GGI lectures on quantum Hall systems and topological insulators

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IIa Haldane model: edge modes
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SOURCES

- Book E. Fradkin (2nd edition, 2013)
- Book B.A. Bernevig & T. Hughes (2013)
- Hasan & Kane, RMP 2010
- Qi & Zhang, RMP 2011

+ research papers

IIa Big Picture exp. & th.

$\boxed{\text{quantum Hall systems (qH)}}$

integer qH effect (1980)

TKNN (1982)
'topological quantization'

fractional qH effect (1982)

Laughlin (1983)
'fqH wavefunction'

non-Abelian qH states (1987*)

* to be confirmed

Moore, Read (1991)
'paired qH state'

Haldane (1988)
'quantum anomalous Hall effect'

Kane-Mele (2005)
'quantum spin Hall effect'

$\boxed{\text{Topological Insulators}}$

Bernevig, Hughes, Zhang (2006)
'qSHe in HgTe'

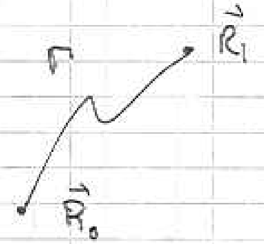
observed 2007 ↦

Fu, Kane (2007)
'3D Z_2 TI'

observed in BiSb, .., 2008 ↦

IIb Berry, Chern, Pontryagin, Hall

$H(\vec R) \Rightarrow |n(\vec R)\rangle$



Berry phase $e^{i\gamma_n}$

$$\gamma_n = i\int_\Gamma \langle n|\nabla_{\vec R}|n\rangle \cdot d\vec R$$

$$= \int_\Gamma d\vec R \cdot \vec{\mathcal{A}}^{(n)}(\vec R)$$

connection

$$\mathcal{F}^{(n)}_i = \varepsilon_{ijl}\partial_j \mathcal{A}^{(n)}_l$$

curvature

example $\quad H = \vec R\cdot\vec\sigma \Rightarrow \vec{\mathcal{F}}^{(\pm)} = \pm\frac{1}{2}\frac{\vec R}{R^3}$

Dirac fermion

$$H = \vec k\cdot\vec\sigma + m\,\sigma_z = \vec h(\vec k)\cdot\vec\sigma$$

Chern number $\Leftrightarrow \frac{1}{4\pi}\iint d^2k\,(\partial_x\hat h\wedge\partial_y\hat h)\cdot\hat h$

(Pontryagin index)

2D band structure

$$\psi_m = u_m(\vec k)\,e^{i\vec k\cdot\vec x} \qquad m = 1..M$$

$$\mathcal{A}^{(m)}_i = (-i)\langle u_m|\partial_{k_i}|u_m\rangle \qquad \mathcal{F}^{(m)} = \varepsilon_{ij}\partial_i\mathcal{A}^{(m)}_j$$

$$\frac{1}{2\pi}\iint_{BZ} d^2k\,\mathcal{F}^{(m)}(\vec k) \overset{!}{=} N_m$$

(integer)

'Chern number'

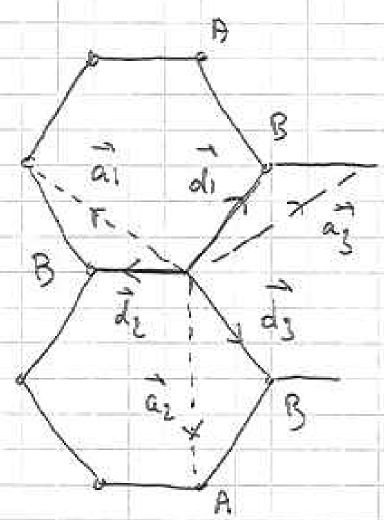
$$\boxed{\text{Hall conductance } (m^{th}\text{ band}) \qquad \sigma_{xy} = \frac{e^2}{h}N_m}$$

for vicinity of Dirac point

$$Q = \tfrac{1}{2}\,\mathrm{sgn}(m) \Rightarrow \sigma_{xy} = \frac{e^2}{h}\frac{1}{2}\,\mathrm{sgn}(m)$$

'Dirac points come in pairs'

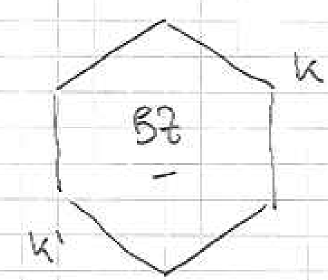
Ic Haldane model



NN hopping on graphene (along $\vec{d}_i$)

$$E^{\pm} = \pm t_1 \sqrt{\left| e^{i\vec{k}\cdot\vec{d}_1} + e^{i\vec{k}\cdot\vec{d}_2} + e^{i\vec{k}\cdot\vec{d}_3} \right|^2}$$

linearize around k, k' $(a = 1, 2)$



$$H_0 = \int d^2x \sum_{a=1,2} \psi_a^\dagger v_F \left(i\sigma_1 \partial_1 + i\sigma_2 \partial_2 \right) \psi_a$$

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \begin{matrix} \leftarrow \text{A sites} \\ \leftarrow \text{B sites} \end{matrix}$$

Haldane (1988) : extra terms :

1) potential $\pm \varepsilon$ on A, B sites

2) hopping $t_2 e^{i\phi}$ along $\vec{a}_i$ (staggered flux)

$\Downarrow$

mass terms

$$\int d^2x \, \psi_a^\dagger m_a \sigma_z \psi_a$$

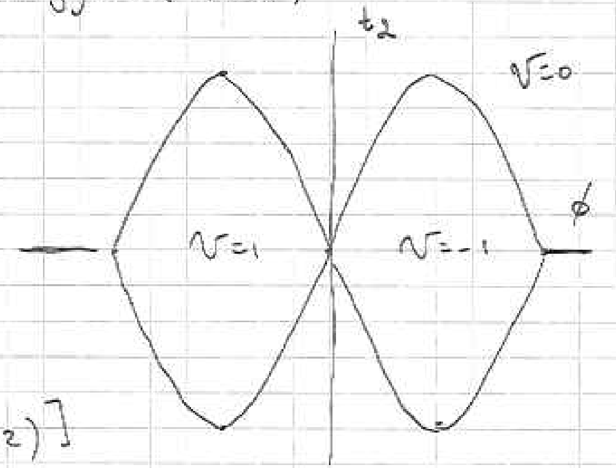
in H_0, m_a depend on $\{ t_1 ; \varepsilon, t_2, \phi \}$

$\Longrightarrow$

gap at k, k'

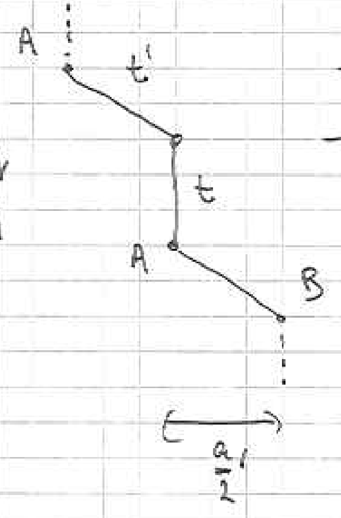
$$\sigma_{xy} = \nu \frac{e^2}{h}$$

$$\nu = \frac{1}{2} \left[\operatorname{sgn}(m_1) + \operatorname{sgn}(m_2) \right]$$



IIa Haldane model : edge modes

cylinder : (k_x, y)



effective chain hopping

$$\begin{pmatrix} 0 & t' & & \\ t' & 0 & t & \\ & t & 0 & \\ & & & \ddots \end{pmatrix}$$

$$t' = 2t\cos\left(k_x \frac{a_1}{2}\right)$$

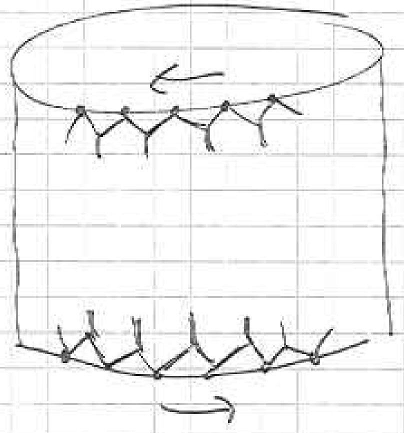
$\Longrightarrow$ zero mode (N_y odd)

$$\left(1 \;\; 0 \;\; -t'/t \;\; 0 \;\; (-t'/t)^2 \;\; 0 \;\; \ldots\right)$$

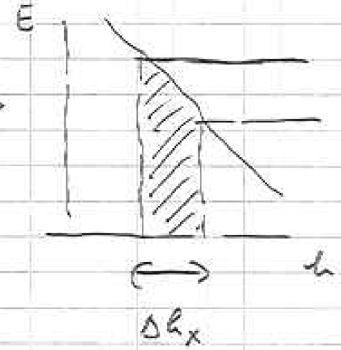
$|t'/t| < 1$ upper edge

$|t'/t| > 1$ lower edge

$$\boxed{\text{adding } \varepsilon,\ t_2 e^{i\phi}\text{, find that edge mode crosses gap in topological phase where } \operatorname{sgn}(m_1) + \operatorname{sgn}(m_2) \neq 0}$$



$\Longrightarrow$



$\#$ (extra occupied states)

$$= \Delta k_x \Big/ \left(\frac{2\pi}{L_x}\right)$$

current/state : $\left(\frac{e}{L_x}\right) \cdot \frac{1}{\hbar}\frac{\partial E}{\partial k}$

$$\Delta E = eV_H \qquad \Delta k_x = \Delta E \Big/ \frac{\partial E}{\partial k}$$

$$\Longrightarrow \quad I = \frac{e^2}{h} V_H$$

IIb Lattice qH ⇒ Hofstadter bands

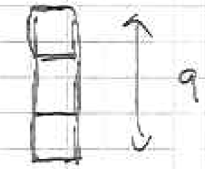
2D square lattice

$\Phi = {}^{p}/_{q}\,\Phi_0$ per plaquette

$$H = -t \sum_{\vec{x}, j} |\vec{x}\rangle\, e^{i\frac{e}{\hbar c}A_j} \langle \vec{x} + \vec{e}_j| + h.c.$$

$$\sum A_j = B \qquad \text{choose} \quad \begin{cases} A_1 = -Bx \\ A_2 = 0 \end{cases}$$

unit cell



Magnetic translations : $\{\hat{T}_1, \hat{T}_2^{\,q}\}$

find q bands w/ each L_1L_2/q states (Hofstadter)

$$\psi^{(j)}_r(k_1, k_2)$$

$-\pi \le k_1 \le \pi$

$-\frac{\pi}{q} \le k_2 \le \pi/q$

$j = 1, 2, \ldots q$ band index

$r = 1 \ldots q$ "vector" index

$\boxed{\text{Harper equation}}$

Recipe for Chern numbers & σ_{xy}

$$\sigma_{xy} = \frac{e^2}{h} \sum_{j=1}^{t} I_j = \frac{e^2}{h} \sum_{j=1}^{t} (\ell_j - \ell_{j-1}) = \frac{e^2}{h}\ell_t$$

ℓ_j such that

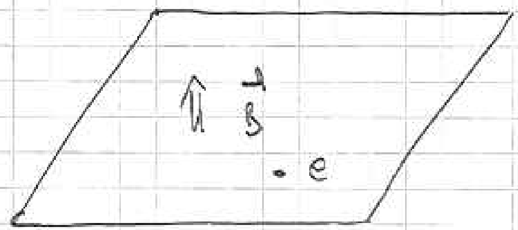
$$j = q\, s_j + p\, \ell_j \qquad |\ell_j| \le \frac{q}{2}$$

example : $p = 2,\ q = 5$

$I_1 = I_3 = I_5 = -2 \qquad I_2 = I_4 = +3$

II_c Continuum QH → Landau levels

Landau problem



$$H = \frac{1}{2M}\left(i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 \qquad \ell_0^2 = \frac{\hbar c}{eB}$$

$z = x_1 + i x_2$

$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$

etc.

symmetric gauge

$A_i = -\frac{1}{2} B\, \varepsilon_{ij} x_j$

$\psi(z,\bar{z})$

$\Leftrightarrow$ $= f(z,\bar{z})\, e^{-|z|^2/4\ell_0^2}$

---

angular momentum

$L_z = \hbar\left(z\partial_z - \bar{z}\partial_{\bar{z}}\right)$

---

spectrum $\quad E_n = \hbar\omega_c\left(n + \frac{1}{2}\right)$

(n^{th} Landau level)

$\omega_c = eB/mc$

lowest Landau level (LLL)

$\psi_n \propto z^n\, e^{-|z|^2/4\ell_0^2}$

⇙

peaks near

$r_n = \ell_0\sqrt{2n}$

⇓

area

$\pi r_n^2 = 2\pi\ell_0^2 \cdot n$

⇗

area/orbital $= 2\pi\ell_0^2$

⇓

flux/orbital $= 2\pi\ell_0^2 B = \Phi_0$

⇓

$N_{\text{orbital}} = N_\Phi$

IIIa Filling the LLL : iqHe

filling the LLL : $\Psi(z_1 \dots z_N) = \prod_{i<j} (z_i - z_j)\, e^{-\sum |z_i|^2/4\ell_0^2}$

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topological quantization of σ_{xy}?

no 2D BZ for LLL $\Rightarrow$ TKNN not directly applicable ...

<u>instead</u> generalized PBC

$$\Psi(x_1 + L_1, x_2) = e^{i\left(\frac{e}{\hbar c}\beta_1 + \theta_1\right)} \Psi(x_1, x_2)$$

etc.

$\beta_1 \leftrightarrow$ magnetic field
$\theta_1 \leftrightarrow$ electric field

Niu-Thouless-Wu :

$$(\sigma_{xy})^{\Psi} = \int_0^{2\pi} \frac{d\theta_1}{2\pi} \int_0^{2\pi} \frac{d\theta_2}{2\pi} \left(\frac{e^2}{\hbar}\right)(2i) \left[\partial_2 \langle\Psi|\partial_1|\Psi\rangle - \partial_1 \langle\Psi|\partial_2|\Psi\rangle \right]$$

explicit $\Psi^{(\text{many})}$ lead to $\sigma_{xy} = \frac{e^2}{h}$

(Haldane - Rezayi)

edge modes ?

cylinder geometry $\quad A_x = -By$, $A_y = 0$

$$\Psi(x,y) = e^{i k_x x} f_{k_x}(y)$$

$-L_y \leq y \leq 0$

H.O. potential

$$\frac{1}{2} m \omega_c^2 \left(y + k_x \ell_0^2\right)^2$$

<u>plus</u> hard walls @ $y = -L_y, 0$

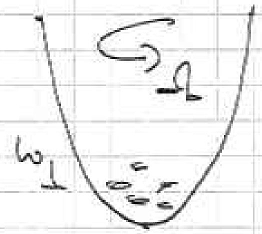
$\Downarrow$

LLL energies curve up near edges and cross gap!

III b Interactions in the LLL, bosons

alternative settings for LL

- × cold atoms in artificial gauge field
- × cold atoms in rotating trap

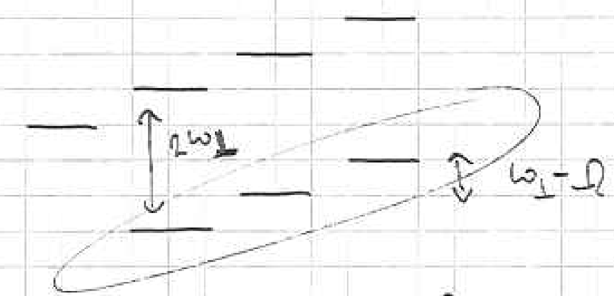


rotating frame:

$$\frac{1}{2M}\left(\vec{p} - M\vec{\Omega}\wedge\vec{z}\right)^2$$

$$\frac{e}{c}\vec{A}$$

<u>1-body spectrum</u>



LLL in limit $\Omega \to \omega_\perp$

<u>interactions</u>

$$g\sum_{i<j}\delta(i-j)$$

$\boxed{\text{N interacting bosons in the LLL}}$

$$H = (\omega_\perp - \Omega)L_z + g\sum_{i<j}\delta(i-j)$$

<u>$L_z = 0$</u> $\quad \hat{\Psi} = 1 \quad \mapsto$ BEC

$E_{int} \sim g\frac{1}{2}N(N-1)$

<u>$L_z = N$</u> $\quad \hat{\Psi} = \prod(z_i - z_c) \qquad z_c = \frac{1}{N}(z_1 + \dots z_N)$

vortex, $E_{int} \sim g\frac{1}{4}N(N-2)$

<u>$L_z = N(N-1)$</u> $\quad \hat{\Psi} = \prod(z_i - z_j)^2 \qquad E_{int} = 0$

$\Downarrow$

Laughlin wavefunction (exact!)

filling (density) $= N/N_\Phi = 1/2$.

III$_c$ Laughlin states & CFT

$$\Psi_{\text{Laughlin}}(z_1, z_2, \dots z_N) = \prod_{i<j} (z_i - z_j)^m \, e^{-\sum_i |z_i|^2/4\ell_0^2}$$

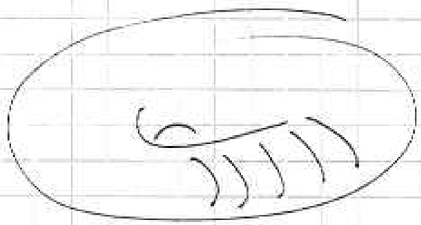
Laughlin (1983) : good overlap w/ ground state for Coulomb interaction at $\nu = 1/3$ $(m=3)$

Expand :

$$\Psi_L^{(m=3)} = z_1^0 z_2^3 z_3^6 \dots + \dots$$

100100 ...

on torus



$|gs_1\rangle \simeq 100100 ..$
$|gs_2\rangle \simeq 010010 ..$
$|gs_3\rangle \simeq 001001 ..$

$\sigma_H = \frac{1}{3} \frac{e^2}{h}$

3-fold degenerate

$\tilde{\Psi}_L$ from CFT

CFT operator $V_\alpha = e^{i\alpha\phi}$

$\Psi_e \to V_{\alpha = \sqrt{m}}$

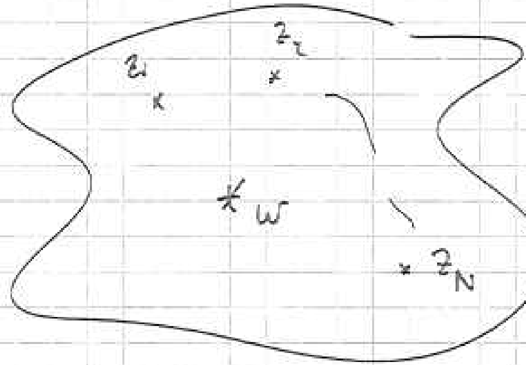
identity (QH-CFT)

$$\prod (z_i - z_j)^m = \lim_{z_\infty \to \infty} z_\infty^{mN^2} \left\langle V_{\sqrt{m}}(z_1) \dots V_{\sqrt{m}}(z_N) V_{-N\sqrt{m}}(z_\infty) \right\rangle$$

IV. qH-CFT connection, bulk

chiral scalar field ($c=1$ CFT)

charge operator $Q_{CFT} = \oint \frac{dz}{2\pi i}\, i\partial\phi$



Vertex operator $V_\alpha(z)$

1) $Q_{CFT} V_\alpha = \alpha V_\alpha$

2) OPE: $V_\alpha(z) V_\beta(w) \sim (z-w)^{\alpha\beta} V_{\alpha+\beta}(w)$

$\boxed{\text{qH-CFT}}$

electron $\rightarrow \psi_e \sim V_{\sqrt{m}}$ $\qquad Q = \frac{e}{m} Q_{CFT}$

defect $\rightarrow \phi = V_\alpha$

$\psi_e(z)\phi(w) \simeq (z-w)^{\alpha\sqrt{m}}$

choosing $\alpha = \frac{1}{\sqrt{m}} \Rightarrow$ quasi-hole ϕ_{qh}

$Q\phi_{qh} = \frac{e}{m}\phi_{qh}$ $\qquad$ fractional charge

$\tilde{\psi}(w; z_1 \ldots z_N)$

$= \prod_i (z_i - w) \cdot \prod_{i<j} (z_i - z_j)^m$

"flux quantum at w"

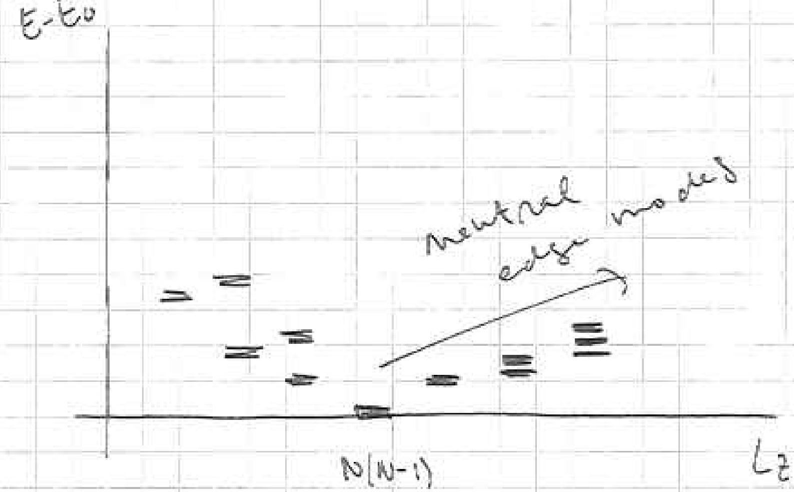
$\tilde{\psi}(w_1, w_2; z_1 \ldots) \simeq (w_1 - w_2)^{1/m}$

fractional statistics $\quad e^{\frac{\pi i}{m}}$

III qH - CFT connection, edge

LLL bosons

$$H = (\omega_\perp - \Omega) L_z + g \sum_{i<j} \delta(i-j)$$



$\mapsto$ $E_{int} = 0$

wavefunctions w/ $E_{int} = 0$

$$\tilde{\psi}(z_1, \dots z_N) = P_{sym}(z_1, \dots z_N) \prod_{i<j} (z_i - z_j)^2$$

$P = 1$ $\quad L_z = N(N-1)$

$P = (z_1 + \dots z_N)$ $\quad \Delta L_z = 1$

$\left.\begin{array}{l} P = (z_1^2 + z_2^2 + \dots) \\ (z_1 z_2 + \dots) \end{array}\right\} \quad \Delta L_z = 2$

$\vdots$ etc

$$Z_{disc} = \mathrm{Tr}\left(q^{L_z}\right) = q^{N(N-1)} (1 + q + \dots)$$

CFT correspondence:

$$gs \ @ \ L_z = N(N-1) \Leftrightarrow |0\rangle_{CFT}$$

$$(i\partial\phi)(z) = \sum_n j_n z^{-n-1}$$

$$j_{-n} \Leftrightarrow (z_1^n + z_2^n + \dots)$$

$$Z_{disc} \Leftrightarrow Z_{CFT} = \mathrm{tr}\left(q^{L_0}\right)$$

(details: Ardonne–? 2006)

Pf_k : $\mathbb{Z}_k$ parafermions , $c = 2\,\frac{k-1}{k+2}$

$\{ \psi_1, \psi_2, \dots \psi_{k-1} = \psi_1^+ \}$ $\quad h_{\psi_i} = \frac{i(k-i)}{k}$ $\quad$ plus Pf spin fields $\varepsilon, \sigma_i, \dots$

$\underline{k=2}$ $\quad \{ 1, \psi, \sigma \}$

$h_\psi = \frac{1}{2}, \; h_\sigma = \frac{1}{16}$

$\times$	σ	ψ
σ	$1+\psi$	
ψ	σ	1

$\underline{k=3}$ $\quad \{ 1, \psi_1, \psi_2, \sigma_1, \sigma_2, \varepsilon \}$ $\quad \Longrightarrow$

$h_\psi = \frac{2}{3}, \; h_\sigma = \frac{1}{15}, \; h_\varepsilon = \frac{2}{5}$

$\times$	σ_1	σ_2	ε	ψ_1	ψ_2
σ_1	$\psi_1+\sigma_2$				
σ_2	$1+\varepsilon$	$\psi_2+\sigma_1$			
ε	$\psi_2+\sigma_1$	$\psi_1+\sigma_2$	$1+\varepsilon$		
ψ_1	ε	σ_1	σ_2	ψ_2	
ψ_2	σ_2	ε	σ_1	1	ψ_1

$\varepsilon \times \varepsilon = 1 + \varepsilon$

IV. more CFT

combining w/ chiral scalar field

$$\psi_B \Leftarrow \begin{cases} J^+(z) \propto \psi_1(z)\, e^{i\sqrt{\frac{2}{k}}\phi(z)} \\ J^-(z) \propto \psi_1^+(z)\, e^{-i\sqrt{\frac{2}{k}}\phi(z)} \\ J^0(z) \propto \partial\phi(z) \end{cases}$$

$$\psi_e \Leftarrow \begin{cases} G^+(z) \propto \psi_1(z)\, e^{i\sqrt{\frac{2+k}{k}}\phi(z)} \\ G^-(z) \propto \psi_1^+(z)\, e^{i\sqrt{\frac{2+k}{k}}\phi(z)} \\ J^0(z) \propto \partial\phi \end{cases}$$

bosonic, $SU(2)_k$ WZW $\qquad$ fermionic, $(N=2)_k$ superconformal

IIa Paired qH states

Moore-Read (1991) : $\Psi_e = \psi\, e^{i\sqrt{2}\phi}$

$\Downarrow$

$$\hat{\Psi}_{MR} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^2$$

Greiter-Wen-Wilczek (1991/1992)

view Pf as BCS wave fn

Cappelli-Georgiev-Todorov (2001)

$$\hat{\Psi}_{MR} \doteq \mathcal{S}\left[\prod (u_i - u_j)^2 \prod (v_i - v_j)^2 \right] \prod (z_i - z_j)$$

$$\{z_i\} = \{u_i, v_j\}$$

Pairing in the LLL

$$H_{int} \sim g \sum_{i,j,k} \delta(i-j)\,\delta(i-k)$$

bosonic $\hat{\Psi}_{MR,B} = \mathrm{Pf}(\,)\prod(z_i - z_j)$

unique w.f. with

1) $H_{int}\,\psi = 0$

2) $L_z \psi = \frac{1}{2}N(N-2)$

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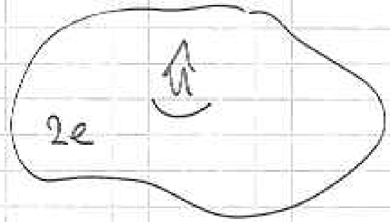
quasi-holes over MR condensate



flux quantum

through charge 2e

condensate : $\Phi_0/2$ $\Leftrightarrow$

$$e^* = \nu e \cdot \frac{1}{2} = \frac{e}{4}$$

CFT

$$\phi_{qh} \to \sigma\, e^{i\frac{1}{2\sqrt{2}}\phi}$$

II.6 Beyond pairing

using Pf_k ⇒ k-clustered state (Read-Rezayi 1999)

bosons ⇒ $\nu = \frac{k}{2}$ fermions ⇒ $\nu = \frac{k}{k+2}$

<u>example k=3, fermions, $\nu = 3/5$</u>

$$\tilde{\Psi}_{RR} = \langle \psi_1(z_1) \dots \psi_1(z_N) \rangle \prod_{i<j} (z_i - z_j)^{5/3}$$

$$\stackrel{!}{=} \left[S \prod (u_i-u_j)^2 \prod (v_i-v_j)^2 \prod (w_i-w_j)^2 \right] \prod (z_i - z_j) \qquad \{z_i\} = \{u_i, v_j, w_k\}$$

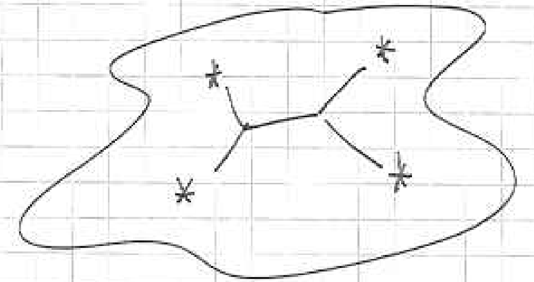
quasi-hole operator?

$$\psi_e = \psi_1 e^{i\sqrt{\frac{5}{3}}\phi} \quad , \quad \text{try} \quad \Phi_{qh} = \sigma_1 e^{i\alpha\phi}$$

using $\sigma_1(z)\psi_1(w) \simeq (z-w)^{-1/3} \varepsilon(w)$

find $\alpha = \frac{1}{\sqrt{15}}$ ⇒ $Q\Phi_{qh} = \frac{e}{5}$ agrees w/ insertion of flux $\Phi_0/3$.

II. Fusion & braiding ; TQC



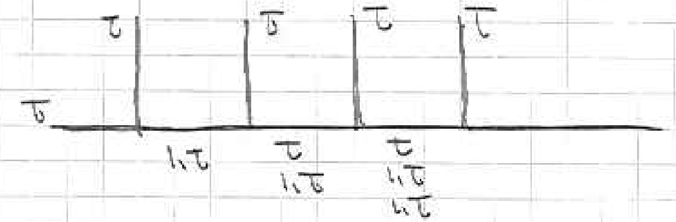
$k=2$ / $k=3$

$$\Psi^{2}_{MR/RR}(w_1, w_2, w_3, w_4 ; z_1, z_2, \dots z_N)$$

stands for 2 wavefn.

counting $k=2$ $\quad n$ qh $\Rightarrow$ # channels $= 2^{\frac{n}{2}-1}$

$k=3$



channels $= (\text{Fibo})_{n-1}$

('Fibonacci anyons')

braiding

represented by

2×2 unitary U_{ij}

$k=2$ $\quad U_{13} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (Hadamard)

$k=3$ $\quad U_{13} =$ [illegible] $\begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}$ $\quad \tau = \frac{1}{2}(\sqrt{5}-1)$

$k=3$ RR and $k=2$ NASS states universal for TQC !