integer)

Free action:

$$S = \frac{m}{4\pi}\sum_P \int d\tau\, dx \left[iP\,\partial_\tau \phi_P \;\; \partial_x \phi_P + v\,(\partial_x \phi_P)^2 \right]$$

($P = \pm 1$ for R, L movers, respectively)

Free Hamiltonian:

$$H_0 = \frac{vm}{4\pi} \int dx \left[(\partial_x \phi_R)^2 + (\partial_x \phi_L)^2 \right]$$

Commutation relations:

$$[\phi_P(x), \phi_P(x')] = \frac{i\pi P}{m}\,\mathrm{sgn}(x'-x)$$

$$[\phi_R(x), \phi_L(x')] = \frac{i\pi}{m}$$

The density of right / left movers is:

$$\rho_P = \frac{1}{2\pi}\,\partial_x \phi_P$$

The electron operator at the edge is

$$\psi_{R,L} \sim e^{\pm i m \phi_{R,L}}$$

Q.P. operator:

$$\chi_{R,L} \sim e^{\pm i \phi_{R,L}}$$

Note about notations

I am using a different convention here relative to previous lectures:

$$\underbrace{\phi_R}_{\text{used here}} = \underbrace{\frac{\varphi}{\sqrt{m}}}_{\text{Schoutens, Regnault}}$$

OPE of vertex operators:

$$e^{i\alpha \phi_R(z)}\, e^{-i\alpha \phi_R(w)} \sim \frac{1}{(z-w)^{\alpha^2/m}}$$

$$\left(\langle \phi_R(z)\, \phi_R(w) \rangle \sim -\frac{1}{m} \ln (z-w) \right)$$

$$\langle e^{\alpha i (\phi_R(z) - \phi_R(w))} \rangle \sim e^{-\alpha^2 [\langle \phi_R^2(0)\rangle - \langle \phi_R(z)\, \phi_R(w)\rangle]}$$

$$\langle e \qquad\qquad \rangle \sim e$$

$$\sim e^{-\frac{\alpha^2}{m} \ln(z-m)}$$

$$= \frac{1}{(z-m)^{\alpha^2/m}}$$

Extra terms that arise in SC / FM regions:

S.C. pair tunneling term

$$-g_s(x)\, \psi_R^+ \psi_L^+ + h.c. \sim -g_s(x)\, e^{-mi(\phi_R - \phi_L)} + h.c.$$

$$= -2g_s \cos m(\phi_R - \phi_L)$$

Assume g_s real

F.M. Single electron tunneling

$$-g_F(x)\, \psi_R^+ \psi_L + h.c. \sim -2g_F \cos m(\phi_R + \phi_L)$$

Define $\phi(x) = \frac{1}{2}(\phi_R - \phi_L)$

$$\theta(x) = \frac{1}{2}(\phi_R + \phi_L)$$

$$[\phi(x), \theta(x')] = \frac{1}{4}\left[\phi_R(x) - \phi_L(x),\ \phi_R(x') + \phi_L(x')\right]$$

$$= \frac{1}{4}\left(2\frac{i\pi}{m}\,\mathrm{sgn}(x'-x) + 2\frac{i\pi}{m}\right) = \frac{i\pi}{m}\underbrace{\Theta}_{\text{step function}}(x'-x)$$

Spin and charge densities:

$$\rho(x) = \frac{1}{2\pi}(\partial_x \phi_R + \partial_x \phi_L) = \frac{1}{\pi}\partial_x \theta$$

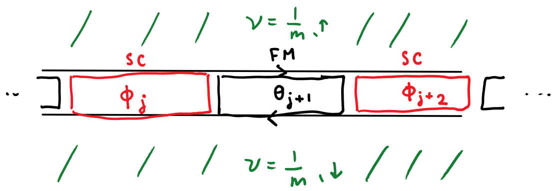
$$s^z(x) = \frac{1}{2\pi}(\partial_x \phi_R - \partial_x \phi_L) = \frac{1}{\pi}\partial_x \phi = J \text{ (current density)}$$

Hamiltonian: $H = H_0 - \int dx \left[g_s(x) \cos(2m\phi) + g_F(x)\cos(2m\theta)\right]$

$$H_0 = \frac{mv}{2\pi}\int dx\, K(\partial_x\theta)^2 + \frac{1}{K}(\partial_x\phi)^2$$

K - Luttinger parameter

Suppose that both the FM and SC regions are in their gapped phase (g_s, g_F large enough)



ϕ_j "pinned" in SC regions to cosine minima: $\phi_j = \frac{n_j \pi}{m}$

θ_ℓ pinned in FM regions: $\theta_\ell = \frac{n_\ell \pi}{m}$

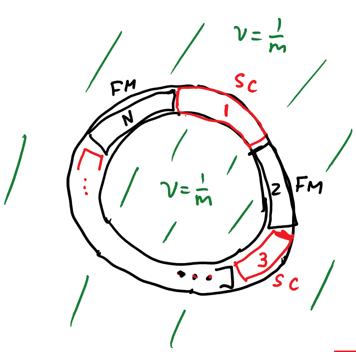
$$e^{i\phi_j} e^{i\theta_\ell} = e^{[i\phi_j, i\theta_\ell]} e^{i\theta_\ell} e^{i\phi_j}$$

$$= e^{-\frac{i\pi}{m}\Theta_{\ell,j}} e^{i\theta_\ell} e^{i\phi_j}$$

$$\Theta_{\ell,j} = \begin{cases} 1 & \ell > j \\ 0 & \ell \le j \end{cases}$$

Therefore, θ_ℓ, ϕ_j $(\ell > j)$ cannot be pinned together. We can choose a basis where $\langle e^{i\phi_j}\rangle \neq 0$ and then $\langle e^{i\theta_\ell}\rangle = 0$, or vice versa.

Suppose there are N domains:



We can choose a basis in which

$$\langle e^{i\theta_{2\ell}}\rangle = |A|\, e^{i\frac{\pi}{m} n_{2\ell}} \qquad n_{2\ell} = 0 \ldots 2m-1$$

$$\ell = 1 \ldots \frac{N}{2}$$

$$\Downarrow$$

$$\boxed{N_{gs} \propto (2m)^{\frac{N}{2}} = \left(\sqrt{2m}\right)^N}$$

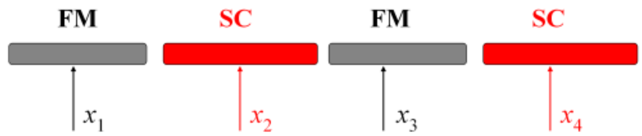
"Anyon" with quantum dimension $d = \sqrt{2m}$ at every interface.

$m = 1$: $d = \sqrt{2}$ (Majorana zero mode!)

Physical interpretation: Charge and Spin operators

Q and S operators

In terms of the ϕ, θ fields, one can define the Q, S operators:



$$e^{i\pi Q_2} = e^{i\int_{x_1}^{x_3} dx\partial_x\theta} = e^{i[\theta(x_3)-\theta(x_1)]}$$

$$e^{i\pi S_3} = e^{i\int_{x_2}^{x_4} dx\partial_x\phi} = e^{i[\phi(x_4)-\phi(x_2)]}$$

$$\boxed{e^{i\pi S_i}e^{i\pi Q_j} = e^{i\frac{\pi}{m}(\delta_{i,j+1}-\delta_{i,j-1})}e^{i\pi Q_j}e^{i\pi S_i}}$$

Zero modes at interfaces

Just as in the Majorana case, there is a "zero mode" at the interface between the SC and FM regions.

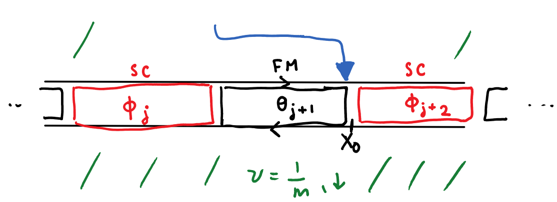
To see this, notice that in the $m=1$ case, the operators

$$\psi_{R,L} \sim e^{i(\theta \pm \phi)}$$

are essentially the Majorana zero modes at the interfaces.

To understand this, imagine acting with ψ_R or ψ_L near one of the interfaces:

$$e^{i[\phi(x_0) \pm \theta(x_0)]}$$



To understand what this does, imagine "splitting" the operator

$$e^{i\phi(x_0)}e^{\pm i\theta(x_0)} \rightarrow e^{i\phi(x_0+\varepsilon)}e^{\pm i\theta(x_0-\varepsilon)}$$

where ε is of the order of the correlation length ξ.

In the FM region: $e^{\pm i\theta(x_0-\varepsilon)} \approx \langle e^{i\theta_{j+1}}\rangle = |A|\, e^{i\frac{\pi}{M}n_j}$

In the SC region: $e^{i\phi(x_0+\varepsilon)}$ shifts $e^{i\theta_\ell} \rightarrow e^{i\theta_\ell + i\frac{\pi}{M}}$ for $\ell > i+1$

(you can check that this changes the charge of the $j+2$ s.c. :

$$\langle e^{i\pi Q_{j+2}}\rangle \to \langle e^{i\pi (Q_{j+2} - \frac{1}{m})}\rangle$$

These arguments carry over for any m. For $m \geq 3$, the zero mode operator $e^{i(\phi \pm \theta)}$ is a Laughlin q.p.;

i.e., at the interface fractional Laughlin q.p.'s can be absorbed with zero energy cost.

L3b

Thursday, February 12, 2015 9:27 PM

Explicit representation of zero mode operators on low-energy subspace

Now, we can construct explicitly a representation of the zero of the zero modes.

choose a basis such that $\theta_{2j+1} = \frac{\pi}{m} n_{2j+1}$, $n_{2j+1} = 0 \ldots 2m-1$.

A state is represented as

$| n_1, n_3, \ldots \rangle$

The zero mode operator $\chi_{j\uparrow,\downarrow} \sim e^{i\phi_j} e^{\pm i\theta_{j+1}}$ acts as follows:

$$\chi_{j\uparrow,\downarrow} = e^{\pm i \frac{\pi}{m} n_{j+1}} | \ldots n_{j+1}, n_{j+3}+1, n_{j+5}+1, \ldots \rangle$$

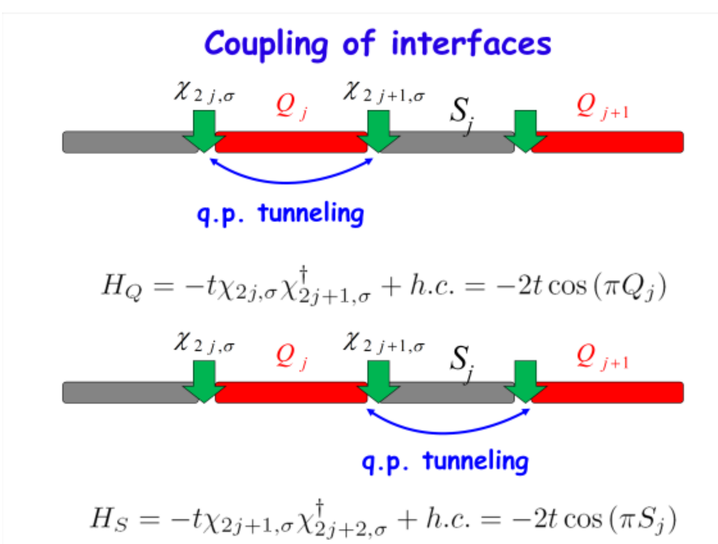
These zero mode operators satisfy the following relations:

$$(\chi_{j,\sigma=\uparrow,\downarrow})^{2m} = 1$$

$$\chi_{j,\uparrow} \chi_{k,\sigma} = e^{-\frac{i\pi}{m}} \chi_{k,\sigma} \chi_{j,\uparrow}$$

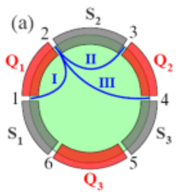
$$\chi_{j,\downarrow} \chi_{k,\sigma} = e^{i\frac{\pi}{m}} \chi_{k\sigma} \chi_{j\downarrow} \qquad (j<k)$$

"Parafermionic exchange relations" (Fendley 2012)



Braiding

Braiding domain walls 3 and 4:



$$U_{34} = \exp\left(i\frac{\pi m}{2}\hat{Q}_2^2\right) = \exp\left(i\frac{\pi}{2m}q_2^2\right)$$

$$Q_2 = \frac{1}{m}q_2, \quad q_2 = 0, \ldots, 2m-1$$

Example: m=3 $\quad q_2 = 2p + 3q \quad (p = 0, 1, 2, \; q = 0, 1)$

$$U_{34} = \exp\left(i\frac{\pi}{6}q_2^2\right) = \exp\left(-i\frac{\pi}{2}q^2\right)\exp\left(i\frac{2\pi}{3}p^2\right)$$

(Majorana) ⊗ (Something new!)

Alternative form of the braiding matrix

$$\hat{U}_{34}^{(k)} = \exp\left[\frac{i\pi m}{2}\left(\hat{Q}_2 + \frac{k}{m}\right)^2\right]. \qquad (24)$$

Alternatively, using the identity [48] $e^{i(\pi/2m)q^2} = \sqrt{\frac{1}{2m}}\sum_{p=0}^{2m-1} e^{i(\pi/m)[pq-(p^2/2)]+i(\pi/4)}$, one can write

$$\hat{U}_{34}^{(k)} = \sqrt{\frac{1}{2m}}\sum_{p=0}^{2m-1} e^{-(i\pi/2m)(p-k)^2+i(\pi/4)}(e^{i\pi\hat{Q}_2})^p. \qquad (25)$$

The braiding matrices satisfy the Yang Baxter equation:

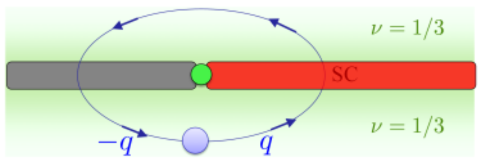
$$U_{12}^{(k_1)} U_{23}^{(k_2)} U_{12}^{(k_1)} = U_{23}^{(k_2)} U_{12}^{(k_1)} U_{23}^{(k_2)}.$$

In the $\nu = \frac{1}{m}$ FQH state, the "topological spin" of the q.p. is $\theta = \frac{\pi}{m}$ (the statistical phase obtained when the particle is exchanged with itself)

$\Rightarrow$ The exchange phase of two defects $= \frac{1}{2}$ (top. spin of total "fusion charge").

Fractionalized zero modes at "twist defects" in topological phases

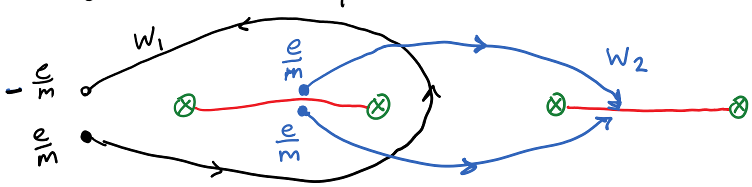
Ends of line defects that interchange anyon types
("topological symmetry")



The "defect line" can permute anyon types.

Barkeshli, Jian, Qi (2013); Fidkowski, Lindner, Kitaev (unpublished)

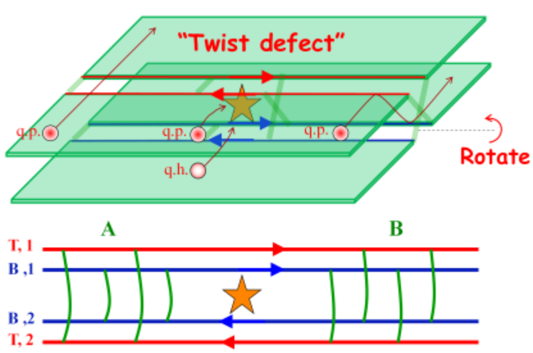
Relation to quantum dimension:



$W_1 W_2 = e^{-\frac{4i\pi}{m}} W_2 W_1 \Rightarrow$ G.S. is at least m-fold degenerate!

Fractionalized zero modes at "twist defects" in topological phases

ν=1/3 bilayer

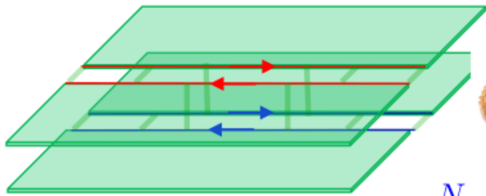


Fractionalized zero modes at "twist defects" in topological phases

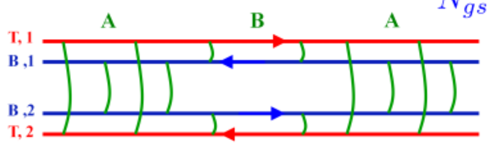
Alternate A,B domains:

Parafermions without superconductivity!

High genus surface



$N_{gs} = 3^{N_{\text{holes}}}$



Fractionalized zero modes at "twist defects" in topological phases

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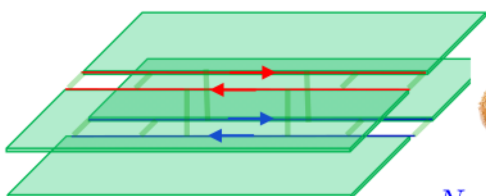

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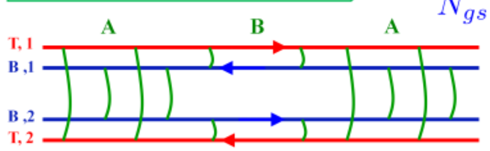
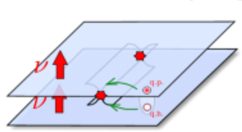
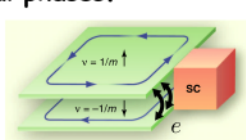


Conclusion

New paradigm for realizing non-abelian anyons: defects on edges of two-dimensional topological phases.





Future directions:

Classification of 1D gapped edge states of 2D topological theories?

Experimental signatures?

Thank you.