1. Fock space

(a) Assume that the vacuum $|0\rangle$ is normalized such that $\langle 0|0\rangle = 1$. Calculate the product $\langle n'|n\rangle$, where $|n\rangle = \left(a^\dagger\right)^n |0\rangle$, with a and $a^\dagger$ defined as in the lectures, i.e., such that $a|n\rangle = n|n-1\rangle$ and $a^\dagger|n\rangle = |n+1\rangle$.

(b) The projection state $\langle \mathscr{P}|$ is defined by the property that $\langle \mathscr{P}|n\rangle = 1$ for all possible values of n. Show that $\langle \mathscr{P}| = \langle 0|e^a$.

2. Generalized reactions

Consider a population of individuals A with the following dynamics:

$$\underbrace{A + A + \cdots + A}_{k} \xrightarrow{\lambda} \underbrace{A + A + \cdots + A}_{l}. \tag{1}$$

Note that the case of lethal competition discussed in the lectures corresponds to $k = 2$ and $l = 1$, while the branching process corresponds to $k = 1$ and $l = 2$.

(a) Write the rate equation for the evolution of the (average) number of particles, based only on Eq. (1), as we did in the lectures for the Lotka-Volterra model. Solve this equation and discuss its qualitative features depending on l and k.

(b) Which are the absorbing states of the dynamics in Eq. (1) with $k = 2$ and $l = 0$?

(c) For generic k and l, write down the master equation for the probability $P(n,t)$ to have n individuals at time t. Compare this result with the cases discussed in the lectures.

(d) From the master equation derived above, write down the evolution equation for $\langle n\rangle$ and show that, within the mean-field approximation $\langle n^2\rangle \simeq \langle n\rangle^2$, $\langle n^3\rangle \simeq \langle n\rangle^3$, etc., and $\langle n\rangle \gg k$, the evolution equation reduces to the one determined at point (a).

(e) Based on the result at point (c), determine the expression of the Hamiltonian H for Eq. (1) in terms of the operators a and $a^\dagger$. Discuss its relevant features.