Critical Phenomena at Classical and Quantum Transitions

Main ideas, some applications and overview of results

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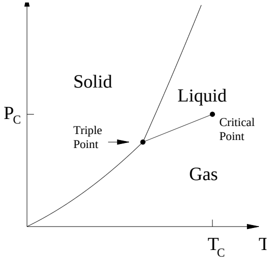
Phase transitions and Critical phenomena in many physical systems

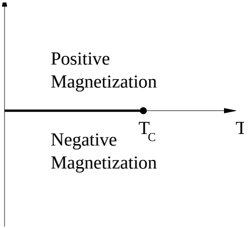
There are two broad classes of phase transitions:

first order → discontinuity in thermodynamic quantities, such as the energy density, magnetization, etc...

continuous → nonanalytic behavior due to a diverging length scale characterizing the physical correlations

Examples of first-order and continuous transitions are found in the phase diagrams of magnets, liquids, etc...





- first general framework was proposed by Landau (1937) → mean-field approx
- satisfactory understanding by the renormalization-group theory (Wilson 1971)

First-order transitions

- discontinuous energy density (latent heat) and/or magnetization

- No diverging length scale is developed approaching T_c after the thermodynamic limit, i.e.

$$\lim_{T\to T_c^\pm} \lim_{L\to\infty} \xi \;\equiv\; \xi^\pm < \infty$$

- However $\lim_{L\to\infty} \lim_{T\to T_c}$ may develop a diverging length scale related to the mixed phase (*persistence* length scale) $\longrightarrow$ nontrivial FSS

Continuous transitions develop a diverging length scale

$$\lim_{T\to T_c^\pm} \lim_{L\to\infty} \xi = \infty$$

giving generally rise to power-law behaviors

$$\xi \sim |T - T_c|^{-\nu}, \quad \chi \sim \xi^{2-\eta}, \quad C_H \sim a + c|T - T_c|^{-\alpha}$$

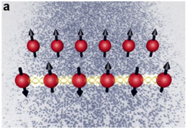
Simple interactions may give rise to complex phenomena with long-distance correlations, after an appropriate tuning of the thermodynamic parameters.

Critical phenomena observed in many different materials have several features in common: Universality.

• Ferromagnetism, Curie transition: • magnetization from the spin of electrons in the imcomplete atomic shells of metal atoms: each electron carries one Bohr magneton • interactions among spins due to exchange effects

$$H_{\rm spin} = -J \sum_{\langle ij\rangle} \vec{s}_i \cdot \vec{s}_j - \sum_i \vec{h} \cdot \vec{s}_i,$$

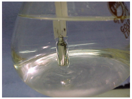
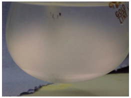
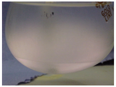
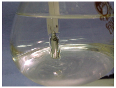
$$Z = \sum_{\{\vec{s}_i\}} e^{-H/T}, \qquad F = -\frac{T}{V} \ln Z$$



Ising model with $s_i = \pm 1$ describes uniaxial magnets

• Liquid-vapor transition (density instead of magnetization, μ instead of h): $H_{\rm lattice\ \ gas} = -J \sum_{\langle ij\rangle} \rho_i \rho_j - \mu \sum_i \rho_i$, with $\rho_i = (1 + s_i)/2 = 0, 1$.

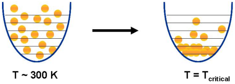
Ex.: critical opalescence in liquid systems, at their liquid-gas continuous transition, light diffusion when the correlation length increases from 10^{-9} m to 10^{-6} m

BEC transition in a perfect Bose gas when

$$\lambda_{\rm DB} = \left(\frac{2\pi\hbar^2}{mT}\right)^{1/2} \approx \text{average distance of atoms} = (N/V)^{-1/3}$$

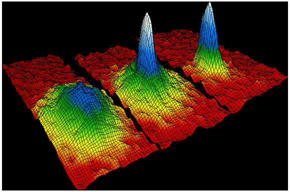
Below $T_c \approx \hbar^2(N/V)^{2/3}/m$, a macroscopic number of atoms condenses to the lowest state, in a free gas $N_0/N = 1-(T/T_c)^{3/2}$.



Interactions, even weak, give rise to a power-law critical behavior characterized by the U(1) symmetry related to the phase of the condensate wave function.

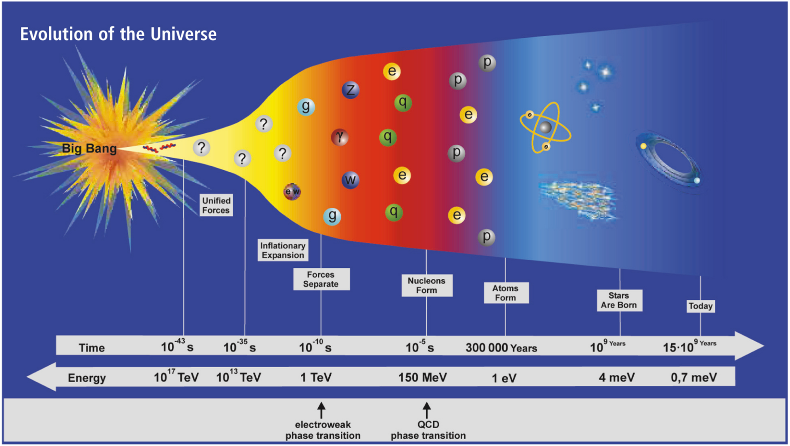
BEC recently observed in weakly interacting boson gases, made of alkali atoms, rubidium, sodium, lithium

velocity distribution of rubidium atoms



BEC-like phenomena in the normal-to-superfluid transition of ^{4}He, which is a liquid rather than a gas, thus strongly interacting (only 9% atoms condense for $T\to 0$, providing the superfluid component)

Some phase transitions characterize the earlier universe evolution



Continuous phase transitions are characterized by power-law behaviors

Phases related by a spontaneous SB, such as $O(N) \Longrightarrow O(N-1)$

- Disordered (symmetric) phase ($t \equiv T/T_c - 1 > 0$, $H = 0$):

$$\xi \sim t^{-\nu}, \quad C_H \sim t^{-\alpha}, \quad \chi \sim t^{-\gamma} \sim \xi^{2-\eta}, \quad \widetilde{G}(q) \approx \frac{1}{q^2 + m^2}$$

- Ordered (broken) phase ($t < 0$, $H = 0^+$):

$$M \sim |t|^{\beta}, \quad C_H \sim |t|^{-\alpha}, \quad \widetilde{G}_T(q) \approx \frac{M^2}{MH + \rho_s q^2}, \quad \widetilde{G}_L(q) \sim q^{d-4}$$

massless Goldstone (spin wave) modes appear, singular $\widetilde{G}_T$ for $H \to 0$

- Critical isotherm ($t = 0$, $h > 0$): $\quad \xi \sim |h|^{-\nu/\beta\delta}, \quad \widetilde{G}(q) \sim q^{-2+\eta}$

- Scaling equation of state: $h = t^{\beta\delta} F(z)$, $z = M t^{-\beta}$ in magnets,

$\Longrightarrow$ in fluids $h \to \rho - \rho_c$ and $M \to \mu - \mu_c$,
$\Longrightarrow$ at the chiral transition of QCD, condensate $\langle \bar{\psi}\psi \rangle$ and quark masses

- There are also notable critical behaviors with exponential approaches:

• 2D Kosterlitz-Thouless transitions where $\xi \sim \exp[c/\sqrt{T - T_c}]$
and also • LATTICE QCD where $\xi \sim \exp(c\beta)$ • 2D σ models

Main ideas to describe the critical behavior at a continuous transition

• **Order parameter** which effectively describes the critical modes

• **Scaling hypothesis**: singularities arise from the long-range correlations of the order parameter, diverging length scale

• **Universality**: the critical behavior is essentially determined by a few global properties: the space dimensionality, the nature and the symmetry of the order parameter, the symmetry breaking, microscopic interaction range

RENORMALIZATION-GROUP THEORY

• RG flow in a Hamiltonian space

• the critical behavior is associated with a fixed point of the RG flow

• only a few perturbations are relevant, the corresponding positive eigenvalues are related to the critical exponents ν, η, etc...

The free-energy density obeys a scaling law at continuous transitions

$$\mathcal{F}_{\rm sing}(u_1, u_2, \ldots, u_k, \ldots) = b^{-d}\mathcal{F}_{\rm sing}(b^{y_1}u_1, b^{y_2}u_2, \ldots, b^{y_k}u_k, \ldots)$$

u_i are nonlinear scaling fields (analytic functions of the model parameters)

Two relevant fields u_i with $y_i > 0$: $\quad u_t \sim t, \;\; u_h \sim H, \;$ for $\; t, h \to 0$

- Setting $b^{y_t}|u_t| = 1 \to \mathcal{F}_{\rm sing} = |u_t|^{d/y_t}\mathcal{F}_{\rm sing}(u_h|u_t|^{-y_h/y_t}, v_i|u_t|^{-y_i/y_t})$

- Since $v_i|u_t|^{-y_i/y_1} \to 0$ for $t \to 0$,

$$\mathcal{F}_{\rm sing} \approx |t|^{d/y_t} f(|h||t|^{-y_h/y_t}) + |t|^{d/y_t+\Delta_i} f_{(1,i)}(|h||t|^{-y_h/y_t}) + \ldots$$

$y_t = 1/\nu$, $y_h = (\beta+\gamma)/\nu$, $\Delta_i = -y_i/y_t > 0$.

FSS: scaling allowing for finite size L of the system, scaling as $L \to L/b$

$$\mathcal{F}_{\rm s}(u_i, L) = b^{-d}\mathcal{F}_{\rm s}(b^{y_i}u_i, L/b) \Longrightarrow \;\; \mathcal{F}_{\rm s}(u_t, u_h) = L^{-d}\mathcal{F}_{\rm s}(L^{y_t}u_t, L^{y_h}u_h)$$

Scaling phenomena emerge also at first-order transitions in finite systems

They are essentially related to the coexistence of different phases

Finite-size scaling at first-order transitions like continuous transitions.

- related to the diverging persistence length of the coexisting phases

- Simple effective exponents to be consistent with the discontinuities in the thermodynamic limit: $F(T_c^+) = F(T_c^-), \quad \partial F/\partial T|_{T_c^+} \neq \partial F/\partial T|_{T_c^-}$

$\longrightarrow$ Assuming $\xi \sim |T - T_c|^{-\nu}$ and $F_{\rm sing} = A_\pm \xi^{-d}$, consistency requires $\nu = 1/d$ (this generally holds for systems with box-like shape)

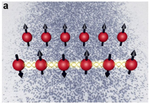
Note that $\nu > 1/d$ at continuous transitions.

- also derived by phenomenological analyses using double-Gaussian distributions

- RG description in terms of a discontinuous fixed point

Heuristic derivation of QFTs to describe critical phenomena

Ex.: d-dim lattice Ising model $H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$,
$\sigma_i = \pm 1$, $\quad Z = \sum_{\{\sigma_i\}} \exp(-H/T)$



- The critical behavior is due to the long-range modes, with $l \gg a$.

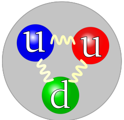
- Blocking over $b \ll l \to$ coarse-grained variable $\varphi(x) = \sum_{i \in b_x} \sigma_i \in \Re$, with $H_{\varphi^4} = b^{d-2} \sum_{x,\mu} (\varphi_{x+b\mu} - \varphi_x)^2 + u b^d \sum_x (\varphi_x^2 - v^2)^2$ with $\mathbb{Z}_2$ sym

- The limit $a \to 0$ of H_{φ^4} should not change the long-distance behavior, then $\mathcal{H}(\varphi) = \int d^d x \left[(\partial_\mu \varphi)^2 + r\varphi^2 + u\varphi^4 \right]$, where $r - r_c \propto T - T_c$

- Effectively $Z = \int [d\varphi] \exp[-\mathcal{H}(\varphi)] \to \mathrm{QFT}$ with $\mathcal{H}(\varphi) \to \mathcal{L}(\varphi)$

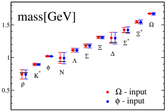
- RG flow by a set of RG equations for the correlation functions

$\Longrightarrow$ The way back provides a nonperturbative formulation of an Euclidean QFT, from the critical behavior of a statistical model.



QCD defined from the critical regime of 4D statistical systems

from quarks to baryons



Effective Landau-Ginzburg-Wilson Φ^4 theories of the critical modes

Many critical phenomena are described by LGW Φ^4 theories

$\Longrightarrow$ They are constructed by introducing an order-parameter field Φ_i capturing the main features of the critical modes, and imposing a few global properties of the system, keeping terms up to 4th order:

$$\mathcal{L} = \sum_{i=1}^{N} [(\partial_\mu \Phi_i)^2 + r_i \Phi_i^2] + \sum_{ijkl=1}^{N} u_{ijkl}\, \Phi_i \Phi_j \Phi_k \Phi_l$$

associated with different UNIVERSALITY CLASSES sharing the same
• spatial dimension • nature of critical modes and their order parameter
• symmetry and symmetry-breaking pattern • range of the interactions

Ex: Bose-Einstein condensation of an 3D atomic gas: D=3, quantum amplitude of the condensate as order parameter, U(1) symmetry $\Longrightarrow$ 3D XY UNIVERSALITY CLASS:
$\mathcal{L} = |\partial_\mu \varphi|^2 + r\,|\varphi|^2 + u\,|\varphi|^4$ with a complex field φ

The RG flow of the LGW Φ^4 theory controls the universal features of the critical behavior, such as the critical exponents

LGW Φ^4 theories

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{N}(\partial_\mu\varphi_i)^2 + r_i\varphi_i^2] + \frac{1}{4!}\sum_{ijkl=1}^{N} u_{ijkl}\,\varphi_i\varphi_j\varphi_k\varphi_l$$

r_i and u_{ijkl} depend on the symmetry

• If criticality is driven by one T-like parameter, and all φ_i become critical, $\sum_i \varphi_i^2$ must be the only invariant quadratic term. Thus $r_i = r$, $\sum_i u_{iikl} \propto \delta_{kl}$, etc...

• The simplest case is the $O(N)$ model, $\mathcal{L} = (\partial_\mu\vec{\varphi})^2 + r\vec{\varphi}^{\,2} + u\,(\vec{\varphi}^{\,2})^2$

• In the absence of a large symmetry, several quartic couplings must be considered.

• Several physically interesting systems require multi-parameter Φ^4 theories, disorder and frustrated systems, spin-wave density models, finite-T transition in hadronic matter, competing orderings, etc...

• all Φ^4 theories are trivial for $D = 4$, as O(N) models

RG flow can be studied by perturbative approaches in QFTs

We are interested in the critical behavior of the "bare" correlation functions $\Gamma_n(p; r, u, \Lambda)$ of the theory $\mathcal{L} = (\partial_\mu \vec{\varphi})^2 + r\vec{\varphi}^{\,2} + u\,(\vec{\varphi}^{\,2})^2$

Note that in high-energy physics the physical correlations are the renormalized ones

- Massive (disordered phase) zero-momentum renormalization scheme

$$\Gamma_2(p) = Z_\varphi^{-1}[m^2 + p^2 + O(p^4)], \quad \Gamma_4(0) = Z_\varphi^{-2} m^{4-d} g, \quad \Gamma_{2,1}(0) = Z_t^{-1}$$

which relate the renormalized parameters m, g to the bare r, u.

Renormalizable for $d = 4$, super-renormalizable for $d < 4$
(the 3D Φ^4 theory is finite when written in terms of u/m^{4-d}, neverthless the renormalized quartic coupling g turns out to be useful even in 3D)

- The critical limit $m \to 0$ can be studied by the Callan-Symanzik RG equations for $\Gamma_n^{(r)}(p; m, g) = Z_\varphi^{n/2}\Gamma_n(p; r, u, \Lambda)$, i.e.

$$\left[m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} - \frac{1}{2}n\eta_\varphi(g)\right]\Gamma_n^{(r)}(p) = [2 - \eta_\varphi(g)]m^2\Gamma_{n,1}^{(r)}(p; 0)$$

- The RG functions $\beta(g) = m\partial g/\partial m$ and $\eta_{\varphi,t}(g) = \partial \ln Z_{\varphi,t}/\partial \ln m$ can be perturbatively computed as power series of g
(computed up to six, seven loops by Nickel etal for O(N) models, requiring the calculation of $O(10^3)$ Feynman diagrams; then extended to more complicated Φ^4 theories with multiparameter quartic potentials)

- when $m \to 0$ the coupling g is driven toward an infrared-stable fixed point, i.e., a zero g^* of the β-function $\beta(g) \approx -\omega(g^* - g)$

- Using the RG eqs, one identifies: $\eta = \eta_\varphi(g^*)$, $1/\nu = 2 - \eta_\varphi(g^*) + \eta_t(g^*)$, and compute the critical exponents perturbatively

- The perturbative FT expansions are asymptotic: they must be resummed before evaluating at g^*, exploiting Borel summability and knowledge of the large-order behavior (appropriate instanton solutions)

- Other perturbative schemes have been exploited, such as the $\overline{\mathrm{MS}}$ renormalization scheme defined at the critical point $T = T_c$ (massless theory), $\epsilon \equiv 4 - d$ expansion, etc... (computed up to five loops for O(N) models; then extended to more complicated Φ^4 models)

Φ^4 theory for the 3D Ising model

ex. : Ising model $H_{\rm is} = -J \sum_{\langle ij \rangle} s_i s_j, \quad s_i = \pm 1, \qquad Z = \sum_s e^{-H_{\rm is}/T}$

$\longrightarrow$ Z_2 symmetry (which may arise dynamically, e.g. liquid-vapor transitions)

Universal critical properties encoded by the most general Φ^4 theory with the same symmetry, up to fourth order in the fields

$$\longrightarrow \quad \mathcal{L}_{\rm LGW} = (\partial_\mu \varphi)^2 + r\varphi^2 + u\varphi^4, \qquad Z = \int D\varphi \, e^{-\int d^3x \, \mathcal{L}_{\rm LGW}}$$

only renormalizable terms are RG relevant, higher-order terms are irrelevant

critical behavior controlled by the stable fixed point of the RG flow

	ν	α	η	β
PQFT: 6,7-l MZM	0.6304(13)	0.109(4)	0.034(3)	0.326(1)
PQFT: $O(\epsilon^5)$ exp	0.6290(25)	0.113(7)	0.036(5)	0.326(3)
Experiments: liquid-vapour	0.6297(4)	0.111(1)	0.042(6)	0.324(2)
Experiments: fluid mixtures	0.6297(7)	0.111(2)	0.038(3)	0.327(3)
Experiments: uniaxial magnets	0.6300(17)	0.110(5)		0.325(2)
Lattice: high-T expansion	0.63012(16)	0.1096(5)	0.0364(2)	0.3265(1)
Lattice: Monte Carlo	0.63020(12)	0.1094(4)	0.0368(2)	0.3267(1)
Lattice: Monte Carlo	0.63002(10)			
CFT: conformal bootstrap	0.629(1)		0.0341(5)	
CFT: conformal bootstrap	0.629977(24)	0.11007(7)	0.036302(12)	0.326423(16)

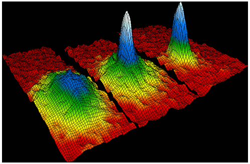
Bose-Einstein condensation in atomic gases and liquid ^{4}He

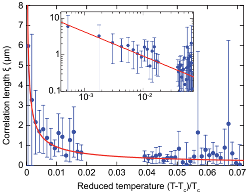
BEC in gases when

$$\lambda_{\text{de Broglie}} = \left(\frac{2\pi\hbar^2}{mT}\right)^{1/2} \approx d_{\text{atoms}} = (N/V)^{-1/3}$$

recently observed in weakly interacting gases, made of alkali atoms, rubidium, sodium, lithium

velocity distribution of rubidium atoms $\longrightarrow$





evidence of a critical behavior

$\longleftarrow$ the correlation length of critical modes at BEC of a trapped atomic gas

(Donner etal, Science 2007)

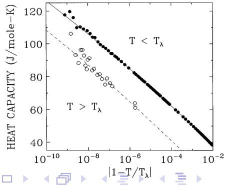
BEC at the normal-to-superfluid transition of ^{4}He

strongly interacting (9% atoms condense for $T \to 0$)

specific heat of liquid ^{4}He up to a few nK from T_c $\longrightarrow$

$\alpha = -0.0127(3)$ from $C_v \approx a + b|t|^\alpha$

(space-shuttle experiment in micoigravity conditions, Lipa etal, PRL 1996 + 2000 + PRB 2003)



Example of BEC transitions: The Bose-Hubbard model on a cubic lattice

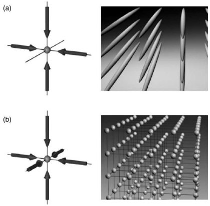
Bosonic gas defined from the lattice annihilation-creation operators $b_{\mathbf{x}}, b^\dagger_{\mathbf{x}}$ and

$$H_{\rm BH} = -\frac{J}{2}\sum_{\langle \mathbf{xy}\rangle}(b^\dagger_{\mathbf{x}} b_{\mathbf{y}} + \text{h.c.}) + U\sum_{\mathbf{x}} n_{\mathbf{x}}(n_{\mathbf{x}}-1) - \mu\sum_{\mathbf{x}} n_{\mathbf{x}}$$

$$[b_{\mathbf{x}}, b^\dagger_{\mathbf{y}}] = \delta_{\mathbf{xy}}, \qquad n_{\mathbf{x}} \equiv b^\dagger_{\mathbf{x}} b_{\mathbf{x}}, \qquad \text{U}(1)\ \ \text{sym}: \quad b_{\mathbf{x}} \to e^{i\theta} b_{\mathbf{x}}$$

BH models describe bosonic atoms loaded in optical lattices (arrays of microscopic potentials induced by ac Stark effects of interfering laser beams) (D. Jaksch etal, PRL 1998)

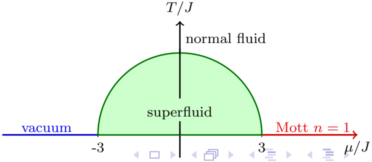
The capability of varying the confining potential allows to vary the spatial geometry, achieving also quasi-1D geometries



finite-T normal-to-superfluid transitions arising from BEC

$\langle b^\dagger_{\mathbf{x}} b_{\mathbf{y}}\rangle \to M^2$ at large distance

phase diagram of the 3D BH model in the hard-core $U \to \infty$ limit



Landau-Ginzburg-Wilson Φ^4 QFT approach to BEC transitions

Universal behaviors from LGW Φ^4 QFT sharing global properties:
dimensionality (3D at finite T), order-parameter nature, symmetry breaking

BEC gives rise to the symmetry breaking U(1) $\rightarrow$ Z_2

$b_{\mathbf{x}} \longrightarrow \varphi(\mathbf{x})$ complex order-parameter field of the LGW theory

$$Z = \int [d\varphi(\mathbf{x})] \exp[-\int d^d x\, \mathcal{L}_{\rm LGW}], \qquad \mathcal{L}_{\rm LGW} = |\partial_\mu \varphi|^2 + r\,|\varphi|^2 + u\,|\varphi|^4$$

BEC of atomic gases, superfluid transition in ^{4}He, superconductors, liquid crystals, easy-plane magnets, etc...

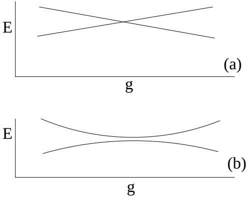
Some of the best results for the U(1)-symmetric 3D XY universality class.
$*$ marks results obtained using the hyperscaling relation $2 - 3\nu = \alpha$

			ν	η
Experiment	^{4}He	Lipa etal, 1996, 2000, 2003	0.6709(1)*	
Φ^4 QFT	6,7-loops MZM	Nickel, Zinn-Justin,...,1976, 1998	0.6703(15)	0.035(3)
	5-loop $O(\epsilon^5)$ exp	Guida, Zinn-Justin, 1998	0.6680(35)	0.038(5)
Lattice	MC+HT	CHPV 2006	0.6717(1)	0.0381(2)
	MC	Burovski etal, 2006	0.6717(3)	

$\longrightarrow$ note however a significant discrepancy between experiments and lattice results driven by quantum fluctuations $\longrightarrow$ singular low-energy properties

$\Longrightarrow$ Nonanaliticity of the ground-state energy with respect to one of model parameters, where the gap Δ vanishes in the large-volume limit

Consider a quantum many-body theory described by the Hamiltonian $H = H_0 + gH_1$. Two scenarios:



• Level crossing if $[H_0, H_1] = 0$.

• More interestingly: avoided level crossing between the ground state and the first excited state, which closes approaching the infinite volume limit

$\Longrightarrow$ leading to a nonanaliticity at $g = g_c$

Quantum critical behavior describes the interplay between quantum and thermal fluctuations at low T around the $T = 0$ quantum critical point

Continuous QPT $\longrightarrow$ diverging length scale ξ, and scaling properties, described by the RG scaling theory

Example: the quantum Ising chain in a transverse field

$$Z = \mathrm{Tr}\, e^{-H/T}, \qquad H = -J \sum_i \sigma_i^x \sigma_{i+1}^x - g \sum_i \sigma_i^z - h \sum_i \sigma_i^x,$$

For $h = 0$:

- $g \to +\infty \longrightarrow$ GS=$\prod_i |\uparrow_i\rangle$ ($|\uparrow_i\rangle$ and $\downarrow_i\rangle$ are eigenstate of σ_i^z)
- $g = 0 \longrightarrow$ two degenerate ground states $\prod_i |\rightarrow_i\rangle$ and $\prod_i |\leftarrow_i\rangle$

Such phases extended to finite g, giving rise to two different quantum ordered and disordered phases.

- Continuous QPT at $g = g_c = J$ and $h = 0$ between the quantum paramagnetic ($g > g_c$) phase and the ordered ($g < g_c$) phase breaking Z_2

2D Ising universality class: $\Delta \sim \xi^{-1} \sim |g - g_c|$

- First-order transition driven by h at the coexistence curve $g < g_c$, with a discontinuity of the magnetization $\langle \sigma_i^x \rangle$ of the ground state.

Universality and RG theory extend to continuous QPT

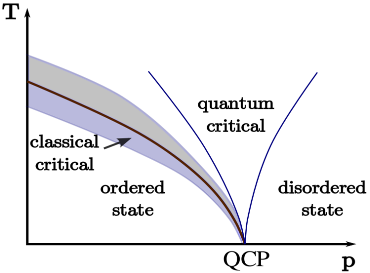
Consider a CQPT characterized by a relevant parameter g to be tuned to approach the quantum transition point, with $y_g \equiv 1/\nu$

$$\bar{g} = g - g_c \Longrightarrow \qquad \xi \sim |\bar{g}|^{-\nu}, \qquad \Delta \sim |\bar{g}|^{z\nu} \sim \xi^{-z}$$

The temperature represents a *relevant* parameter: $\xi \sim T^{-1/z}$ at g_c.

Scaling law of the free energy $F(\mu, T) = b^{-(d+z)} F(\bar{g} b^{1/\nu}, T b^z)$

Typical 3D phase diagram $\longrightarrow$



• classical description at the finite-T transition line, when $\hbar\omega_{\text{crit}} < k_B T$

• Quantum scaling laws describe the critical behavior around the QCP, arising from the interplay between thermal and quantum fluctuations

Like classical transitions, global features determine the critical behavior, such as spatial dimensions, symmetry breaking pattern, etc

RG scaling and finite-size scaling at a continuous quantum transition

- Assuming the quantum critical point at $T = 0,\ \mu_c = 0,\ h = 0$,

$$F(L,T,\mu,h) = F_{\rm reg}(L,T,\mu,h) + F_{\rm sing}(u_l,u_t,u_\mu,u_h,\{v_i\})$$

$u_\mu = \mu + O(\mu^2)$, $u_h = h + b_h\mu h + \ldots$, $u_t \sim T$, $u_l = L^{-1} + bL^{-2} + \ldots$
are nonlinear scaling fields.

- Homogenous scaling law of the free-energy density

$$F_{\rm sing}(u_l,u_t,u_\mu,u_h,\{v_i\}) = b^{-(d+z)} F_{\rm sing}(b\,u_l, b^z u_t, b^{y_\mu} u_\mu, b^{y_h} u_h, \{b^{y_i} v_i\})$$

usually $y_\mu = 1/\nu$ and $y_h = (d+z+2-\eta)/2$

- Taking $b = 1/u_l \approx L \longrightarrow F_{\rm sing} \approx L^{-(d+z)} \mathcal{F}[TL^z, u_\mu L^{y_\mu}, u_h L^{y_h}, \{v_i L^{y_i}\}]$

- The singular part of the free-energy density can be expanded as

$$\begin{aligned} F_{\rm sing} \quad &\approx \quad L^{-(d+z)} \mathcal{F}_0(TL^z, u_\mu L^{y_\mu}, u_h L^{y_h}) + \\ &+ \quad v_1 L^{-(d+z+\omega)} \mathcal{F}_\omega(TL^z, u_\mu L^{y_\mu}, u_h L^{y_h}) + \ldots \end{aligned}$$

Quantum-to-classical mapping

• The temperature may be represented as a further Euclidean dimension of size $1/T$ by techniques analogous to Path Integral

• Quantum-to-classical mapping from d-dim quantum many-body systems to $(d+1)$-dim classical statistical systems

$$Z = \mathrm{Tr}\, e^{-H/T} \quad \longrightarrow \quad Z = \int D\phi\, e^{-\int d^dx\, d\tau\, \mathcal{L}(\phi)}$$

• Again global features, such as the summetry and symmetry breaking pattern, determine the critical behavior

• When $z=1$, the classical statistical model can be chosen isotropic, i.e. such that $\xi_\parallel = \xi$ with rotation $O(d+1)$ (relativistic) invariance.

Ex.: d-dim quantum Ising/Heisenberg models lead to the $(d+1)$-dim QFT $\mathcal{L}_{\mathrm{LGW}} = \int d^d\mathbf{x}\, d\tau\, [(\partial_\mu \vec{\varphi})^2 + r\vec{\varphi}^{\,2} + u(\vec{\varphi}^{\,2})^2]$, thus sharing the same universal features of the $(d+1)$-dim classical Ising/Heisenberg models, such as the critical exponents

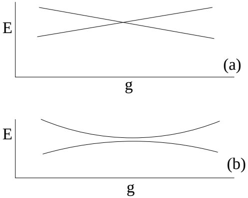
• When the dynamic exponent $z \neq 1$, the corresponding $(d+1)$-dim classical model is anisotropic, i.e. $\xi_\parallel \sim \xi^z$ (the Mott-to-superfluid quantum transition of lattice gases has $z=2$, it is described by a nonrelativistic Lagrangian)

• Simplest scenario: the lowest energy states show a level crossing and observables such as the energy density and magnetization change discontinuously (in the infinite volume limit)

Like classical FOT, no diverging correlation lengths characterize FOQTs in the infinite volume limit (however limits do not commute)

• A level crossing generally occurs in the infinite-volume limit.

• In a finite system, nonvanishing matrix elements among lowest states lift the degeneracy $\longrightarrow$ avoided level crossing.



• Finite-size scaling at FOQTs similar to that at continuous transitions, as expected from the quantum-to-classical mapping of d-dimensional quantum systems onto a classical anisotropic (sometime very anisotropic) $(d+1)$-dimensional systems

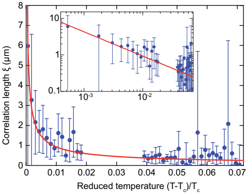
Quantum transitions in several physical contexts

- Quantum magnetism and criticality
- Magnetic excitations of the insulator $LiHoF_4$, quantum Ising transition
- Quantum Heisenberg antiferromagnets, the insulator La_2CuO_4
- High-T superconductors
- Quantum particle systems
- BCS to BEC transition in Fermi atomic systems
- new matter states, such as spin liquids, disorder down to very low T
- Localization transitions, topological phases

......

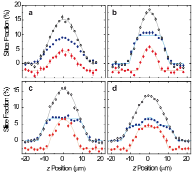
experiments with trapped cold atoms

Experiments with trapped cold atoms show an increasing correlation length compatible with a continuous BEC transition (Donner, etal, Science 2007).
$\nu = 0.67(13)$ from $\xi \sim |T - T_c|^{-\nu}$
(to be compared with $\nu = 0.6717(1)$)



Evidence of BKT transitions (Hung etal, Nature 2010)

Quantum Mott insulator to superfluid transitions and different Mott phases in many experiments (e.g., Fölling etal 2006, Inguscio etal, PRL 2009)



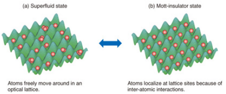
A common feature is a confining harmonic potential, which can be varied to achieve different spatial geometries, allowing also to effectively reduce the spatial dims

Trapping potentials give rise to inhomogeneous conditions which significantly affect the critical behaviors, universal distortion described by trap-size scaling

Quantum Mott transitions of BH models when $T \to 0$

$$H_{\mathrm{BH}} = -\frac{J}{2}\sum_{\langle \mathbf{xy}\rangle}(b_{\mathbf{x}}^\dagger b_{\mathbf{y}} + \mathrm{h.c.}) + U\sum_{\mathbf{x}} n_{\mathbf{x}}(n_{\mathbf{x}}-1) - \mu\sum_{\mathbf{x}} n_{\mathbf{x}}$$

Mott insulators $\langle n_{\mathbf{x}}\rangle = 0, 1, 2, \ldots$ (incompressible $\partial\langle n_i\rangle/\partial\mu = 0$) and superfluid phases

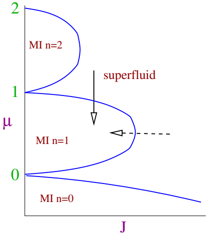


quantum-to-classical mapping: to a $d+1$ QFT

at the transitions driven by μ, nonrelativistic QFT

$$Z = \int [D\phi]\exp(-\int_0^{1/T} dt\, d^d x\, \mathcal{L})$$

$$\mathcal{L} = \phi^*\partial_t\phi + \frac{1}{2m}|\nabla\phi|^2 + r|\phi|^2 + u|\phi|^4$$



(Fisher etal, PRB 1989)

- The upper critical dimension is $d_c = 2$ • Mean field for $d > 2$.
- for $d = 2$ the QFT is free (apart from logs), thus $z = 2, \;\; \nu = 1/2$.
- 1D equivalent to nonrelativistic spinless fermions, thus $z = 2, \;\; \nu = 1/2$.
- The special transitions at fixed integer density (along the dashed line) belongs to the $d+1$ XY universality class (relativistic QFT), thus $z = 1, \;\; \nu = \nu_{XY}$.

Critical phenomena requiring multiparameter LGW Φ^4 theories

associated with more complex order parameters, even matrix-like, with more complex symmetry-breaking patterns

They are constructed by introducing an order-parameter field Φ_i capturing the main features of the critical modes, and imposing a few global properties of the system, keeping terms up to 4th order:

$$\mathcal{L} = \sum_{i=1}^{N} [(\partial_\mu \Phi_i)^2 + r_i \Phi_i^2] + \sum_{ijkl=1}^{N} u_{ijkl}\, \Phi_i \Phi_j \Phi_k \Phi_l$$

UNIVERSALITY CLASSES of critical behaviors determined by • spatial dimension • nature of the critical modes and order parameter • symmetry and symmetry-breaking pattern

For some interesting cases we may have several quadratic and quartic parameters, corresponding to more complex symmetry breakings

The RG flow of the LGW Φ^4 theory controls the universal features of the critical behavior, such as the critical exponents

- MN **model** with a real $M \times N$ matrix field ϕ_{ai}

$$\mathcal{L} = \sum_{i,a} \left[(\partial_\mu \phi_{ai})^2 + r\phi_{ai}^2 \right] + \sum_{ij,ab} \left(u_0 + v_0 \delta_{ij} \right) \phi_{ai}^2 \phi_{bj}^2$$

- **O(M)$\otimes$O(N) model**, with a real $M \times N$ matrix field ϕ_{ai} (M sets of N-comp vectors ϕ_a)

$$\mathcal{L} = \sum_a [(\partial_\mu \phi_a)^2 + r\phi_a^2] + u_0 (\sum_a \phi_a^2)^2 + v_0 \sum_{a,b} (\phi_a \cdot \phi_b)^2$$

- **Spin-density wave model** (Φ_a are complex N-comp vectors)

$$|\partial_\mu \Phi_1|^2 + |\partial_\mu \Phi_2|^2 + r(|\Phi_1|^2 + |\Phi_2|^2) + u_{1,0}(|\Phi_1|^4 + |\Phi_2|^4)$$
$$+ u_{2,0}(|\Phi_1^2|^2 + |\Phi_2^2|^2) + w_{1,0}|\Phi_1|^2|\Phi_2|^2 + w_{2,0}|\Phi_1 \cdot \Phi_2|^2 + w_{3,0}|\Phi_1^* \cdot \Phi_2|^2$$

Several physically interesting examples

• **U(N)⊗U(N) models** (Φ is a complex N×N matrix)

$$\mathcal{L}_U = \mathrm{Tr}\partial_\mu\Phi^\dagger\partial_\mu\Phi + r\mathrm{Tr}\Phi^\dagger\Phi + u_0\left(\mathrm{Tr}\Phi^\dagger\Phi\right)^2 + v_0\mathrm{Tr}\left(\Phi^\dagger\Phi\right)^2$$

• **SU(N)⊗SU(N) models**: $\mathcal{L}_{SU} = \mathcal{L}_U + w_0\left(\det\Phi^\dagger + \det\Phi\right) + x_0\left(\mathrm{Tr}\Phi^\dagger\Phi\right)\left(\det\Phi^\dagger + \det\Phi\right) + y_0\left[(\det\Phi^\dagger)^2 + (\det\Phi)^2\right]$

• The **competition of different orderings** described by more complicated LGW Φ^4 theories containing more quadratic invariants.
Ex.: **O(n_1)⊕O(n_2) theory** with two O(n_1) and O(n_2) vector fields

$$\mathcal{L} = (\partial_\mu\vec{\phi}_1)^2 + (\partial_\mu\vec{\phi}_2)^2 + r_1\vec{\phi}_1^2 + r_2\vec{\phi}_2^2 + u_1(\vec{\phi}_1^2)^2 + u_2(\vec{\phi}_2^2)^2 + w\vec{\phi}_1^2\vec{\phi}_2^2$$

The multicritical behavior is determined by the stable FP when both r_i are tuned to their critical values.

Perturbative calculations in multiparameters LGW Φ^4 theories

RG flow, critical exponents, etc..., by perturbative QFT methods

$$\mathcal{L} = \sum_i [(\partial_\mu \varphi_i)^2 + r_i \varphi_i^2] + \sum_{ijkl} u_{ijkl}\, \varphi_i \varphi_j \varphi_k \varphi_l$$

• Massive (disordered-phase) MZM scheme: expansion in powers of the MZM quartic couplings g_{ijkl}

$$\Gamma^{(2)}_{ij}(p) = \delta_{ij} Z_\varphi^{-1} \left[m^2 + p^2 + O(p^4) \right], \qquad \Gamma^{(4)}_{ijkl}(0) = m\, Z_\varphi^{-2}\, g_{ijkl}$$

• Massless (critical) $\overline{\rm MS}$ scheme: Minimal subtraction within the dimensional regularization, ϵ expansion, $d = 3$ $\overline{\rm MS}$ exp

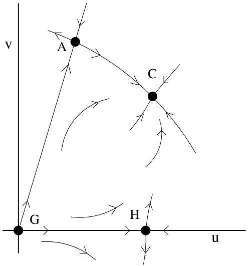
• High-order computations for several LGW Φ^4 theories, to six loops (requiring the calculation $O(10^3)$ diagrams)

• Resummation exploiting Borel summability and calculation of the large-order behavior, by instanton semiclassical calculation

• The comparison of MZM and $\overline{\rm MS}$ expansions checks the results

• The RG flow is determined by the FPs, i.e. common zeroes g^*_{ijkl} of $\beta_{ijkl}(g_{abcd}) \equiv m\partial g_{ijkl}/\partial m$ (MZM), $\beta_{ijkl}(g_{abcd}) \equiv \mu\partial g_{ijkl}/\partial\mu$ ($\overline{\mathrm{MS}}$)

• A FP is stable if all eigenvalues of its stability matrix $S_{ij} = \partial\beta_i/\partial g_j|_{g=g^*}$ have positive real part



• The existence of a stable FP implies that • physical systems with the given global properties can undergo a continuous transition, • the asymptotic behavior in continuous transitions is controlled by the stable FP (apart from cases requiring further tunings)

• The absence of a stable FP predicts 1st-order transitions between the disordered and ordered phases in all systems

• Systems that are outside the attraction domain of the stable FP undergo 1st-order transitions

BEC phenomena in mixtures of bosonic gases

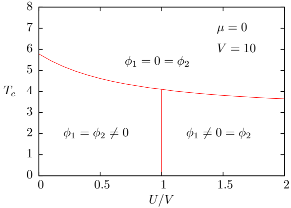
in particular, two-component bosonic gases, such as the BH model

$$-t\sum_s \sum_{\langle \mathbf{xy}\rangle} (b^\dagger_{s\mathbf{x}} b_{s\mathbf{y}} + \mathrm{h.c}) - \mu \sum_{s\mathbf{x}} n_{s\mathbf{x}} + \frac{1}{2} V \sum_{s\mathbf{x}} n_{s\mathbf{x}}(n_{s\mathbf{x}} - 1) + U \sum_{\mathbf{x}} n_{1\mathbf{x}} n_{2\mathbf{x}}$$

symmetric under the transformations $b_{s\mathbf{x}} \to e^{i\alpha_s} b_{s\mathbf{x}}$ and $b_1 \leftrightarrow b_2$

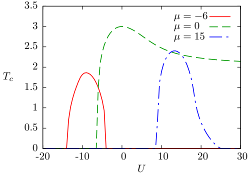
Global symmetry: $\mathbb{Z}_{2,e} \otimes [\mathrm{U}(1) \oplus \mathrm{U}(1)]$

Complex phase diagram in the space of the model parameters



mean-field

hard-core limit $\to$



We may observe the simultaneous condensation of both components or the asymmetric condensation of only one component

Symmetry-breaking patterns at BEC transitions

At the finite-T BEC transition, two-component bosonic systems, such as

$$-t\sum_s \sum_{\langle \mathbf{xy}\rangle} (b^\dagger_{s\mathbf{x}} b_{s\mathbf{y}} + \text{h.c}) - \mu \sum_{s\mathbf{x}} n_{s\mathbf{x}} + \frac{1}{2} V \sum_{s\mathbf{x}} n_{s\mathbf{x}}(n_{s\mathbf{x}}-1) + U\sum_{\mathbf{x}} n_{1\mathbf{x}} n_{2\mathbf{x}}$$

show different spontaneous symmetry breaking of $\mathbb{Z}_{2,e}\otimes[\mathrm{U}(1)\oplus\mathrm{U}(1)]$

$$\mathbb{Z}_{2,e}\otimes[\mathrm{U}(1)\oplus\mathrm{U}(1)] \to \mathbb{Z}_{2,e}\otimes[\mathbb{Z}_2\oplus\mathbb{Z}_2]$$

when both components condense simultaneously, $\langle b_{1\mathbf{x}}\rangle = \langle b_{2\mathbf{x}}\rangle \neq 0$

$$\mathbb{Z}_{2,e}\otimes[\mathrm{U}(1)\oplus\mathrm{U}(1)] \to \mathrm{U}(1)\oplus\mathbb{Z}_2$$

when only one component condenses, e.g. $\langle b_{1\mathbf{x}}\rangle \neq 0, \;\; \langle b_{2\mathbf{x}}\rangle = 0$

Different symmetry-breaking patterns generally imply distinct universality classes of the critical behaviors

LGW Φ^4 QFT of two-comp bosonic gases with sym $\mathbb{Z}_{2,e} \otimes [\mathrm{U}(1) \oplus \mathrm{U}(1)]$

- a complex scalar field $\varphi_s(\mathbf{x})$ with each bosonic component
- most general Φ^4 theory consistent with symmetry $\mathbb{Z}_{2,e} \otimes [\mathrm{U}(1) \oplus \mathrm{U}(1)]$

$$\mathcal{L}_{\mathrm{LGW}} = \sum_{s,\mu} |\partial_\mu \varphi_s|^2 + r \sum_s |\varphi_s|^2 + g \sum_s |\varphi_s|^4 + 2u\, |\varphi_1|^2 |\varphi_2|^2$$

the minima of its potential lead to the symmetry breakings

$$\mathbb{Z}_{2,e} \otimes [\mathrm{U}(1) \oplus \mathrm{U}(1)] \to \mathbb{Z}_{2,e} \otimes [\mathbb{Z}_2 \oplus \mathbb{Z}_2] \qquad \text{for } g - u > 0$$

when $\langle \varphi_1 \rangle = \langle \varphi_2 \rangle \neq 0$ for $r < 0$

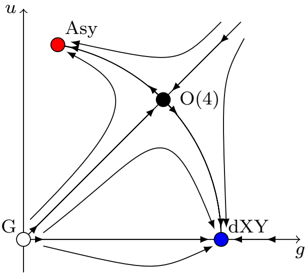
$$\mathbb{Z}_{2,e} \otimes [\mathrm{U}(1) \oplus \mathrm{U}(1)] \to \mathrm{U}(1) \oplus \mathbb{Z}_2 \qquad \text{for } \; g - u < 0$$

when $\langle \varphi_1 \rangle = 0$ and $\langle \varphi_2 \rangle \neq 0$ (or viceversa) for $r < 0$

The two regions are separated by the line $g = u$, equivalent to the O(4) vector model, thus O(4)→O(3)

Critical behaviors inferred from the RG flow of the LGW Φ^4 QFT

$$\mathcal{L}_{\rm LGW} = \sum_{s,\mu} |\partial_\mu \varphi_s|^2 + r \sum_s |\varphi_s|^2 + g \sum_s |\varphi_s|^4 + 2u\, |\varphi_1|^2 |\varphi_2|^2$$



the stable FPs control the critical behaviors
→ universality classes of continuous transitions sharing the same symmetry-breaking pattern

- RG study by computing and resumming high-order (up to six loops) perturbative expansion of the β functions associated with g and u

- both φ_s condense (hard-core regime $V \gtrsim U$): decoupled XY FP
  → asymptotic decoupling of XY critical modes, $\nu = 0.6717$, interactions give rise to slowly-decaying scaling corrections $O(\xi^{-0.022})$

- one φ_s condenses (soft-core $V \lesssim U$): asymmetric coupled FP
  → univ. class of chiral transitions in frustrated spin models, with $\nu \approx 0.6$.

- the Gaussian and O(4) FPs are unstable

- Systems outside the attraction domains of stable FPs undergo FO transitions

Numerical checks for the two-component BH models

$$H_{2\mathrm{BH}} = -t \sum_s \sum_{\langle \mathbf{xy} \rangle} (b^\dagger_{s\mathbf{x}} b_{s\mathbf{y}} + \mathrm{h.c}) - \mu \sum_{s\mathbf{x}} n_{s\mathbf{x}} + \frac{1}{2} V \sum_{s\mathbf{x}} n_{s\mathbf{x}} (n_{s\mathbf{x}} - 1) + U \sum_{\mathbf{x}} n_{1\mathbf{x}} n_{2\mathbf{x}}$$

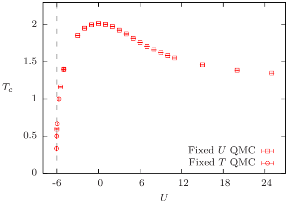
In the hard-core $V \to \infty$ limit, BEC with condensation of the two components

RG predicts critical behaviors controlled by the decoupled 3D XY FP

- diverging length scale $\xi \sim |T - T_c|^{-0.6717}$ of critical correlations
- slowly-decaying corrections $O(\xi^{-0.022})$, much slower than those at the standard 3D XY FP for a single-component model, which are $O(\xi^{-0.78})$

QMC simulations support this scenario

the slowly-decaying corrections turn out to be significant in the attractive $U < 0$ region, while they are small in the repulsive region



Multicritical behaviors in the case of nonidentical components

$$-\sum_s t_s \sum_{\langle \mathbf{xy}\rangle} (b^\dagger_{s\mathbf{x}} b_{s\mathbf{y}} + \mathrm{h.c}) - \sum_s \mu_s \sum_{\mathbf{x}} n_{s\mathbf{x}} + \sum_s \frac{1}{2} V_s \sum_{\mathbf{x}} n_{s\mathbf{x}}(n_{s\mathbf{x}} - 1) + U \sum_{\mathbf{x}} n_{1\mathbf{x}} n_{2\mathbf{x}}.$$

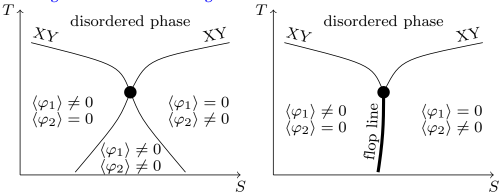
various phases with transition lines where only one component condenses

Multicritical behavior from the competition of the two distinct U(1) orderings at the intersection of their transition lines

$\Longrightarrow$ Most general U(1)$\oplus$U(1) symmetric Φ^4 theory

$$\mathcal{L}_{\mathrm{LGW}} = \sum_{s,\mu} |\partial_\mu \varphi_s|^2 + \sum_s r_s |\varphi_s|^2 + \sum_s g_s |\varphi_s|^4 + 2u\, |\varphi_1|^2 |\varphi_2|^2$$

Multicritical behaviors when r_1 and r_2 are tuned to the intersection of the lines



• in the presence of the mixed phase: the intersection point presents a tetracritical behavior controlled by a decoupled XY FP

• in its absence, bicritical point controlled by another fixed point

Finite-T transition of QCD with N_f light quarks

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \sum_{f=1}^{N_f}\bar\psi_f\,(i\gamma_\mu D_\mu - m_f)\,\psi_f$$

$$Z = {\rm Tr}\,e^{-\beta H} = \int DAD\bar\psi D\psi\;\exp(-S/g^2), \quad S = \int_0^\beta dt\int d^3x\,\mathcal{L}_{\rm QCD}$$

Chiral symmetry for $m_f \to 0$: $\psi_{L,R} \to U(N_f)_{L,R}\psi_{L,R}$

$U(N_f)_L \otimes U(N_f)_R \;\simeq U(1)_V \otimes U(1)_A \otimes SU(N_f)_L \otimes SU(N_f)_R$

where $U(1)_V$ quark-number conservation, $U(1)_A$ broken by the anomaly

- Symm. breaking due to $\langle\bar\psi\psi\rangle \Longrightarrow SU(N_f)_L \otimes SU(N_f)_R \to SU(N_f)_V$

Phase transition at $T_c \simeq 200$ Mev restoring chiral symmetry

- Order parameter $\Longrightarrow \Phi_{ij} = \bar\psi_{L,i}\psi_{R,j}$, a $N_f \times N_f$ complex matrix

The nature of the transition depends on N_f

The nature of the finite-T transition in QCD can be investigated using renormalization-group methods based on universality

(A) Let us assume that the transition is continuous, when $\xi \gg 1/T_c$ the system is effectively 3D, then its critical behavior should belong to a 3D universality class characterized by the same symmetry breaking pattern.

- SU$(N_f)_L \otimes$SU$(N_f)_R \to$ SU$(N_f)_V$
- Complex $N_f \times N_f$ matrix order parameter
- If U$(1)_A$ is effectively restored at $T_c \Longrightarrow$ U$(N_f)_L \otimes$U$(N_f)_R \to$ U$(N_f)_V$

(B) The most general 3D LGW Φ^4 theory compatible with the given symmetry breaking pattern provides an effective theory of the critical modes at T_c.

- Neglecting U$(1)_A$ anomaly,

$$\mathcal{L}_{\mathrm{U}(N)} = \mathrm{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r\mathrm{Tr}\Phi^\dagger\Phi + u_0 \left(\mathrm{Tr}\Phi^\dagger\Phi\right)^2 + v_0 \mathrm{Tr}\left(\Phi^\dagger\Phi\right)^2$$

- if Φ_{ij} is a complex $N \times N$ matrix, $v_0 > 0$, U$(N)_L \otimes$U$(N)_R \rightarrow$U$(N)_V$

- SU$(N_f)_L \otimes$SU$(N_f)_R$ $\rightarrow$SU$(N_f)_V$ due to the anomaly can be realized by adding further terms $\Longrightarrow$ $\mathcal{L}_{\mathrm{SU}(N)} = \mathcal{L}_{\mathrm{U}(N)} + w_0 \left(\det\Phi^\dagger + \det\Phi\right)$

- Nonvanishing quark masses correspond to an external field H_{ij} coupled to Φ_{ij}, by adding $\mathrm{Tr}\left(H\Phi + \mathrm{h.c.}\right)$

(C) A necessary condition of consistency with the initial hypothesis **(A)** of continuous transition is the existence of a stable FP in the corresponding 3D Φ^4 theory

• If no stable FP's exist, the transition of QCD is predicted to be first order

• If a stable 3D FP is found, the transition can be continuous, and its universal critical behavior is determined by the FP. But, it may still be first order if the system is outside the attraction domain of the stable FP.

QCD$_{N_f=2}$, no anomaly

Symmetry breaking: U(2)⊗U(2) $\Longrightarrow$ U(2)

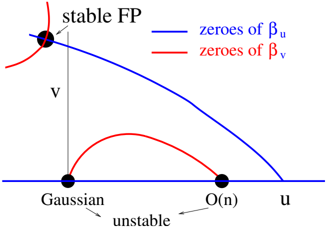
The corresponding universality class exists if there is a stable FP in the 3D U(2)⊗U(2) theory with a complex 2×2 matrix field Φ

$$\mathcal{L}_{\mathrm{U}(N)} = \mathrm{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r\mathrm{Tr}\Phi^\dagger\Phi + u_0 \left(\mathrm{Tr}\Phi^\dagger\Phi\right)^2 + v_0 \mathrm{Tr}\left(\Phi^\dagger\Phi\right)^2$$

RG flow shows that a stable FP exists

$\Longrightarrow$ 3D U(2)⊗U(2)/U(2) universality class exists

It implies that the transition can be continuous with $\nu \approx 0.7$, $\eta \approx 0.1$

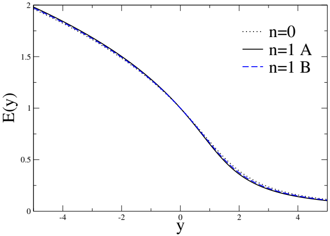


QCD$_{N_f=2}$ taking into account the U(1)$_A$ anomaly

Symmetry breaking $\Longrightarrow$ SU(2)$\otimes$SU(2)/SU(2) $\simeq$ O(4)/O(3)

This corresponds to the O(4) universality class. This means that, if the transition is continuous, it must show the O(4) scaling behavior

$$\vec{M} \propto \vec{H}|H|^{(1-\delta)/\delta}E(y)$$
$$y \propto t|H|^{-1/(\beta+\delta)}$$
$$\langle\bar{\psi}\psi\rangle \propto |M|, \quad m_f \propto |H|$$
$$\delta = 4.789(6), \quad \beta = 0.3882(10)$$



$\Longrightarrow$ The corresponding LGW Φ^4 theory is more complicated

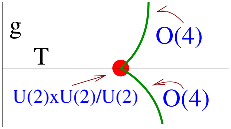
Taking into account the U(1)$_A$ anomaly

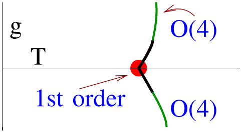
$$\mathcal{L}_{\mathrm{SU}(2)} = \mathcal{L}_{\mathrm{U}(2)} + w_0\left(\det\Phi^\dagger + \det\Phi\right) + x_0\left(\mathrm{Tr}\Phi^\dagger\Phi\right)\left(\det\Phi^\dagger + \det\Phi\right) + y_0\left[(\det\Phi^\dagger)^2 + (\det\Phi)^2\right],$$

where w_0,x_0,$y_0 \sim g \Longrightarrow$ effective breaking of U(1)$_A$

TWO mass terms: transition lines in the T-g plane meeting at a MCP controlled by the U(2)$_L\otimes$U(2)$_R$ theory for $g = 0$

depending whether it is continuous or first order at $g = 0$





O(4) critical behavior if the transition is continuous and $g \neq 0$.

A mean-field behavior may occur for particular values of g in the right case.

If $|g|$ is small (a suppression of anomaly effects around T_c is suggested by MC), we may have crossover effects controlled by the U(2)$\otimes$U(2) MCP
$[\mathcal{F}_{\mathrm{sing}} \approx t^{3\nu} f(g t^{-\phi}),\ \nu \approx 0.7,\ \phi \approx 1.5]$

- QCD with $N_f > 2$ light flavors

$$\mathcal{L} = \mathrm{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r\mathrm{Tr}\Phi^\dagger\Phi + \\ u_0 \left(\mathrm{Tr}\Phi^\dagger\Phi\right)^2 + v_0 \mathrm{Tr}\left(\Phi^\dagger\Phi\right)^2 + w_0 \left(\det\Phi^\dagger + \det\Phi\right)$$

The high-order analysis does not show any stable FP for $N \geq 3$, therefore the transition in QCD with $N_f \geq 3$ is predicted to be first order

Summary of predictions for the finite-T transition of QCD

	QCD	no anomaly, $N_c \to \infty$
	$\mathrm{SU}(N_f) \otimes \mathrm{SU}(N_f)$	$\mathrm{U}(N_f) \otimes \mathrm{U}(N_f)$
$N_f = 1$	crossover or first order	O(2) or first order
$N_f = 2$	O(4) or first order	U(2)$_L \otimes$U(2)$_R$/U(2)$_V$ or first order
$N_f \geq 3$	first order	first order

- In th case of two light flavors: • MC simulations show a crossover at $m_f > 0$ around their physical values; • the transition is still controversial in the chiral limit.
- First-order for $N_f \geq 2$.