1. Master equation and non-symmetric Hamiltonians

Consider the generic master equation

$$\frac{\partial}{\partial t}P(C,t) = \sum_{C'} \{W(C' \to C)P(C',t) - W(C \to C')P(C,t)\} \equiv -\sum_{C'} H_{CC'}P(C',t). \quad (1)$$

Show that:

(a) the "Hamiltonian" H *is not* symmetric;

(b) if the condition of detailed balance

$$P_{eq}(C)W(C \to C') = P_{eq}(C')W(C' \to C) \quad (2)$$

is satisfied, then a suitable "change of basis" M in the space of configurations can be done such that the transformed Hamiltonian $\widetilde{H} \equiv MHM^{-1}$ is symmetric. Determine the form of M.

2. Branching and decay process

Consider the branching and decay process in zero spatial dimension, with

$$A \xrightarrow{\sigma} A + A \quad \text{and} \quad A \xrightarrow{\mu} \emptyset, \quad (3)$$

assuming that the population consists of n_0 individuals at time $t = 0$.

(a) Write down the master equation for the evolution of the probability $P_n(t)$ of this process to have n individuals in the population at time t (see lecture).

Introduce the generating function $g(x,t) = \langle x^n \rangle_t$, where $\langle \cdot \rangle_t$ is calculated at time t. From the master equation, derive the evolution equation for

(b) $g(x,t)$ and solve it with the method of characteristics;

(c) $\langle n \rangle$ and solve it.

Assume, now, that a lethal competition occurs among individuals, with

$$A + A \xrightarrow{\lambda} A. \quad (4)$$

(d) Write down the master equation corresponding to Eqs. (3) and (4). Is the corresponding Hamiltonian H symmetric (see Problem 1)? Are there values of σ, μ, and λ for which this happens? Why?

(e) As at point (c) above, from the master equation derive the evolution equation for $\langle n \rangle$ and show that it involves higher-moments of the variable n. Within a *mean-field approximation* $\langle n^2 \rangle \simeq \langle n \rangle^2$ etc. solve the evolution equation and discuss the qualitative features of the process (e.g., $\langle n \rangle$ at long times) for fixed λ as μ and σ vary.