From quantum to classical integrability: applications to field theories

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CONTENTS

I. INTRODUCTION TO CLASSICAL INTEGRABLE MODELS

A. Classical phase space

We start considering a classical system with a finite dimension n. In this case, the phase space $\mathcal{S}$ is a $2n$-dimensional manifold parametrized in the usual way by $2n$ real variables organized into two subsets:

$$p_i \quad i = 1, \dots, n \tag{1}$$

$$q_i \quad i = 1, \dots, n \tag{2}$$

respectively the generalized momenta and positions. We will use bold symbols to label group of variables, e.g. $\boldsymbol{q}$ and $\boldsymbol{p}$. Any function F on the phase space $\mathcal{S}$ is therefore parameterized as a function over the two sets: $F(\boldsymbol{q}, \boldsymbol{p})$. The phase space is promoted to a symplectic manifold introducing the Poisson bracket, defined for two functions $F(\boldsymbol{q}, \boldsymbol{p})$ and $G(\boldsymbol{q}, \boldsymbol{p})$ as

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right) \tag{3}$$

which is consistent with the canonical Poisson bracket:

$$\{p_i, q_j\} = \delta_{ij} \ , \qquad \forall i, j = 1, \dots, n \tag{4}$$

We can introduce the Hamiltonian as a particular function $H(\boldsymbol{q}, \boldsymbol{p})$ which is associated with the time-evolution according to

$$\dot{F} = \{H, F\} \tag{5}$$

It is clear from this equation that any function F such that $\{H, F\} = 0$ will remain constant in time. This leads immediately to the notion of Liouville integrability that we explain in the next section.

B. Liouville Integrability

The definition of integrability according to Liouville is associated with the existence of conserved quantities.

Definition I.1. *A classical system (with a $2n$-dimensional phase space) is said to be Liouville integrable if it has n independent conserved quantities F_i in involution:*

$$\{F_i, F_j\} = 0 \quad \forall i, j = 1, \dots, n \tag{6}$$

For independence we imply that the set defined by the simultaneous conditions $F_i = f_i$ for $i = 1, \dots, n$ and $f_i \in \mathbb{R}$ define a n-dimensional sub-manifold of the phase space.

Excercise I.1 ($\star$)**.** *Argue that the number of conserved quantities can never be larger than n and therefore that integrability corresponds to the situation where the number of conserved quantities is maximal.*

It follows that the hamiltonian H has to be dependent on the conserved quantities F_i, so it is a function of them $H = H(\boldsymbol{F})$.

The important point behind this definition of integrability by Liouville is that it is possible to put in direct relation with the notion of solvability. This is the result of the following theorem.

Theorem I.1 (the Liouville theorem)**.** *The solution of the equation of motion of a Liouville integrable system can be obtained by "quadrature".*

Solvability by quadrature means that instead of facing the difficulty of solving the equation of motion, which are partial differential equations, we can find a change of coordinates such that the dynamics trivializes. Then, the only difficulty lies in the change of variable itself. The idea behind the theorem is that one passes from the set $\{\boldsymbol{p}, \boldsymbol{q}\}$ to the new canonical coordinates $\{\boldsymbol{F}, \boldsymbol{\Psi}\}$, still satisfying

$$\{F_i, \Psi_j\} = \delta_{ij} \tag{7}$$

In this coordinate, the dynamics is rather simple as

$$\dot{F}_i = \{H, F_i\} = 0 \ , \qquad \dot{\Psi}_i = \{H, \Psi_i\} = \frac{\partial H}{\partial F_i} = \text{constant} \tag{8}$$

The notion of Liouville integrability already provides the set of functions $F_i = F_i(\boldsymbol{q}, \boldsymbol{p})$ in terms of the original coordinates $\{\boldsymbol{p}, \boldsymbol{q}\}$. So the difficulties lies in the determination of the functions $\Psi_i(\boldsymbol{q}, \boldsymbol{p})$. This is easily done introducing the action, restricted to the submanifold of constant $F_i(\boldsymbol{q}, \boldsymbol{p}) = f_i$, as a differential form:

$$dS|_F = \sum_i p_i dq_i \ . \tag{9}$$

We show that Liouville integrability implies $dS|_F$ is a closed differential form:

$$d^2 S|_F = 0 \quad \text{i.e.} \quad \left.\frac{\partial p_i}{\partial q_j}\right|_F = \left.\frac{\partial p_j}{\partial q_i}\right|_F \tag{10}$$

In order to do so, we introduce two matrices:

$$M_{kj} = \frac{\partial F_k}{\partial p_j} \ , \qquad N_{kj} = \frac{\partial F_k}{\partial q_j} \tag{11}$$

This allows us to rewrite

$$\{F_k, F_l\} = 0 = (NM^t - MN^t)_{k,l} \quad \Rightarrow \quad M^{-1}N = (M^{-1}N)^t \tag{12}$$

Then, on a surface of constant $F_k = f_k$ are constant, we have (thanks to Dini theorem)

$$0 = dF_k = \sum_j \frac{\partial F_k}{\partial p_j} dp_j + \frac{\partial F_k}{\partial q_j} dq_j = 0 \quad \Rightarrow \quad \left.\frac{\partial p_i}{\partial q_j}\right|_F = -(M^{-1}N)_{ij} \tag{13}$$

Being $dS|_F$ a closed form, one can (at least locally) integrate it and define the action S:

$$S(F, q) = \int_{q_0}^{q} \sum_i p_i(F, q) dq_i \tag{14}$$

and we introduce the variables Ψ_i simply as

$$\Psi_j = \left.\frac{\partial S}{\partial F_j}\right|_q \tag{15}$$

Excercise I.2 ($\star\star$)**.** *Show that the coordinates $\{\boldsymbol{F}, \boldsymbol{\Psi}\}$ are canonical*

$$\{F_i, \Psi_j\} = \delta_{ij} \quad \forall i, j = 1, \dots, n \tag{16}$$

(Hint: you need to convert the derivatives with respect to $\boldsymbol{q}$ and $\boldsymbol{p}$ in the Poisson bracket into derivatives with respect to $\boldsymbol{F}, \boldsymbol{q}$ and then use that $\{F_i, F_j\} = 0$.)

Remark I.1. *We see that the difficulty of the problem lies in two main steps*

1. *Finding the functions $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{F}, \boldsymbol{q})$ by inverting the definitions of the conserved quantities $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{q}, \boldsymbol{p})$.*
2. *Obtaining the expressions $\boldsymbol{\Psi} = \boldsymbol{\Psi}(\boldsymbol{F}, \boldsymbol{q}) = \frac{\partial S(\boldsymbol{F}, \boldsymbol{q})}{\partial \boldsymbol{F}}$ and inverting them to obtain $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{F}, \boldsymbol{\Psi})$.*

This gives us a first insight on why integrable systems allow bargaining the difficulty of solving partial differential equations with the difficulty of finding "direct" and "inverse" transformations between the physical coordinates $\{\boldsymbol{p}, \boldsymbol{q}\}$ and the coordinates that trivialize the dynamics $\{\boldsymbol{F}, \boldsymbol{\Psi}\}$.

Remark I.2 (Action-angle variables)**.** *As we discussed before, the Hamiltonian does not play a special role for an integrable system. One can use any conserved quantity F_i to define a time variable evolution. Since there are n of them and they are independent, if the phase space has to be compact one can understand that the phase space can be foliated by n-cycles tori: every time we fix the variables $F_i(\boldsymbol{q}, \boldsymbol{p}) = f_i$, we obtain a n-dymensiona submanifold, wich must have the structure of a n-cycles torus. As a consequence, rather than using the variables $\boldsymbol{F}$, one can combine them into the set*

$$I_j = \frac{1}{2\pi} \int_{c_j} dS|_F \tag{17}$$

obtained integrating the action on each cycle c_j. As $dS|_F$ is a closed form, the integral does not depend on continuous deformation of the cycle and therefore $I_j = I_j(\boldsymbol{F})$, so they are also conserved. Together with the conjugate

$$\theta_j = \left.\frac{\partial S}{\partial I_j}\right|_q \tag{18}$$

one obtains a new set of coordinates called "action-angle" variables $\{\boldsymbol{I}, \boldsymbol{\theta}\}$. In practice, these coordinates $\{\boldsymbol{I}, \boldsymbol{\theta}\}$ have the particular property that once the $I_j(\boldsymbol{q}, \boldsymbol{p}) = i_j$ have been fixed, the resulting manifold is a torus and the conjugate variables θ_j parametrizes the angular value on each cycle of the torus.

One of the main difficulty related to integrable models is that there is no general procedure to say if a given Hamiltonian is integrable or not. According to definition I.1, one has to search for a sufficiently large number of conserved quantities although no general procedure to construct them exist, given the Hamiltonian of the model. For this reason, there are generically two situations where one can effectively use this approach with success:

1. In cases where the number of degrees of freedom is small, one can look for symmetries of the model which give rise to conserved quantities. It may happen that their number equals the number of degrees of freedom; this is the situation of simple problems in classical mechanics;

2. When the number of degrees of freedom becomes large (even infinite as we will discuss in the next section), finding conserved quantities by brute force becomes impossible. However, one follows a different procedure: integrable models are introduced in a constructive way that ensures the existence of the conserved quantities. In this case, one is therefore sure that the model is integrable and all the difficulty is restricted to the inverse problem.

C. Lax pairs

The basic idea is to formulate the dynamics in such a way that the presence of conserved quantities becomes automatic. The basic concept is the one of Lax pair. We consider two matrices L and M, whose entries are functions of the phase space coordinates, i.e. $L_{ij} = L_{ij}(\boldsymbol{q}, \boldsymbol{p})$. Then we introduce a dynamical system whose dynamics is given by

$$\dot{L} = [M, L] = ML - LM \tag{19}$$

where the ML represents the standard matrix product. Note that as this equation only involves a commutator, it is well defined for an abstract Lie algebra and automatically extends to any of its representations. It is very easy to get a formal solution of this equation in the form

$$L(t) = g(t)L(0)g^{-1}(t) \tag{20}$$

where the matrix $g(t)$ is obtained from

$$\dot{g}(t) = Mg(t) \quad \Rightarrow \quad g(t) = \mathcal{T}\left\{\exp \int_0^t M(t)\right\} \tag{21}$$

It follows immediately from (20) that one can obtain conserved quantities by tracing powers of L in the matrix space:

$$F_n(t) \equiv \mathrm{Tr}[L^n(t)] = \mathrm{Tr}[L^n(0)] = F_n(0) \tag{22}$$

Powers of a matrix determine its characteristic polynomial and therefore, its eigenvalues are also conserved quantities.

So we see that any pair of matrices involving functions in the phase space can be used to construct a dynamical system which preserves several conserved quantities. However, this is still not enough to arrive to Liouville integrabiliy. We would still need

1. A Hamiltonian evolution: nothing grants that the dynamics obtained from (19) is indeed equivalent to one coming from a Hamiltonian;

2. Conserved quantities have to be independent and their number must equal the number of degrees of freedom;

3. Conserved quantities have to be in involution among themselves.

For specific examples one can verify that this is the case.

Excercise I.3 ($\star$)**.** *Show that the Lax pair:*

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & -\omega/2 \\ \omega/2 & 0 \end{pmatrix} \tag{23}$$

leads to the dynamics of the harmonic oscillator, with $H = \frac{1}{4}\,\mathrm{Tr}\,L^2$.

II. CLASSICAL FIELD THEORIES

We now pass to consider the situation where the system has an infinite number of degrees of freedom. We will focus on the cases where the dynamics is described by a partial differential equation. The paradigmatic example will be the sinh-Gordon equation:

$$\partial_t^2\phi - \partial_x^2\phi = -\frac{m^2}{g}\sinh g\phi\,. \tag{24}$$

One can imagine this equation as derived in a context similar to the previous one, where now the degrees of freedom q_i, p_i labeled by the index i, have been replaced by $\phi(x), \Pi(x)$ labeled by the continuous index x. This means that the phase space in this case is an infinitely dimensional manifold whose coordinates are the two functions $\phi(x), \Pi(x)$. Indeed, one has similarly the Poisson structure:

$$\{\Pi(x), \phi(x')\} = \delta(x - x') \ . \tag{25}$$

which is extended to arbitrary functions through the product rule

$$\{FG, H\} = F\{G, H\} + \{F, H\}G \ , \tag{26}$$

with F, G, H functionals of $\phi(x)$ and $\Pi(x)$. Of course, this limiting procedure can lead to problems as points close in space cannot vary freely otherwise the derivative in (24) would make no sense. For this reason, we restrict the space of functions to differentiable ones which vanish at infinity. In this way, Eq. (24) can be shown to be the equation of motion associated to the Hamiltonian:

$$H = \int_{-\infty}^{\infty} dx \left(\frac{1}{2}\Pi(x)^2 + \frac{1}{2}(\partial_x \phi(x))^2 + \frac{m^2}{g^2}(\cosh(g\phi) - 1) \right) \ . \tag{27}$$

where one can use integration by parts and neglect boundary terms. In this case, we see that the Hamiltonian is written as the integral of a local density.

Definition II.1 (Local field theory)**.** *A local field theory is defined by a phase space which is the space of functions* $\Pi(x), \phi(x)$ *and a Hamiltonian which is written as the integral of a local density:*

$$H = \int_{-\infty}^{\infty} dx \ h(x) \tag{28}$$

where the local energy density $h(x)$ *is expressed in terms of* $\Pi(x)$*,* $\phi(x)$ *and (a finite number of) their derivatives.*

Remark II.1. *It is clear that a direct conversion of the notion of integrability à-la-Liouville can lead to some problems: as the degrees of freedom are infinitely many, we must require an infinite number of conserved quantities. However, counting such infinities is somehow ambiguous. Once again, an important role is played by locality.*

Definition II.2. *A local field theory is integrable if there are infinitely many independent local conserved quantities in involution:*

$$F_i = \int_{-\infty}^{\infty} dx \ f_i(x) \ , \qquad \{F_i, F_j\} = 0 \tag{29}$$

Remark II.2. *We see that from this definition integrability ends up intimately related with space-time. In particular, it turns out that integrable field theory which are non-trivial exist only in* $1+1$ *space-time theories.*

As we discussed before, it is clearly very difficult to find conserved quantities for a given Hamiltonian H. Invariance under translation ensures that the total momentum of the theory is conserved

$$P = \int dx \ \Pi(x)\partial_x \phi(x) \tag{30}$$

In general, it seems that if few conserved quantities are found (beyond those implied by symmetries as the momentum), the theory ends up being integrable in practice. One can check that

$$Q = 2(\phi_{xt}^2 + \phi_{xx}^2) + \frac{3g^2\phi_t^2\phi_x^2}{4} + \frac{g^2(\phi_t^4 + \phi_x^4)}{8} + \frac{m^2}{2}(\phi_t^2 + \phi_x^2)\cosh(g\phi) + \frac{m^2}{2g^2}\sinh(g\phi)(m^2\sinh(g\phi) - 4g\phi_{xx}) \tag{31}$$

is indeed conserved and gives a strong indication that the model described by (27) is indeed integrable. However, finding an infinite sequence of them require a different approach.

Let's start with something simple.

A. Klein-Gordon field theory

In the limit $g \to 0$, the interaction term in Eq. (27) becomes simply quadratic. The resulting Hamiltonian goes under the name of Klein-Gordon theory and it corresponds to a free model

$$\hat{H} = \frac{1}{2}\int_{-\infty}^{\infty}\left[\Pi(x)^2 + \phi_x(x)^2 + m^2\phi(x)^2\right]dx \ , \tag{32}$$

It is the easiest example we can consider to show how the conserved charges can be constructed. The total momentum carried by the field is

$$\hat{P} = -\int \Pi(x)\,\phi_x(x)\,dx \ . \tag{33}$$

The field $\phi(x)$ and the conjugate momentum $\Pi(x)$ have a the canonical commutator

$$\{\Pi(x,t),\phi(x,t)\} = \delta(x-y) \ . \tag{34}$$

1. Solution of the dynamics

The equation of motion for the field operator is simply

$$(\partial_t^2 - \partial_x^2 + m^2)\hat{\phi}(x,t) = 0 \ . \tag{35}$$

It can be solved going in Fourier transform

$$\phi(x,t) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{1}{\sqrt{2\omega(k)}}\left[A(k)\,e^{-i\omega t+ikx} + A^*(k)\,e^{i\omega t-ikx}\right] \ , \tag{36}$$

and consequently for $\Pi(x,t) = \partial_t\phi(x,t)$, with $\omega(k) = \sqrt{m^2+k^2}$. The modes $A(k)$ and $A^*(k)$ define a new system of coordinates, for which Eq. (34), becomes

$$\{A^*(k), A(k')\} = 2\pi\imath\delta(k-k') \ . \tag{37}$$

Substituting the solution of the equation of motion into the Hamiltonian, one has

$$H = \int \frac{dk}{2\pi}\,\omega(k)|A(k)|^2 \ . \tag{38}$$

The time evolution of this quantities is simply obtained in the standard way using the Poisson bracket and the product rule

$$\dot{A}(k) = \{H, A(k)\} = \int_{-\infty}^{\infty}\frac{dk'}{2\pi}\,\omega(k)\{A^*(k'), A(k)\}A(k') = \imath\omega(k)A(k) \ . \tag{39}$$

which is easily solved by $A(k,t) = A(k)e^{\imath\omega(k)t}$. So in terms of these variables the dynamics trivializes and clearly $|A(k)|^2$ is a conserved quantity. At the classical level, we can see that the Fourier transform acts essentially as the scattering transform: the field variables $\phi(x,t)$, $\pi(x,t)$ are transformed into the new set $A(k), A^*(k)$, whose dynamics is trivial.

2. Local conserved quantities

In the previous section, by solving the dynamics through Fourier transform, we obtained as a byproduct a set of conserved quantities $|A(k)|^2$. Nevertheless, to assess the integrability of the model, we need to find local conserved quantities as introduced in (29). Indeed, the Klein-Gordon theory admits an infinite number of local conservation laws that can be easily derived using the light-cone coordinates τ and σ defined by

$$t = \frac{1}{2}(\tau-\sigma) \ , \qquad x = \frac{1}{2}(\tau+\sigma). \tag{40}$$

In the light-cone variables the equation of motion becomes

$$\phi_{\sigma\tau} = \frac{m^2}{4}\,\phi \ . \tag{41}$$

Taking τ as "time" variable, we have the infinite chain of conservation laws coming from the equation of motion ($n = 1, 2, \cdots$)

$$\partial_\tau \phi^2_{n\sigma} = \frac{m^2}{4}\,\partial_\sigma \phi^2_{(n-1)\sigma} \ , \qquad \partial_\sigma \phi^2_{n\tau} = \frac{m^2}{4}\partial_\tau \phi^2_{(n-1)\tau} \tag{42}$$

where $\phi_{n\sigma} = \frac{\partial^n \phi}{\partial \sigma^n}$ and analogously for $\phi_{n\tau}$. The equations above have the general form $\partial_\tau A = \partial_\sigma B$ and therefore, going back to the original laboratory coordinates (x, t), they can be expressed in terms of the continuity equation

$$\partial_t (A + B) = \partial_x (B - A) \ , \tag{43}$$

so that the conserved charges are $Q = \int dx (A + B)$. For the Klein-Gordon we have then the following set of conserved charges

$$Q_n = \int_{-\infty}^{\infty} dx \left[\phi^2_{n\sigma} + \frac{m^2}{4}\phi^2_{(n-1)\sigma}\right] , \qquad Q_{-n} = \int_{-\infty}^{\infty} dx \left[\phi^2_{n\tau} + \frac{m^2}{4}\phi^2_{(n-1)\tau}\right] . \tag{44}$$

Taking the sum and the difference of these quantities, we can define the even and odd conserved charges

$$\begin{aligned} \mathcal{E}_n &= (Q_n + Q_{-n}) = \int dx \left[\phi^2_{n\sigma} + \phi^2_{n\tau} + \frac{m^2}{4}(\phi^2_{(n-1)\sigma} + \phi^2_{(n-1)\tau})\right] \\ \mathcal{O}_n &= (Q_n - Q_{-n}) = \int dx \left[\phi^2_{n\sigma} - \phi^2_{n\tau} + \frac{m^2}{4}(\phi^2_{(n-1)\sigma} - \phi^2_{(n-1)\tau})\right] \end{aligned} \tag{45}$$

It is now easy to see that they can be expressed in terms of the mode occupation of the field by injecting (36). To do so, it is more convenient to adopt the rapidity variable $k = m \sinh\theta$, arriving to the expressions

$$\mathcal{E}_n = \frac{m^{2n-1}}{2^{2n-2}} \int \frac{d\theta}{2\pi}\, P(\theta)\, \cosh[(2n-1)\theta] \ , \qquad \mathcal{O}_n = \frac{m^{2n-1}}{2^{2n-2}} \int \frac{d\theta}{2\pi}\, P(\theta)\, \sinh[(2n-1)\theta] \tag{46}$$

where we set $P(\theta) = m|A(m\sinh\theta)|^2 \cosh\theta$. You can compare with the derivation that we had done in (90), which match except for few difference of normalization. The first representatives of these expressions correspond to the energy and the momentum of the field. We will see how this expression are interpreted in the quantum field theory interpretation.

B. The inverse scattering method

We have seen how conserved quantities can be found explicitly in the simple example of the free Klein-Gordon equation. In general, the situation is not equally easy and we need a different approach. If we want to apply the idea behind Lax pairs, we must expect that the size of those matrices becomes infinite: only in this way, indeed, the number of conserved quantities can diverge. The formalism of the *inverse scattering method* (ISM) starts from a remarkable observation. The equation of motion is obtained as the compatibility condition of an auxiliary, overdetermined, linear system. To be concrete, one considers the linear system

$$\partial_x F(x,t) = U_\lambda F(x,t) \tag{47a}$$

$$\partial_t F(x,t) = V_\lambda F(x,t) \tag{47b}$$

where $F^t = (f_1, f_2)$ and U_λ and V_λ are linear differential operators. It is easy to check, computing in two possible ways the cross derivative of $F(x,t)$, that for the system to be well defined one needs to impose

$$\partial_t U_\lambda - \partial_x V_\lambda + [U_\lambda, V_\lambda] = 0 \ . \tag{48}$$

If we interpret U_λ, V_λ as a connection on the trivial vector bundle $\mathbb{R}^2 \times \mathbb{C}^2$, we see that (48) is nothing but the *zero-curvature* condition. To state this more explicitly, we introduce the covariant derivative $D_\mu = \partial_\mu - A_\mu$ where $\mu = 0, 1$ and the two values correspond to

$$\begin{cases} \partial_0 - A_0 = \partial_x - U \\ \partial_1 - A_1 = \partial_t - V \end{cases} \tag{49}$$

Then Eq. (48) is equivalent to

$$[D_0, D_1] = 0 \tag{50}$$

or in other words that the differential form $A = A_\mu dx^\mu$ has a vanishing covariant derivative

$$DA = D_\mu A_\nu \, dx^\mu \wedge dx^\nu = 0 \tag{51}$$

An application of the non-Abelian Stokes theorem, then tells us that the Wilson loop associated to the connection A_μ is simply the identity:

$$\mathcal{P}\exp\left[\oint A_\mu dx^\mu\right] = \mathcal{P}\exp\left[\int DA\right] = \mathbf{1} \tag{52}$$

where the symbol $\mathcal{P}\exp$ is the path-ordered exponential, so that operators from right to left corresponds to increasing points in the path.

Let's use (52) in a smart way. Consider a rectangular path as in Fig. 1. We introduce the propagators in the two

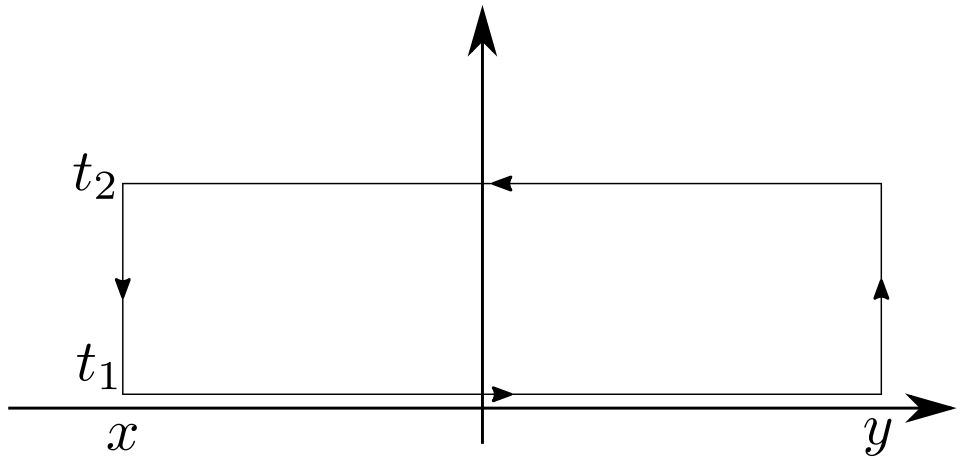

FIG. 1. A closed rectangular loop in the space-time (x, t) plane. The path-ordered exp on this path has to be equal to the identity.

directions:

$$T_\lambda(x, y; t) = \mathcal{P}\exp\int_x^y U_\lambda(x', t)dx' \,, \qquad S_\lambda(t_1, t_2; x) = \mathcal{P}\exp\int_{t_1}^{t_2} V_\lambda(x, t')dt' \tag{53}$$

Clearly the propagator satisfies

$$\partial_y T_\lambda(x, y; t) = U_\lambda(y, t)T(x, y; t) \,, \qquad \partial_x T_\lambda(x, y; t) = T(x, y; t)U_\lambda(x, t) \tag{54}$$

So that the integral over the loop in Fig. 1 becomes simply

$$S_\lambda(t_2, t_1; x)T_\lambda(y, x; t_2)S_\lambda(t_1, t_2; y)T_\lambda(x, y; t_1) = \mathbf{1} \tag{55}$$

Simple inspection of (53) tells us that

$$T_\lambda(x, y; t) = T_\lambda(y, x; t)^{-1} \,, \qquad S_\lambda(t_1, t_2; x) = S_\lambda(t_2, t_1; x)^{-1} \tag{56}$$

which can be used to arrive at

$$S_\lambda(t_1, t_2; y)^{-1}T_\lambda(x, y; t_2)S_\lambda(t_1, t_2; x) = T_\lambda(x, y; t_1) \tag{57}$$

Now we see that if we can find to points x and y such that

$$V_\lambda(x, t) = V_\lambda(y, t) \tag{58}$$

then the operator $L_{x,y}(\lambda) = T_\lambda(x, y; t)$ and the operator $M(\lambda) = V_\lambda(x, t)$ will be a Lax pair for any λ as Eq. (57) coincides with Eq. (20). Moreover $\mathrm{Tr}\, L(\lambda)$ will be time-independent for any λ thus giving an infinite family of conserved quantities. It means that we succeded in the construction of conserved quantities for the dynamics described by (48).

C. Inverse scattering for the sinh-Gordon equation

From the previous section, we conclude that if, for an appropriate choice of U_λ, V_λ, (48) reduces to the equation of motion of a given dynamical system, this turns out to be integrable, i.e. it will admit an infinite set of conserved charges (here for simplicity we ignore the problem of involution between conserved quantities).

Excercise II.1 ($\star$). *Show that setting*

$$U_\lambda = \begin{pmatrix} \frac{g(\phi_x+\phi_t)}{4} & \frac{\lambda^2-e^{g\phi}m^2}{4\lambda} \\ -\frac{\lambda^2-e^{-g\phi}m^2}{4\lambda} & -\frac{g(\phi_x+\phi_t)}{4} \end{pmatrix} \tag{59}$$

$$V_\lambda = \begin{pmatrix} \frac{g(\phi_x+\phi_t)}{4} & \frac{\lambda^2+e^{g\phi}m^2}{4\lambda} \\ -\frac{\lambda^2+e^{-g\phi}m^2}{4\lambda} & -\frac{g(\phi_x+\phi_t)}{4} \end{pmatrix} \tag{60}$$

for a field $\phi(x,t)$ (and its derivatives $\phi_x(x,t) = \partial_x\phi(x,t)$ and $\phi_t(x,t) = \partial_t\phi(x,t)$) Eq. (48) *reduces to* (24) *for the field $\phi(x,t)$*

From this excercise, we see that if $\phi(x,t)$ (and its derivatives) are choosen from a solution of the sinh-Gordon equation given in (24) then (48) will be satisfied at any (x,t). As a consequence, we can construct a family of conserved quantities, according to the procedure explained below Eq. (58). In order to do this, we have to fix the boundary conditions of the problem. Simplest choices are the following:

1. the whole line case: $|x|^\alpha \frac{d^k\phi(x,t)}{dx^k} \to 0$ as $|x| \to \infty$, for any $\alpha > 0$ and $k \in \mathbb{N}$;
2. the periodic case $\phi(x+2L,t) = \phi(x,t)$, where the function can be defined on the restricted spacial domain $[-L, L]$.

D. The whole line case

As we discussed in the Liouville theorem I.1, the main point about integrability lies in the possibility of constructing a new set of coordinates for which the dynamics becomes much simpler. This is essentially what the action-angle variables do. We will now see how this can be achieved for an integrable field theory going to the scattering variables. We consider the linear problem (47a)

$$\partial_x F(x,t) = U_\lambda(x,t)F(x,t) \tag{61}$$

where $F(x,t)$ is a two-dimensional vector defined for $x \in \mathbb{R}$ and $\phi(x,t=0), \phi_t(x,t=0)$ are rapidly decreasing functions. Given any initial vector $F(x_0,t)$, we can get a solution of (61) applying the propagator $T(x_0,x;t)$:

$$F(x,t) = T(x_0,x;t)F(x_0,t)\ . \tag{62}$$

Since $T(x_0,x_0;t) = \mathbf{1}$, we can see that Eq. (61) is satisfied because of (54). Being a first order bidimensional system, the space of solutions is 2-dimensional. It is useful to fix a basis by fixing the behavior of the solution at $x \to \pm\infty$. For $x \to \infty$, we have

$$\mathcal{U}_\lambda = \lim_{x\to\infty} U_\lambda = \frac{i(\lambda^2-m^2)}{4\lambda}\sigma_y \tag{63}$$

for which we can identify a basis of asymptotic solutions ($x \to \infty$) as

$$\psi_\pm(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} e^{\pm ikx} \tag{64}$$

where we introduced the standard parametrization for the rapidity and the momentum, $\lambda = me^\theta$ and $k = \frac{m\sinh\theta}{2} = \frac{1}{4}(\lambda - m^2\lambda^{-1})$. From this, we can identify two basis of solutions $f_\pm, g_\pm$ of (47a), said *Jost solutions*, according to their asymptotic behavior at $x \to +\infty$ or $x \to -\infty$

$$\mathbf{f}_\pm(x,t) \overset{x\to+\infty}{\longrightarrow} \psi_\pm(x) \tag{65}$$

$$\mathbf{g}_\pm(x,t) \overset{x\to-\infty}{\longrightarrow} \psi_\pm(x)\ . \tag{66}$$

These are known as Jost solution in scattering theory. We can arrange the two vectors $\mathbf{f}_\pm$ as the two columns of a matrix (and similarly for $\mathbf{g}_\pm$):

$$\mathfrak{F}(x,t) = \begin{pmatrix}\mathbf{f}_+(x,t) & \mathbf{f}_-(x,t)\end{pmatrix}, \qquad \mathfrak{G}(x,t) = \begin{pmatrix}\mathbf{g}_+(x,t) & \mathbf{g}_-(x,t)\end{pmatrix}, \qquad \mathbf{\Psi}(x) = \begin{pmatrix}\psi_+(x) & \psi_-(x)\end{pmatrix} \tag{67}$$

Note that, since $U_\lambda(x,t)$ is a real matrix and $\psi_\pm(x)^* = \psi_\mp(x)$, we simply deduce (for real λ)

$$\begin{cases} \mathbf{f}_+ = \mathbf{f}_-^* \\ \mathbf{g}_+ = \mathbf{g}_-^* \end{cases} \Rightarrow \begin{cases} \mathcal{F}(x,t)^* = \mathcal{F}(x,t)\sigma_x \\ \mathcal{G}(x,t)^* = \mathcal{G}(x,t)\sigma_x \end{cases} \tag{68}$$

using (62), we can write explicitly this functions as

$$\mathfrak{F}(x,t) = \lim_{x_0\to\infty} T(x_0,x;t)\mathbf{\Psi}(x_0) \tag{69}$$

$$\mathfrak{G}(x,t) = \lim_{x_0\to-\infty} T(x_0,x;t)\mathbf{\Psi}(x_0) \tag{70}$$

Since both are basis for the space of solutions, it is possible to define a change of basis between the two

$$\mathfrak{F}(x,t) = \mathfrak{G}(x,t)\mathcal{T}(\lambda,t) \tag{71}$$

which using (69) leads to

$$\mathcal{T}(\lambda,t) = \mathfrak{G}^{-1}(x,t)\mathfrak{F}(x,t) = \lim_{\substack{x_0\to\infty \\ y_0\to-\infty}} \mathbf{\Psi}(y_0)^{-1} T(x_0,y_0;t)\mathbf{\Psi}(x_0)\,, \tag{72}$$

This operator converts Jost solutions one into the other and can therefore be interpreted as a classical version of the S-matrix.

1. *Evolution of the scattering data*

Let's write explicitly the coefficients of the transfer matrix by studying their properties. It is useful the following

Excercise II.2 ($\star$)**.** *Show that for real λ, the transfer matrix $\mathcal{T}(\lambda)$ can be written in the form*

$$\mathcal{T}(\lambda,t) = \begin{pmatrix} a(\lambda,t) & b^*(\lambda,t) \\ b(\lambda,t) & a^*(\lambda,t)\end{pmatrix} \tag{73}$$

The functions $a(\lambda), b(\lambda)$ give the so-called scattering data. It is useful to investigate what is their time evolution. In order to do so, we take $t_1 = t_2 + \delta t$ in (57) to obtain

$$\frac{dT(x,y;t)}{dt} = V(y,t)T(x,y;t) - T(x,y;t)V(x,t) \tag{74}$$

Now, in order to take for $x, y \to \infty$, we note that

$$\lim_{x\to\pm\infty} V(x,t) = \frac{\imath(\lambda^2+m^2)}{4\lambda}\sigma_y \quad \Rightarrow \quad \lim_{x\to\pm\infty} V(x,t)\mathbf{\Psi}(x) = \lim_{x\to\pm\infty} \frac{\imath(\lambda^2+m^2)}{4\lambda}\mathbf{\Psi}(x)\sigma_z \tag{75}$$

which leads to the equation of evolution for the transfer matrix:

$$\frac{d\mathcal{T}(\lambda,t)}{dt} = \frac{\imath(\lambda^2+m^2)}{4\lambda}\left(\sigma_z\mathcal{T}(\lambda,t) - \mathcal{T}(\lambda,t)\sigma_z\right) \tag{76}$$

which translates into the evolution equations for the coefficiens $a(\lambda), b(\lambda)$:

$$\frac{da(\lambda,t)}{dt} = 0, \qquad \frac{db(\lambda,t)}{dt} = \frac{i(\lambda^2+m^2)b(\lambda,t)}{2\lambda}\,. \tag{77}$$

which immediately tells that $a(\lambda,t) = a(\lambda)$ is constant in time. Moreover, as $\operatorname{Tr} U_\lambda(x,t) = 0$, we have the equality

$$\det\mathcal{T}(\lambda,t) = |a(\lambda)|^2 - |b(\lambda)|^2 = 1\,, \qquad \forall\lambda\in\mathbb{R} \tag{78}$$

which allows us to conclude that also $|b(\lambda,t)|$ is constant in time and that only the argument of $b(\lambda)$ evolves:

$$\frac{d\arg b(\lambda,t)}{dt} = \frac{\lambda^2+m^2}{2\lambda}\,. \tag{79}$$

2. Analytic properties

From the previous consideration, we see that the combinations $t(\lambda) = |1/a(\lambda)|^2$ and $r(\lambda) = |b(\lambda)/a(\lambda)|^2$ can be realized, respectively, as transmission and reflection coefficients satisfying

$$t(\lambda) + r(\lambda) = 1 \,, \qquad \forall \lambda \in \mathbb{R} \tag{80}$$

This shows the spectrum of (47a) is continuous and coincides with the real line in the λ-complex plane. In general, one can extend these definitions on the complex λ plane; the function $a(\lambda)$ is analytical for $\Im\lambda > 0$, except for simple zeros, corresponding to the position of the spectrum where a bound state occurs. For the sinh-Gordon case, no bound states are present. It follows that also $\log a(\lambda)$ will be analytical for $\Im\lambda > 0$ and therefore, $\arg a(\lambda)$ is fixed by its modulus by the dispersion relation

$$\arg a(\lambda) = -\frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{d\lambda' \ln|a(\lambda')|}{\lambda' - \lambda} \tag{81}$$

where $p.v.$ stays for the Cauchy principal value. Using the Sochocki-Plemelj formula

$$\frac{1}{\lambda' - \lambda - \imath\epsilon} = p.v. \frac{1}{\lambda' - \lambda} + \pi\imath\delta(\lambda' - \lambda) \tag{82}$$

we arrive at the "dispersion relation"

$$\log a(\lambda) = \frac{1}{\pi\imath} \int_{-\infty}^{\infty} \frac{d\lambda' \ln|a(\lambda')|}{\lambda' - \lambda - \imath\epsilon} \tag{83}$$

Excercise II.3 $(*)$**.** *Show that for real λ one has*

$$\mathcal{T}(-\lambda) = \mathcal{T}^*(\lambda) \tag{84}$$

(Hint: note that $\sigma_z T_\lambda(x,y;t)\sigma_z = T_{-\lambda}(x,y;t)$.)

Since $a(\lambda)$ is conserved, it can be used to generate conserved quantities, which can be obtained from the Laurent series of $a(\lambda)$ around a certain point. The dispersion relation suggests two possible expansion points $\lambda = 0$ and $\lambda = \infty$:

$$\log a(\lambda) \overset{\lambda\to\infty}{=} \frac{i}{\pi} \sum_{n=0}^{\infty} \lambda^{-n-1} \int_{-\infty}^{\infty} d\lambda' \, (\lambda')^n \log|a(\lambda')| \,, \tag{85}$$

$$\log a(\lambda) \overset{\lambda\to 0}{=} -\frac{i}{\pi} \sum_{n=0}^{\infty} \lambda^{n} \int_{-\infty}^{\infty} d\lambda' \, (\lambda')^{-n-1} \log|a(\lambda')| \,. \tag{86}$$

From, the Excercise II.3, we see that $|a(\lambda)| = |a(-\lambda)|$ and therefore only the odd n give non-trivial term in the sum. Setting

$$I_{2n+1} = \operatorname{sign}(2n+1) \frac{2i}{\pi} \int_0^{\infty} d\lambda \, (\lambda')^{2n} \log|a(\lambda)| \,. \tag{87}$$

which can be combined into

$$E_n = \frac{2i}{g^2}\left[m^{2n+2} I_{-2n-1} - \frac{I_{2n+1}}{m^{2n}}\right], \qquad O_n = \frac{2i}{g^2}\left[\frac{I_{2n+1}}{m^{2n}} + m^{2n+2} I_{-2n-1}\right], \qquad n \geq 0 \tag{88}$$

We now go to the rapidity variable $\lambda = me^{\theta}$ and we introduced the weight

$$P(\theta) = \frac{8}{\pi g^2} \ln|a(\lambda(\theta))| \,. \tag{89}$$

and we deduce the expressions for $n \geq 0$

$$E_n = \int_{-\infty}^{\infty} d\theta \; P(\theta) m \cosh((2n+1)\theta) \,, \qquad O_n = \int_{-\infty}^{\infty} d\theta \; P(\theta) m \sinh((2n+1)\theta) \tag{90}$$

Excercise II.4 ($\star\star\star$). *Show that the conserved quantities E_n and O_n defined above are local, i.e. they have the form of spacial integral of a density (see (29)). In particular the Hamiltonian and the total momentum are the first conserved quantities*

$$\mathcal{H} \equiv E_0 = \int_{-\infty}^{\infty} dx\ \frac{\Pi(x)^2 + \phi_x(x)^2}{2} + \frac{m^2(\cosh(g\phi(x)) - 1)}{g^2}\ , \tag{91a}$$

$$\mathcal{P} \equiv O_0 = -\int_{-\infty}^{\infty} dx\ \Pi(x)\phi_x(x)\ , \tag{91b}$$

In order to do so, you may find useful to parametrize

$$T(x,y;t) = (1 + W(x))e^{Z(x)}C(y)\ , \tag{92}$$

where the matrix $W(x)$ is off-diagonal, $Z(x)$ is diagonal and $C(y)$ is fixed by the condition at $T(x,x;t) = \mathbf{1}$; then Eq. (54) provides differential equations for $W(x)$ and $Z(x)$ which simplify for $x \to \infty$.

Excercise II.5 ($\star\star\star$). *Consider the construction of the scattering data $a(\lambda), b(\lambda)$ from the field configuration $\phi(x,t), \pi(x,t) = \partial_t\phi(x,t)$ as introduced in Sec. II D 1. As the limit $g \to 0$ of (27) coincides with (32), expand the transfer matrix in (72) to find the relation between $a(\lambda), b(\lambda)$ and $A(k), A^*(k)$ in the small g expansion.*

Remark II.3. *We have seen in this section how the scattering transform puts in place the general picture of classical integrable models. In particular, it transforms the space-time data $\phi(x,t), \partial_x\phi(x,t)$ into a new set of coordinates $a(\lambda), b(\lambda)$ which have the much simpler evolution described in Eq. (77). The final step to complete the solution of the dynamics becomes the inverse scattering transform, that allows to recover the field configuration from the scattering data $a(\lambda), b(\lambda)$. A general theory for this problem exists that allows formulating the problem as the Gelfand-Levitan-Marchenko integral equation*[1].

III. QUANTUM INTEGRABILITY

It seems natural when passing from classical to quantum realm, that the definition I.1 can be used to defined similarly a quantum integrable model. This seems to be a well-defined procedure: as usual the coordinates and the momenta are promoted to operators in an Hilbert space $\mathcal{H}$

$$(q_i, p_i) \quad \to \quad (\hat{q}_i, \hat{p}_i)\ , \qquad [\hat{q}_i, \hat{p}_j] = \imath\hbar\delta_{i,j} \tag{93}$$

In general, all the Poisson brackets would be promoted to commutators; therefore one could say that *a quantum system evolving with the Hamiltonian $\hat{H}$ is integrable if one can finds conserved operators $\hat{F}_i$ in involution*

$$[\hat{F}_i, \hat{F}_j] = 0\ , \quad [\hat{F}_i, \hat{H}] = 0\ , \qquad \forall i,j = 1,\dots,n\ . \tag{94}$$

In this quantum case, we will use the nomenclature "conserved charges" to indicate the operators $\hat{F}_i$. This definition poses two main issues:

1. how many charges do we need? To parallel the classical case, we would like their number to be maximal, which means as many as the number of pairs (q_i, p_i). Counting the degrees of freedom in this way in the quantum case is problematic as the phase-space is not a differentiable manifold and position and momentum cannot be defined simultaneously.

2. how to define independence between charges? In the classical case, independence is again associated to the dimensiono of the submanifold obtained once the value of the charges has been fixed. In the quantum case, this is not possible: charges are operators in an Hilbert space. So we have the obvious notion of linear independence, which is however rather different with respect to the classical one.

The simple approach to overcome the first difficutlty is to deal with a quantum system whose Hilbert space as a finite dimensionality: this is true for spin systems for instance and in general it is not a strong limitation as introduce an infrared and ultraviolet cut-off leads in any case to a finite dimensional spectrum. In this approach all the space of operators operators $\mathcal{O}$ (including the Hamiltonian) simply become composed by finite dimensional matrices $n \times n$:

$$\mathcal{O} = \mathrm{End}(\mathcal{H})\ , \qquad \dim(\mathcal{H}) = n \tag{95}$$

Now, we have two choices, both leading to weird conclusions:

- if we assume that "independence" means "linear independence", the maximal number of conserved quantities is trivially given by n and there is an easy way to find them, for instance:

$$\hat{F}_n = \hat{H}^n \tag{96}$$

  This would lead to the unpleasant conclusion that any quantum system is integrable in practice. Note that this choice of powers of the Hamiltonian do not produce independent conserved quantities at the classical level, as $H(\boldsymbol{q},\boldsymbol{p})^n = h_n$ does not provide an indipendent constraint for different powers of n (the resulting set is either empty or equivalent to $H(\boldsymbol{q},\boldsymbol{p}) = h$ (iff $h_n = h^n$).

- if instead we assume that functional "independence" is meant, then it easy to check that any operator commuting with H will be functionally dependent on a single operator $\hat{Q}$:

$$\forall \hat{F} \quad [\hat{F},\hat{H}] = 0 \quad \Rightarrow \quad \hat{F} = f_F(\hat{Q}) \, . \tag{97}$$

  If the spectrum of $\hat{H}$ is non-degenerate, one can actually take $\hat{Q} = \hat{H}$.

In conclusion, the näive generalization of the classical Liouville definition of integrability leads to a problematic definition, which essentially makes any system integrable. This is also consistent with the idea of solvability: in the classical case, this is identified with quadrature, which allows replacing differential equations with algebraic ones. However, for a quantum system, the dynamics is essentially always an algebraic problem, which is reduced to the diagonalization of the Hamiltonian

$$|\Psi(t)\rangle = e^{-\imath \hat{H} t} \, |\Psi_0\rangle = \sum_n e^{-\imath E_n t} \, \langle n|\Psi_0\rangle \, |n\rangle \ , \qquad \hat{H} \, |n\rangle = E_n \, |n\rangle \tag{98}$$

So this idea of solvability as quadrature leads to nowhere. The main point is however that the size of the Hilbert space $\mathcal{H}$ scales exponentially with the number of its components; take for instance a quantum system composed by N spin 1/2. Then, one has $\dim \mathcal{H} = 2^N$. This makes the algebraic solution of the problem in terms of diagonalization of the Hamiltonian rapidly impossible in practice for any system with large N. These ideas suggest that a better way to interpret the concepts of integrability/solvability is to relate them with the complexity of the problem. We will not pursue this direction in these lectures; we refer the interested reader to[5], for a summary of the problems related to the definition of "quantum integrability" and for an approach based on complexity.

Here, we will instead follow a different approach:

1. We always associate the notion of integrability with the one of locality

2. We consider quantum integrable models that have a corresponding classical analogous.

A. Quantum integrable field theories

There are two different ways to arrive to a quantum integrable field theory and it is useful to summarize them as they allow a deeper level of understanding.

- The first idea is to start from a classical field theory and quantize it. We can for instance start from the sinh-Gordon Hamiltonian introduced in Eq. (27). The Poisson bracket is promoted to the canonical commutation relation

$$[\phi(x), \Pi(x')] = \imath \delta(x - x') \tag{99}$$

  This kind of approach is quite standard but it has one main difficulty: the resulting theory suffers of many divergences, which result from the possibility of strong fluctuation for a field at points close in space $x \simeq x'$. Regularizing these divergences requires renormalization techniques.

- A second possibility is to start with a well-behaved lattice theory. One simple example can be a spin 1/2 model. Consider for instance the Hamiltonian of the anisotropic Heisenberg spin chain (XXZ model)

$$\hat{H} = J \sum_i \hat{s}_i^x \hat{s}_i^x + \hat{s}_i^y \hat{s}_i^y + \Delta \hat{s}_i^z \hat{s}_{i+1}^z \tag{100}$$

The groundstate of this Hamiltonian $|E_0\rangle$ for $\Delta > 1$ is gapped; this fact manifests itself into the large-separation decay of correlation functions, i.e.

$$\langle E_0| s_y^\ell s_y^k |E_0\rangle \simeq e^{-\frac{|\ell-k|}{\xi(\Delta)}} , \quad |\ell - k| \to \infty \tag{101}$$

while this relation is exactly computable for this model, it is true for a very generic class of models in the gapped regime. The quantity $\xi(\Delta)$ defines a characteristic length, called "correlation length". The gap closes as $\Delta \to 1^+$ and simultaneously

$$\Delta \to 1^+ \quad \Rightarrow \quad \xi(\Delta) \to \infty . \tag{102}$$

It is intuitive that in this limit, all the details of the lattice will be washed out as the correlation length becomes larger and larger and only long-scale behavior matters. We can therefore hope that a continuous description emerges, so that

$$\lim_{\Delta\to 1^+} \xi(\Delta)^{2d} \langle E_0| \hat{s}^y_{mx\xi(\Delta)} \hat{s}^y_{mx'\xi(\Delta)} |E_0\rangle = \langle \text{vac}| \hat{\mathcal{O}}(x)\hat{\mathcal{O}}(x') |\text{vac}\rangle \tag{103}$$

where the limit $\Delta \to 1^+$ is taken so that mx and mx' are fixed and the dimension d is the unique number (depending on the operator $\hat{s}^x_i$, in this case) that makes the limit of the left-hand side finite. Quantities like the one on the right-hand side can be described through a field theory. Integrable lattice models naturally lead to quantum integrable field theory: in this case the XXZ model leads to the sin-Gordon field theory, related to the sinh-Gordon model in Eq. (27) by a (rather non-trivial) analytic continuation $\phi \to \imath\phi$. In this approach the idea behind renormalization is easily understood in terms of conversion from lattice variables.

- Finally, a quantum field theory is a theory for the scattering between relativistic particles. In this formulation, one assumes to have: i) asymptotic states, where the particles are infinitely far-apart one from the other and evolve freely; ii) a finite region of space where the interactions occur as described by an appropriate scattering matrix.

We will work on the combination of these three different aspects of a quantum field theory. Note that although a field theory can be defined even in presence of Galilean invariance, here we focus on the relativistic case. The non-relativistic one can be recovered as an approriate limit $c \to \infty$. We introduce a Quantum Field Theory (QFT) from the following set of ingredients

Definition III.1 (Quantum field theory)**.** *A quantum field theory is defined by*

- ***Hilbert space.***— *an Hilbert space $\mathcal{H}$;*
- ***Hamiltonian.***— *an operator $\hat{H}$, acting on $\mathcal{H}$ called the Hamiltonian. Its lowest energy state defines the vacuum and the corresponding eigenvalue can be set to zero*

$$\hat{H} |vac\rangle = 0 \tag{104}$$

- ***Relativistic invariance.***— *The momentum $\hat{P}$ and boost $\hat{B}$ operators satisfying the 1d Poincaré algebra*

$$[\hat{H}, \hat{P}] = 0 , \qquad [\hat{B}, \hat{P}] = \imath\hat{H} , \qquad [\hat{B}, \hat{H}] = \imath\hat{P} \tag{105}$$

$$\hat{H} |vac\rangle = \hat{P} |vac\rangle = \hat{B} |vac\rangle = 0 \tag{106}$$

- ***Locality.***— *The Hamiltonian and the momentum are spacial integrals of local fields (densities)*

$$\hat{H} = \int_{-\infty}^{\infty} dx\, \hat{h}(x) , \qquad \hat{H} = \int_{-\infty}^{\infty} dx\, \hat{p}(x) \tag{107}$$

where for any local field $\hat{\mathcal{O}}(x)$:

$$[\hat{P}, \hat{\mathcal{O}}](x) = \imath \frac{d}{dx}\hat{\mathcal{O}}(x) \quad \Rightarrow \quad \hat{\mathcal{O}}(x) = e^{\imath\hat{P}x}\hat{\mathcal{O}}(0)e^{-\imath\hat{P}x} \tag{108}$$

with $[\hat{\mathcal{O}}(x), \hat{h}(x')] = 0$ for $x \neq x'$.

Note that this is similar to saying that a local operator $\hat{\mathcal{O}}(x)$ depends on some fundamental fields $\hat{\phi}(x), \hat{\pi}(x)$ and possibly a finite number of their derivatives around x.

From this definition of a QFT it is now natural to introduce the concept of quantum integrable field theory (QIFT) by simply extending to the quantum dictionary the definition in II.2.

Definition III.2. *A quantum field theory is integrable if it supports infinitely many independent local conserved charges in involution:*

$$\hat{F}_i = \int_{-\infty}^{\infty} dx\ \hat{f}_i(x)\ , \qquad [\hat{F}_i, \hat{F}_j] = 0\ , \qquad [\hat{H}, \hat{F}_i] = 0 \tag{109}$$

B. Asymptotic states

The standard way to go beyond free theory is to introduce asymptotic states. This is the standard approach when dealing with QFT as theories for relativistic particles in interaction. In practice one assumes that the Hamiltonian can be splitted

$$\hat{H} = \hat{H}_0 + \hat{V} \tag{110}$$

where $\hat{H}_0$ contains the free propagation and $\hat{V}$ all the interaction. Note that, $\hat{H}_0$ can in principle be differen from what one could obtain from $\hat{H}$ setting the coupling constants to zero: the masses of the particles can be renormalized by the interaction or new particles (bound-states) can be generated by the interaction and need to be added to the "free" $\hat{H}_0$ written above. Once this splitting has been performed, one can define "in" and "out" states $|\Psi\rangle^{(\mathrm{i/o})}$. As $\hat{H}_0$ is a free Hamiltonian, its eigenstates are labeled by the set of particles and the corresponding quantum numbers it contains

$$|\Phi\rangle^{(0)} = |\theta_1, \dots, \theta_n\rangle^{(0)}_{a_1,\dots,a_n} \tag{111}$$

where we used the superscript 0 to indicate that they are eigenstates of $\hat{H}_0$. We are also using the rapidities to label the energy and momentum of each particle:

$$\varepsilon(\theta) = m_a \cosh\theta\ , \qquad p_a(\theta) = m_a \sinh\theta\ . \tag{112}$$

The label a is used to indicate all the other possible quantum numbers associated to the particle. Then, the in/out states are the eigenstates $|\Psi\rangle^{(\mathrm{in/out})}$ of $\hat{H}$ which reduce to $|\Phi\rangle^{(0)}$ at time $t \to \pm\infty$:

$$e^{-\imath \hat{H} \tau}\, |\Psi\rangle^{(\mathrm{i/o})} \quad \overset{t\to\mp\infty}{\sim} \quad e^{-\imath \hat{H}_0 \tau}\, |\Phi\rangle^{(0)}\ . \tag{113}$$

"In" states are such that in the infinite past they behaved as free-states. Note that, strictly speaking, this equation cannot make sense as it is: eigenstates of $\hat{H}$ would not evolve at all and so the limits $\tau \to \infty$ are ineffective. The correct way, to interpret the "$\sim$", is to imagine that eigenstates have been replaced by smooth superpositions in energies, so that each particle is a wave-packet localized in space. For more details about this procedure, see for instance Chapter 3 in [6]. It is useful to represent "in" and "out" states by writing explicitly their rapidities and to sort them:

$$|\theta_1, \dots, \theta_n\rangle^{(\mathrm{i/o})}_{a_1,\dots,a_n}\ , \begin{cases} \theta_1 > \dots > \theta_n \in \mathbb{R} & (\mathrm{in}) \\ \theta_1 < \dots < \theta_n \in \mathbb{R} & (\mathrm{out}) \end{cases}\ . \tag{114}$$

In this way, the ordering of the rapidities in a in/out state corresponds to the ordering of the particles on the real line in far past/future (the corresponding eigenstate of $\hat{H}_0$).

Both the *in* and *out* states define an orthonormalized basis. Their overlap instead is highly non-trivial and contains all the scattering information:

$$S^{b_1,\dots,b_{n'}}_{a_1,\dots,a_n}(\theta_1, \dots, \theta_n; \theta'_1, \dots, \theta'_{n'}) = {}^{(\mathrm{i})}_{a_1,\dots,a_n}\langle \theta_1, \dots, \theta_n | \theta'_1, \dots, \theta'_{n'} \rangle^{(\mathrm{o})}_{b_1,\dots,b_{n'}}\ . \tag{115}$$

There is a well-defined procedure to pass from a Hamiltonian formulation as in (107) to the definition of asymptotic states and of the scattering matrix S. The formal procedure employs the LSZ reducton formula, named after the three German physicists Harry Lehmann, Kurt Symanzik and Wolfhart Zimmermann[6], which allows defining S-matrix elements in terms of time-ordered correlation functions. However, the inverse problem, i.e. recovering correlation functions from the S-matrix, is not easy in general. For integrable models, it becomes possible and leads to the theory of form factors.

C. Effects of integrability on the scattering

1. Elastic scattering

We now consider how the presence of integrability can be used to constrain the scattering matrix. Let's see the implications of the existence of infinitely many local conserved quantities. Since they are local in space and they remain conserved in the evolution, we know that for any in or out state, they must be additive on the ensemble of particles contained in the state. Moreover, as seen for the Klein-Gordon equation and for the sinh-Gordon in the classical case, the charges are well organized as functions of the rapidity; it is useful to take combination of them so that they transform trivially under the boost operator of the theory $\hat{B}$. We use the integer label s (called spin as the Boost effectively acts as a rotation in energy-momentum space) so that

$$\hat{Q}_s \left|\theta_1, \dots, \theta_n\right\rangle^{(\mathrm{i/o})}_{a_1,\dots,a_n} = \sum_{k=1}^{n} \chi_{a_k} e^{s\theta_k} \left|\theta_1, \dots, \theta_n\right\rangle^{(\mathrm{i/o})}_{a_1,\dots,a_n} \tag{116}$$

Note that not all the integers s can be realized as local conserved quanties for a given model. For instance, we have seen in Eq. (46) that, for the Klein-Gordon model, only odd $s = 2n+1$ are realized. However, the presence of infinite different values of s is already enough to draw strong conclusions. Let's indicate with $\mathcal{I}$ the set of available values of s for which local conserved charges exist in the model under exam. As both "in" and "out" states are eigenstates of $\hat{Q}_s$, they can have a finite overlap only if the eigenvalue is the same:

$$\sum_{k=1}^{n} \chi^{(s)}_{a_k} e^{s\theta_k} = \sum_{k=1}^{n'} \chi^{(s)}_{b_k} e^{s\theta'_k} , \qquad \forall\ s \in \mathcal{I} \tag{117}$$

For general values of the rapidity set, the combination of these equations imply that

- No particle production/annihilation can occur: $n = n'$.
- The set of rapidities must be the same: $\{\theta_k\} = \{\theta'_k\}$.
- Permutations of rapidities can occur only among particles belonging to a symmetry multiplet, so that $\chi_{a_k} = \chi_{b'_k}$. In particular, rapidities can be exchanged only between particles of the same masses.

These properties are summarized by saying that scattering has to be **elastic**.

2. Factorized scattering

One can go even beyond the conclusion of the previous section. We will now see that actually the scattering matrices can be factorized into only two-body processes. A rigorous argument is hard, but we will focus on a more physical intuition. Suppose we have a charge corresponding to a certain value of s and $-s$ (always true if the theory has parity invariance) and let's take the odd combination:

$$\hat{O}_s = \hat{Q}_s - \hat{Q}_{-s} \quad \Rightarrow \quad \hat{O}_s \left|\theta_1, \dots, \theta_n\right\rangle^{(\mathrm{i/o})}_{a_1,\dots,a_n} = \sum_{k=1}^{n} \chi^{(s)} \sinh(s\theta_k) \left|\theta_1, \dots, \theta_n\right\rangle^{(\mathrm{i/o})}_{a_1,\dots,a_n} \tag{118}$$

where to simplify the argument we assumed $\chi^{(s)}$ does not depend on the particle type. Suppose for simplicity that $s = 3$; then we can reparametrize the eigenvalue in terms of momentum:

$$p(\theta) = m \sinh\theta, \qquad q_3(\theta) = \chi^{(3)} \sinh(3\theta) = \frac{3m^2 p + 4p^3}{m^3} = q_3(p) \tag{119}$$

Now suppose we take an "in" state and we consider small wave-packets which are localized both in space and momentum. This can be done by taking the superposition weighted by the single-particle wave-function

$$\psi(x_0, p_0) \simeq \int_{-\infty}^{\infty} dp e^{-a^2 (p-p_0)^2} e^{\imath p(x-x_0)} \tag{120}$$

where a defines the width of the wave packet in momentum. What happens if we act on this state with unitary transformations generated by the charges? Let's start with the simplest case, i.e. the momentum $\hat{P}$ and consider the transformation

$$\hat{U}_P(u) = e^{\imath u \hat{P}} , \qquad \hat{U}_P(u) \left|\theta_1, \dots, \theta_n\right\rangle^{(\mathrm{i/o})}_{a_1,\dots,a_n} = e^{\imath u \sum_k p_k} \left|\theta_1, \dots, \theta_n\right\rangle^{(\mathrm{i/o})}_{a_1,\dots,a_n} . \tag{121}$$

Inserting this term in the integral in (120), we see that this operator rigidly translates all the particles of the same amount u: this was clearly expected as the momentum is the generator of translations (see Eq. (108)).

However, what happens if we do the same thing with $\hat{O}_3$? Inspecting Eq. (119), we see that the non linear contribution $\sim p^3$, will again translates the position of particles but of an amount which depend on the particle momentum itself: indeed if a is large, the integral in Eq.(120) is dominated by the neighborhood of $p \sim p_0$, therefore inside the integral

$$\int_{-\infty}^{\infty} dp e^{-a^2(p-p_0)^2} e^{\imath p(x-x_0)} e^{\imath u p^3} \simeq \int_{-\infty}^{\infty} dp e^{-a^2(p-p_0)^2} e^{\imath p(x-x_0+up_0^2)} \propto \psi(x_0 + up_0^2, p_0) \tag{122}$$

Different values of s would lead to a similar conclusion. Now let's consider a simple scattering process involving

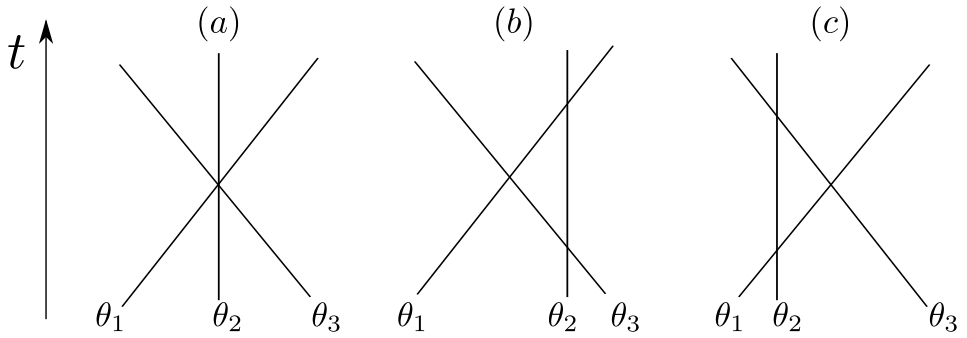

FIG. 2. Three different configurations of "impact parameters" for a $3 \to 3$ process, leading to a different sequence of scattering sequences. The presence of rank $s \geq 2$ conserved quantities imposes that the scattering matrix for the three situation has to be the same, thus implying factorization of the scattering.

three ingoing particles and three outgoing ones. Even though we indicated "in" states simply using their rapidities, the scattering process actually depends on the relative positions of the three particles: this determines the "impact parameter" of the scattering process. We can essentially group the possible scattering processes as the (a), (b) and (c) depicted in Fig. 2. We can write explicitly the resulting scattering matrices for the three cases (sum over repeated indexes is understood):

$$\begin{cases} S^{b_1,b_2,b_3}_{a_1,a_2,a_3}(\theta_1,\theta_2,\theta_3) & \text{(a)} \\ S^{d,c}_{a_2,a_3}(\theta_2,\theta_3) S^{e,b_3}_{a_1,c}(\theta_1,\theta_3) S^{b_1,b_2}_{e,d}(\theta_1,\theta_2) & \text{(b)} \\ S^{d,c}_{a_1,a_2}(\theta_1,\theta_2) S^{b_1,e}_{d,a_3}(\theta_1,\theta_3) S^{b_2,b_3}_{c,e}(\theta_2,\theta_3) & \text{(c)} \end{cases} \tag{123}$$

This is true because locality imposes that scattering can take place only when the wave packets overlap. Now, the remarkable implication of the analysis made before is that these three contributions have to be equal one to the other: indeed, we can act with $e^{\imath u \hat{O}_s}$ to move the wave-packet centers and pass from one configuration to the other; as $[\hat{Q}_s, \hat{H}] = 0$, the resulting scattering matrix has to be the same in the three cases. This argument can be repeated

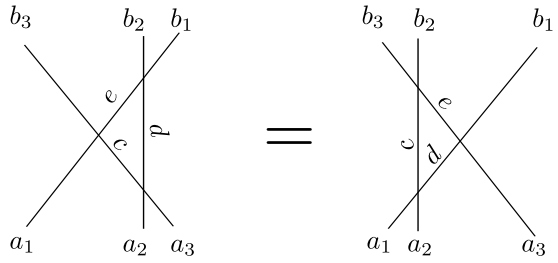

FIG. 3. Graphical transcription of the famous Yang-Baxter equation for the two-body scattering matrices. To simplify the notation rapidities are omitted, but they can be easily reconstructed as a_i goes together with θ_i and then elasticity of the scattering preserves them through the process. See Eq. (**??**) for its translation in a formula.

for more than three particles, leading to the following conclusions:

1. The scattering matrix for a many-body process can be written as product of two-body scattering matrices; e.g. for three particles:

$$S^{b_1,b_2,b_3}_{a_1,a_2,a_3}(\theta_1,\theta_2,\theta_3) = S^{d,c}_{a_2,a_3}(\theta_2,\theta_3)S^{e,b_3}_{a_1,c}(\theta_1,\theta_3)S^{b_1,b_2}_{e,d}(\theta_1,\theta_2) \tag{124}$$

2. Different ways of factorizing the many-body scattering have to be consistent:

$$\boxed{S^{d,c}_{a_2,a_3}(\theta_2,\theta_3)S^{e,b_3}_{a_1,c}(\theta_1,\theta_3)S^{b_1,b_2}_{e,d}(\theta_1,\theta_2) = S^{d,c}_{a_1,a_2}(\theta_1,\theta_2)S^{b_1,e}_{d,a_3}(\theta_1,\theta_3)S^{b_2,b_3}_{c,e}(\theta_2,\theta_3)} \tag{125}$$

This last equation goes under the name of Yang-Baxter equation. It is the fundamental ingredient of any quantum integrable model: $2d$ classical lattice models, integrable spin chains and finally QIFT. It is easily memorized by a sketch, see Fig. 3. In the end, the scattering problem has been reduced to one single matrix function: $S^{b_1,b_2}_{a_1,a_2}(\theta_1,\theta_2)$.

Remark III.1. *From the argument presented in this section, it should become clear why non-trivial integrable field theories exist only in* $1d$*. Indeed, in higher dimensions, repeating the previous construction, one could move particles far apart so that no interactions occur at all: such a statement can be made rigorous and goes under the name of Coleman-Manduja theorem. It tells us that in* $3d$*, whenever a non-trivial conserved quantities exist (still preserving the Lorentz group structure), the S matrix has to be trivial (i.e. non-interacting).*

D. Finding integrable models

We have seen that the existence of conserved quantities imposes several constraints on the possible scatterings, which strongly simplify the problem making it treatable in the end. It is not clear however how one should obtain integrable models in practice: given a Hamiltonion, will it admit conserved charges? Let's try a constructive approach in the simplest case: a single scalar field $\hat{\phi}(x,t)$ with generic interactions

$$\hat{H} = \int_{-\infty}^{\infty} dx \left[\frac{\hat{\Pi}(x)^2}{2} + \frac{\hat{\phi}_x(x)^2}{2} + \frac{m^2\hat{\phi}(x)^2}{2} + \frac{m^2 g_4}{4!}\hat{\phi}(x,t)^4 + \frac{m^2 g_6}{6!}\hat{\phi}(x,t)^6 + \dots \right] \tag{126}$$

where we expanded the potential in powers and we normalized coupling with the mass to make them dimensionless. Note that to reduce the number of terms we assumed parity invariance $\phi \to -\phi$, in the Hamiltonian, so that only even terms can appear. We want to find the conditions on the couplings $g_4, g_6, \dots$ such that the theory is integrable. Being a relativistic theory, in general, one expects to have particle productions. But as we showed in Sec. III C 1, scattering is always elastic: it means that the couplings must satisfy non-trivial relations so that there is no production in the end.

To keep the calculations compact, we work at tree level and, following 7, we consider the diagrams that contribute to the production process $2 \to 4$. It is sufficient to consider the case where the initial particles have precisely the energy to create the four out-coming particles: for the momenta $(p^{(0)}, p^{(1)})$ of the on-shell initial particles in the center of mass reference frame, we have $(2m, \pm\sqrt{3}m)$. The total energy is then $E_t = 4m$ and therefore the four final particles are all at rest, their common value of the momenta being $(m, 0)$. There are three possible diagrams contributing to this amplitude. They are showed in Fig. 6. The free propagator for this theory is simply:

$$G(p) = \frac{\imath}{p^2 - m^2 + \imath 0^+} \,. \tag{127}$$

and we have interactions vertices at any even order $-\imath g_{2n}/(2n!)$. We now proceed to evaluate each diagram, neglecting the contributions of the external legs, which is obviously the same in the three cases.

(a) the quadrimomentum of the inner leg is simply $p_{\text{inn}} = (0, \sqrt{3}m)$. Therefore, counting the number of different ways to realize such diagram, we get:

$$G_{p_{\text{inn}}} \times \frac{1}{2}\left(\frac{-\imath m^2 g_4}{4!}\right)^2 \underbrace{(4\times 4)}_{\text{inner leg}} \times \underbrace{3\times 3}_{\text{in particles}} \times \underbrace{2}_{\text{exch. vertices}} \times \underbrace{4!}_{\text{remaining legs}} = \frac{3\imath m^2 g_4^2}{2} \tag{128}$$

where the cominatorial factor counts the possible ways to construct such diagram.

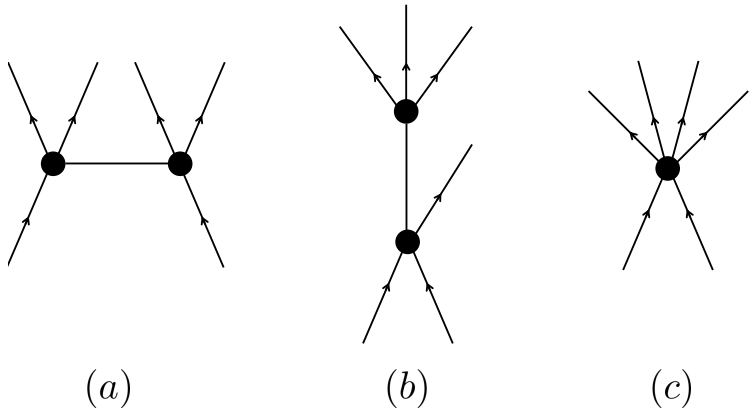

FIG. 4. *Feynman graphs at the tree level for the production process* $2 \to 4$.

(b) the quadrimomentum is $p_{\text{inn}} = (3m, 0)$, which leads to

$$G_{p_{\text{inn}}} \times \frac{1}{2}\left(\frac{-\imath m^2 g_4}{4!}\right)^2 \times \underbrace{(4\times 4)}_{\text{inner leg}} \times \underbrace{3\times 2}_{\text{in particles}} \times \underbrace{2}_{\text{exch. vertices}} \times \underbrace{4!}_{\text{remaining legs}} = -\frac{\imath m^2 g_4^2}{2} \tag{129}$$

(c) Here, we have no inner legs and just one vertex:

$$-\frac{\imath m^2 g_6}{6!} \times 6! \tag{130}$$

If we want that no particle production can occur, the sum of these three diagrams has to vanish

$$(a) + (b) + (c) = \imath m^2 \left(g_4^2 - g_6\right) = 0 \quad \Rightarrow \quad g_6 = g_4^2 \,. \tag{131}$$

Remark III.2. *In the absence of the* ϕ^6 *vertex, the Feynman graphs built only on* Φ^4 *Landau-Ginzburg theory would have given a non-zero value of the production amplitude* $2 \to 4$*, already at the tree level. This implies that in general one cannot have an integrable model whose interaction is a simple polynomial, i.e. involving a finite number of vertices.*

Excercise III.1 $(\star\star\star)$**.** *Using the value of the* g_6 *from* (131)*, show that requiring that the amplitude for the production process* $2 \to 6$*, one has that*

$$g_8 = g_4^3 \tag{132}$$

Generalising this analysis to the higher on-shell production amplitudes $2 \to n$, one finds that the conditions that ensure their cancellation requires the existence of arbitrarily higher couplings whose values are fixed by the equation

$$g_{2n} = (g_4)^{n-1} \,. \tag{133}$$

Relabeling $g_4 \to g$, the series can be resummed, with the final result

$$H = \int_{-\infty}^{\infty} dx \left(\frac{1}{2}\hat{\Pi}(x)^2 + \frac{1}{2}\hat{\phi}_x(x)^2 + \frac{m^2}{g^2}(\cosh(g\hat{\phi}) - 1)\right) . \tag{134}$$

which is again the Sinh-Gordon model!

Remark III.3. *The calculaton presented here should give an intutition on how rare integrable models are. In practice, for a single scalar field with* $\mathbb{Z}_2$ *parity simmetry, there is essentially a unique model: the* sinh*-Gordon, with its closely related partner, i.e. the* sin*-Gordon model, which is obtained for* $g \to \imath g$*. Relaxing the constraint of parity would lead to a different model. These ones essentially exhaust all the possibility with a single scalar field.*

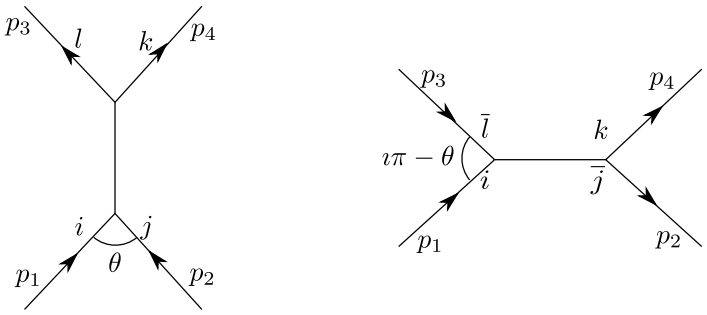

FIG. 5. Two possible channels in the $2 \to 2$ scattering process $S_{ij}^{kl}(p_1, p_2)$. Left: the s-channel, from which we obtain the physical constrain $s > (m_1 + m_2)^2$. Right: the t-channel, which leads to $t < (m_i - m_j)^2$.

E. Analytic properties of the S-matrix

We have seen in the previous section that integrable models are rare and determining them is not easy: already for a single field a lot of Fenyman diagrams are needed, order by order, to fix the interaction potential. However, we have seen that integrability imposes strong constraints on the possible structure of the S-matrix: it turns out that one can exploit this structure to construct S-matrices which correspond to integrable models.

We present here only a brief summary of all the requirements that the S-matrix has to satisfy: further details can be found for instance in 8. In order to study a relativist scattering of type $2 \to 2$, it is useful to use invariant quantities: we introdue the Mandelstam variables, which are a parametrization of the four momenta involved in the scattering process:

$$s = (p_1 + p_2)^2 \,, \qquad t = (p_1 - p_3)^2 \,, u = (p_1 - p_4)^2 \,. \tag{135}$$

As the process is elastic however, the variable u is identically vanishing; instead

$$s(\theta_1, \theta_2) = m_1^2 + m_2^2 + 2m_1 m_2 \cos\theta_{12} \,, \qquad t(\theta_1, \theta_2) = m_1^2 + m_2^2 - 2m_1 m_2 \cos\theta_{12} \,. \tag{136}$$

So as expected from Lorentz invariance, both variables only depend on the rapidity difference and moreover $t(\theta) = s(\imath\pi - \theta)$. Inspecting Fig. 5, we see that this equation has the geometrical interpretation of a $\imath\pi$ "rotation" in the hyperbolic plane. As a first consequence, we obtain that the S-matrix has to be a function of the rapidity difference only:

$$S_{ij}^{kl}(\theta_1, \theta_2) = S_{ij}^{kl}(\theta_1 - \theta_2) \,. \tag{137}$$

So we end up with a matrix of functions of a single variable. It is important to explore the analytic structure of S as a function of θ, when it is promoted to a variable in the complex plane. Clearly, physical values of s and t correspond to the conditions

$$s(\theta) > (m_1 + m_2)^2 \,, \qquad t < (m_i - m_j)^2 \tag{138}$$

One can see that these constraints manifest themselves as branch cuts: the S expressed as a function of the Mandelstam variable s, has branch cuts for $s > (m_i + m_j)^2$ and for $s < (m_i - m_j)^2$, with possible bound states in the middle. Even though both these cuts are square-root ones, infinite sheets are accessible passing through both of them. Such a complicate behavior is better understood going to the θ plane as obtained by the function $s(\theta)$ in (136). In this case, the physical sheet becomes the strip $0 < \Im\theta < \pi$ and all the other sheets are covered by the full θ plane. We can now summarize the properties that $S_{ij}^{kl}(\theta)$ has to satisfy.

- (Unitarity) The transformation of in-states into out-states has to be unitary, as it is simply produced by the time evolution:

$$\sum_{a,b} S_{ij}^{ab}(\theta)(S_{kl}^{ab}(\theta))^* = \delta_{i,k}\delta_{j,l} \tag{139}$$

  Note that if the model is not integrable, unitarity is not so simple: particle production has to be taken into account in the sum ofer all the possible intermediate states.

FIG. 6. Behavior of the S-matrix as a function in the complex plane. Left: $S(s)$ as a function of the Mandelstam variable s; branch cuts appear on the real axis for $s > (m_i + m_j)^2$ and $s < (m_i - m_j)^2$; only the physical sheet is showed here and physical values of s are right above the branch cut $s > (m_i + m_j)^2$, other sheets are accessible through the cuts. Right: $S(\theta)$ as a function of the rapidity difference θ; the physical sheet in the s plane is mapped in the strip $0 < \Im\theta < \pi$.

- (Hermitian analyticity) S is a real analytic function of the variable s:

$$S_{ij}^{kl}(s^*) = (S_{ij}^{kl}(s))^* \Rightarrow (S_{ij}^{kl}(\theta))^* = (S_{lk}^{ji}(-\theta^*)) \tag{140}$$

- (Crossing symmetry) As we discussed before, passing from the s to the t channel corresponds to reading the diagram "from left to right", while changing $\theta \to \imath\pi - \theta$. This gives the relation

$$S_{ij}^{kl}(\theta) = S_{\bar{l}i}^{\bar{j}k}(\imath\pi - \theta) \tag{141}$$

- (Yang-Baxter relation) For consistency of the scattering factorization, as we already discussed, we need to impose Eq. (125).

- (Bound-state consistency) One of the main difficulty is that whenever the $S_{ij}^{kl}(\theta)$ presents a pole on the physical strip $0 < \Im\theta < \pi$, this corresponds to a stable bound state. So it means that this bound state will correspond to a new stable particle, which needs to be added to the list of possible particles as asymptotic states.

F. Possible scattering matrices

The interesting point is that it is possible to find solutions which satisfy all these constraints and correspond to physical models. We focus on the simplest situation: the diagonal case, where no transmutation occurs and particles conserve their quantum numbers:

$$S_{a_2,a_2}^{b_1,b_2}(\theta) = \delta_{a_1}^{b_1}\delta_{a_2}^{b_2}S_{a_1,a_2}(\theta) \tag{142}$$

In this way, the Yang-Baxter equation is trivially satisfied and we just need to care about the analytical properties. In this case, the conditions of Hermitian analiticity, unitarity and crossing symmetry reduce to

$$\begin{cases} (S_{ab}(\theta))^* = S_{ab}(-\theta^*) & \text{Hermitian analiticity} \\ S_{ab}(\theta)S_{ab}(-\theta) = 1 & \text{Unitarity} \\ S_{ab}(\theta) = S_{a\bar{b}}(\imath\pi - \theta) & \text{Crossing} \end{cases} \tag{143}$$

and the most general solution to these conditions is found to be

$$S_{a,b}(\theta) = s_{a,b}^{(0)} \times \prod_{x \in \mathcal{A}_{a,b}} \frac{\sinh(\frac{1}{2}(\theta + \imath\pi x)}{\sinh(\frac{1}{2}(\theta - \imath\pi x)} \tag{144}$$

where the $\mathcal{A}_{ab}$ is a finite set whose elements can be chosen as real and satisfying $-1 < x < 1$. The factor $s_{a,b}^{(0)} = \pm 1$. In principle, its value would be arbitrary, but in practice, all physical models have a fermionic behavior, $S_{a,b}(0) = -1$, which determines such number (for a discussion of the physical consequence of taking instead the bosonic case $S(0) = 1$, see [10]). The function $S_{ab}(\theta)$ has poles for $\theta = \imath\pi x$ and any $x \in \mathcal{A}_{ab}$. Therefore, if some x lies in the physical strip,

i.e. $0 < x < 1$, this corresponds to a bound state which emerges from the scattering of particles a, b. As we discussed in the previous section, bootstrap principle tells us that boundstate must be considered on the same ground of the other asymptotic states: this poses a non-trivial problem as, once the set of particles has been fixed, one has to ensure that the values of $x \in \mathcal{A}_{ab}$, for a given scattering channel $a + b$, do not give place to new particles. We will not deal with this procedure here and we refer to [8, 9] for details. However, if no value of x lies in the physical strip, than we are essentially done. Using crossing, one can then obtain $S_{\bar{a},b}(\theta)$, which induced the relation between the sets

$$\mathcal{A}_{\bar{a}b} = \{1 - x \mid x \in \mathcal{A}_{ab}\} \ . \tag{145}$$

If we require a particle to be neutral, i.e. it is its own antiparticle, then $\mathcal{A}_{\bar{a}b} = \mathcal{A}_{ab}$. The simplest example is the one of a single neutral particle which is obtained taking $\mathcal{A}_{aa} = \{x, 1 - x\}$ and $-1 < x < 0$ (so that no pole is in the physical strip), which corresponds to

$$S(\theta) = \frac{\tanh \frac{1}{2}(\theta + \imath\pi x)}{\tanh \frac{1}{2}(\theta - \imath\pi x)} \tag{146}$$

This corresponds to a theory with a single neutral particle, which satisfies paritiy invariance. So, we can suspect that it is indeed the sinh-Gordon model. Indeed, perturbative calculations show that:

$$x = -B(g) \qquad B(g) = \frac{g^2}{8\pi + g^2} \tag{147}$$

is indeed the scattering matrix of the sinh-Gordon Hamiltonian in Eq. (134):

$$S_{\text{ShG}}(\theta) = \frac{\tanh \frac{1}{2}(\theta - \imath\pi B)}{\tanh \frac{1}{2}(\theta + \imath\pi B)} = \frac{\sinh\theta - \imath \sin(\pi B)}{\sinh\theta + \imath \sin(\pi B)} \tag{148}$$

Remark III.4. *It is interesting to observe that this approach of constructing S-matrix which satisfy a set of constraints, give integrable models but gives no way to identify the corresponding Hamiltonian. As discussed above, in the case of the* sinh*-Gordon case, one has to compare with perturbation theory in order to find the correspondence between the parameter of the model and those in the S-matrix. For more general theory, the problem becomes even more subtle as the bare parameter that appear in the Hamiltonian gets modified by renormalization: in a perturbative approach, this has to be done order by order and the procedure becomes more involuted.*

IV. THERMODYNAMICS OF AN INTEGRABLE FIELD THEORY

A. Bethe-Ansatz equations

We have seen that when a field theory is integrable, there is the possibility of determining its scattering properties. We will now see one of the simplest application of this machinery, which allows to derive exactly the thermodynamical properties of the model. The argument was first introduced by Zamolodchikov in [11] and here we will use it in the case of the sinh-Gordon model with the scattering matrix in (148).

The main difficulty is that, in order to study the thermodynamics, we want to perform the thermodynamic limit of the model: the total energy E has to diverge with the systetm size L, while keeping the ratio E/L fixed. In order to do this we imagine that our field theory is now defined on a circle of length L. This seems to forbid the application of the scattering theory presented before as that required asymptotic states. Nevertheless, we can imagine that, if L is very large, there will be configurations where particles are very far one from the other. Then, they evolve scattering one with the other as described by the scattering matrix (148).

We must stress that relativistic invariance generically forbids any approach of first quantization (so determining a wave-function) simply because it must take into account virtual and real particle creation. In the configuration space, however, there are regions where we have a set of (wave packets of) relativistic particles strongly separated in their space position x_i. In these regions, which we call **free regions**, particles move as free ones and off mass shell effects (virtual particle creation) can be neglected. We can therefore speak about space coordinates x_i and momenta $p_i = m\sinh(\theta_i)$ and introduce a wave function $\Psi(x_1, \dots, x_n)$. The important point is that, as we have seen, scattering has to be **elastic**: the number of particles and the set of their momenta $p_1, \dots, p_n$ must be the same in each free region. We can therefore identify each free region with an ordering of particles (where they are well separated)

$$\{i_1, \dots, i_n\} \quad \Leftrightarrow \quad x_{i_1} \ll x_{i_2} \ll \dots \ll x_{i_n} \ . \tag{149}$$

When we arrive at a transition between two free regions, we will have configurations where particles are close and strongly overlap, giving rise to strong relativistic effects, which make the wave function approach ineffective. Nevertheless, we know what happens after the scattering has occurred thanks to the S-matrix: in particular, for the sinh-Gordon, we will simply have the multiplication times a phase factor:

$$\{i_1,\dots,i_p,i_{p+1},\dots,i_n\}\to\{i_1,\dots,i_{p+1},i_p,\dots,i_n\} \quad \Leftrightarrow \quad S(\theta_{i_p}-\theta_{p+1}) \tag{150}$$

where for the sinh-Gordon $S(\theta)=S_{\rm ShG}(\theta)$ given in (148), but the equation is more general. Behing on a circle, all these connection conditions must be consistent: when a particle has been exchanged with all others it must come back on itself, i.e.

$$e^{\imath p_i L}\prod_{j\neq i}S(\theta_i-\theta_j)=1\ , \qquad i=1,\dots,N \tag{151}$$

or taking logarithms

$$m\sinh\theta_i+\frac{1}{L}\sum_{j\neq i}\Theta(\theta_i-\theta_j)=\frac{2\pi I_i}{L}\ , \qquad S(\theta)=e^{\imath\Theta(\theta)} \tag{152}$$

which go under the name of Bethe-ansatz equations. **In this equation $I_1,\dots,I_n$ are n integer numbers, which have to be different one to the other because of the fermionic nature of the S-matrix ($S(0)=-1$). Essentially, there is a corresponding between the set of integers $\{I_1,\dots,I_n\}$ and the corresponding eigenstate, characterized by the rapidities $\{\theta_1,\dots,\theta_n\}$.** This structure is very common in all quantum integrable systems, both on the lattice and in the continuum: generically, in order to write it, one only needs that the factorization of the scattering is verified and the knowledge of the exact form of the scattering matrix.

Clearly the energy and the momentum for such an eigenstate can be obtained using the free regions, where we simply have

$$E=\sum_{j=1}^{N}m\cosh(\theta_i)\ , \qquad P=\sum_{j=1}^{N}m\sinh(\theta_i)\ . \tag{153}$$

B. Large L limit of Bethe-Ansatz equations

We are now interested in the limit where both the size of the system L and the number of particles N become large. To do so, for a given eigenstate, we introduce the counting function $y(\theta)$ defined by the following properties:

- In correspondence of each rapidity θ_j, it equals the corresponding quantum integer:

$$y(\theta_i)=\frac{2\pi I_i}{L} \tag{154}$$

- It increases monotonically.

Of course, these constraints do not specify a unique function, but the idea is that when N and L will be large (with N/L fixed), rapidities will denser and denser and the function $y(\theta)$ will be completely determined. We can then rewrite

$$m\sinh\theta_i+\frac{1}{L}\sum_{j\neq i}\Theta(\theta_i-\theta_j)=y(\theta_i) \tag{155}$$

For an excited state, with finite energy, not all the available integers are occupied by a value I_j. We can define than a set of "vacancies" (virtual values of rapidities associated to any integer n) using

$$y(\theta_n^v)=\frac{2\pi n}{L} \tag{156}$$

Only a subset of the vacancies $\mathcal{V}=\{\theta_1^v,\dots\}$ corresponds effectively to the rapidity θ_j of a particle. The remaining values are called holes:

$$\{\theta_n^v\}=\{\theta_j\}\cup\{\theta_j^h\} \tag{157}$$

In the limit of large L, we associate densities to these three sets according to

$$\rho(\theta_j) = \lim_{T.L} \frac{1}{L(\theta_{j+1} - _j)} , \qquad \text{density of particles} \tag{158}$$

$$\rho_h(\theta_j^h) = \lim_{T.L} \frac{1}{L(\theta_{j+1}^h - \theta_j^h)} , \qquad \text{density of holes} \tag{159}$$

$$\rho_t(\theta_j) = \lim_{T.L} \frac{1}{L(\theta_{j+1}^v - \theta_j^v)} , \qquad \text{density of vacancies} \tag{160}$$

where $T.L.$ stays for the thermodynamic limit obtained as $N \to \infty, L \to \infty$. Clearly one has $\rho_t(k) = \rho(k) + \rho_h(k)$, therefore $\rho_t(k)$ is the total density, while $\rho(k), \rho_h(k)$ are respectively the particles and holes densities. The derivative of the counting function can be related to the densities using that

$$y'(\theta_n^v) = \lim_{T.L.} \frac{y(\theta_{n+1}^v) - y(\theta_n^v)}{\theta_{n+1}^v - \theta_n^v} = 2\pi\rho_t(\theta_j^v) \tag{161}$$

from which we derive

$$y(\theta) = 2\pi \int_{-\infty}^{k} d\theta\, \rho_t(\theta) . \tag{162}$$

Now, from the thermodynamic limit of (155):

$$y(\theta) = m\sinh\theta - \int_{-\infty}^{\infty} d\theta'\, \rho(\theta')\Theta(k - k') \tag{163}$$

and differentiating with respect to θ leads to the Bethe-Ansatz equations in the thermodynamic limit:

$$2\pi\rho_t(k) = m\cosh(\theta) - \int_{-\infty}^{\infty} d\theta'\, \rho(\theta')\varphi(\theta - \theta') \tag{164}$$

where we introduced the derivative of the scattering phase

$$\varphi(\theta) = \frac{d}{d\varphi}\Theta(\varphi) = \frac{2\sin(\pi B)\cosh(\theta)}{\sin^2(\pi B) + \sinh^2(\theta)} \tag{165}$$

Remark IV.1. *Eq.* (163) *is the thermodynamic limit of the Bethe-Ansatz equation and shall be considered as a relation between the density of occupied modes and the one of available vacancies. It should be compared with what happen for free systems ($B \to 0$), where the density of vacancies is actually fixed and independent on which are occupied. This would correspond to*

$$\rho_t(\theta) = m\cosh(\theta) = \frac{dp(\theta)}{d\theta} \tag{166}$$

which is a flat density in momentum space. In presence of interactions, such distribution is actually modified by the density of occupied rapidities.

C. Computing the free-energy

Now suppose we want to express the partition function of the model. This can be formally written as

$$Z_\beta = \sum_{\{I_1,\dots,I_N\}} e^{-\beta E_I} , \qquad E_I = \sum_{i=1}^{N} m\cosh\theta_i \tag{167}$$

where the sum is over all the configuration of the Bethe Integers, and θ_i are the rapidities obtained solving the Bethe-equations with those integers. When $L \to \infty$

$$E_I = \sum_{i=1}^{N} m\cosh\theta_i = L\int d\theta\, m\cosh\theta\rho(\theta) \equiv L\mathcal{E}[\rho] \tag{168}$$

and so it is defined an energy functional $\mathcal{E}[\rho]$ of the rapidity distribution $\rho(\theta)$. So, we can convert the sum in (167) into a sum over the possible functions $\rho(\theta)$ if we are able to count how many choice of the integers correspond to the same pair $\rho(\theta), \rho_t(\theta)$. In a given interval of rapidity $\delta\theta$ we will have

$$\delta n_v = L\rho_t(\theta)\delta\theta\,, \qquad \text{number of vacancies} \tag{169}$$

$$\delta n = L\rho(\theta)\delta\theta\,, \qquad \text{number of particles} \tag{170}$$

so the number of ways in which we obtain the same $\rho(\theta)$ correspond to the number of ways in which in each interval $\delta\theta$, we choose δn rapidities among the δn_v available vacancies:

$$\#\text{integer configurations corresponding to } \rho(\theta) = \prod_{\text{intervals } \delta\theta} \binom{\delta n}{\delta n_v} \overset{L\to\infty}{\simeq} e^{L\mathcal{S}_{YY}[\rho,\rho_t]} \tag{171}$$

where $S_{YY}[\rho,\rho_t]$ is a functional called the Yang-Yang entropy and by simply performing the Stirling approximation and recalling that $\rho_t(\theta) = \rho(\theta) + \rho_h(\theta)$, one obtains the form

$$\mathcal{S}_{YY}[\rho] = \int d\theta[\rho_t(\theta)\ln\rho_t(\theta) - \rho(\theta)\ln(\rho(\theta)) - \rho_h(\theta)\ln(\rho_h(\theta))] \tag{172}$$

where $\rho(\theta), \rho_h(\theta)$ are related by (164), so there is only one independent function. So we arrive at

$$\mathcal{Z}_\beta = \int \mathcal{D}\rho\, e^{-\beta L\mathcal{E}[\rho]+L\mathcal{S}_{YY}[\rho]} \tag{173}$$

We can now use the saddle-point method to compute this integral for $L\to\infty$. These are better written in terms the pseudo-energy function $\varepsilon(\theta)$:

$$\frac{\rho(\theta)}{\rho_t(\theta)} = \frac{1}{e^{\beta\varepsilon(\theta)}+1} \tag{174}$$

which leads to the integral equation

$$\beta\varepsilon(\theta) = m\beta\cosh(\theta) - \int \varphi(\theta-\theta')\log(1+e^{-\varepsilon(\theta')})\,. \tag{175}$$

These are known as Thermodynamic Bethe-Ansatz (TBA) equations and can be solved by iteration to determine $\varepsilon(\theta)$. Then, $\rho(\theta), \rho_t(\theta)$ can be determined using (174) and (164). Finally, reinjecting into (173), leads to the final expression for the free energy:

$$\beta f = -\frac{1}{L}\log\mathcal{Z}_\beta = -m\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\cosh\theta\log(1+e^{-\beta\varepsilon(\theta)}) \tag{176}$$

which is exactly the same formula one would have got for free fermion systems: the only difference lies in the fact that the single-particle energy appearing in (176) is actually the pseudoenergy obtained from the TBA equation (175).

Excercise IV.1 ($\star\star$)**.** *Derive explicitly Eq.* (175) *and Eq.* (176)*, by performing explicitly the variation of the exponent in* (173) *and using the definition of* $\varepsilon(\theta)$ *in* (174)*.*

Remark IV.2. *The derivation presented here holds similarly for any integrable model. The main difference is that for the* sinh*-Gordon (and its non-relativistic limit, the Lieb-Liniger model) only one stable particle exists. For any other model, bound-states will appear which will need to be treated by introduced other funcitons* $\rho^{(i)}(\theta), \rho_t^{(i)}(\theta)$ *for any particle species* i*. This leads to a set of coupled integral equations, which can still be solved with a similar method.*

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