Binary annihilation process

Consider the binary annihilation process

$$A + A \overset{\lambda}{\mapsto} \emptyset, \tag{1}$$

with diffusion coefficient D, in d spatial dimensions.

(a) For $d = 0$ and an initial state with n_0 particles, discuss the asymptotic state of the process.

(b) On the basis of Eq. (1), write down the (heuristic) rate equations for the average number $n(t)$ of particles at time t and discuss qualitatively the behavior of the population. Which is the phase diagram of the model? Assuming that the population has initially n_0 individuals, determine the long-time behavior and identify possible algebraic behaviors.

(c) On the basis of the master equation for the process, determine the reaction hamiltonian H_R within the Doi-Peliti formalism and the associated field-theoretical action $\mathscr{A}_T$ on the continuum. (See exercise 2 of problem set n. 2.)

(d) Show that the rate equations of point (a) can be obtained by solving the mean-field equations derived from $\mathscr{A}_T$. Expand the theory around the mean-field solution and determine the form of the propagators both in the frequency and in the time domain.

(e) Based on the RG transformations of the Gaussian theory, show that the upper critical dimensionality of this process is 2.

(f) Show that $\Gamma^{(1,1)}$ is not affected by the interaction, i.e., all the diagrammatic corrections to the term $\propto \tilde{\varphi}_< \varphi_<$ in the effective action $\mathscr{A}_{\text{eff}}[\varphi_<, \tilde{\varphi}_<]$ vanish within this theory.

(g) Focus on the two vertices of the theory and show that the only diagrams which contribute to them have the form of a chain of "bubbles". Each bubble can be expressed in terms of

$$B_\ell(q,\omega) \equiv \int \frac{\mathrm{d}\omega'}{2\pi} \int_> \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{i(\omega+\omega') + D(q+k)^2} \, \frac{1}{-i\omega' + Dk^2}, \tag{2}$$

where the integral $\int_>$ is done on the shell $S_\Lambda = \{\vec{k} \mid \Lambda e^{-\ell} < |\vec{k}| < \Lambda\}$ with $\ell > 0$ and q and ω are the total external momentum and frequency entering in the bubble from the external legs. Express each chain of bubbles in terms of $B_\ell(q,\omega)$ and then sum over all possible lengths of the chains.

(h) Determine the effective coupling constants λ' and λ'' after integration (on the basis of the result at point (f)) and after rescaling, respectively. Calculate the *exact* flow equation $\partial_\ell \lambda = \lim_{\ell \to 0}(\lambda'' - \lambda)/\ell$ of the coupling constant.

(i) Discuss the possible fixed points of the theory as a function of the dimensionality d.

(j) Based on the renormalization-group transformations, which is the expected behavior of the average density of particles at long times depending on d?