OPEN QUANTUM SYSTEMS

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— markovian master equation approach —

menu :

I) Quantum operations
- positivity and complete positivity
- Kraus representation
- infinitesimal generators and the Lindblad equations

II) Derivation of master equations
- the assumptions
- Born-Markov approximation and the Redfield equation
- Alternative justification for Lindblad model : repeated interactions protocol
- quantum trajectories

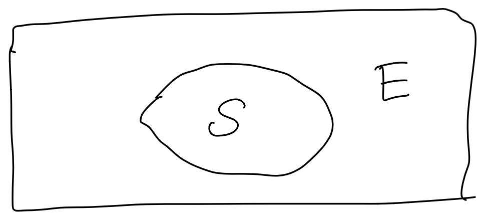
III) Vectorization and superoperator formalism
- Double-Hilbert space approach (aka thermofield dynamics)
  example : open XXZ spin chain
- Operator Fock space approach ("Third quantization")
  examples :
  a) Free fermions, in general non-unitary Bogolyubov tr. & normal master modes
  b) XYh spin chain

IV) Matrix product density operator ansatz and exact solutions of boundary driven interacting open systems
- canonical example : boundary driven open XXZ spin chain
- a cousin from classical STATMECH : boundary driven ASEP
- Factorization of NESS : bulk and boundary relations
- Lax-operator formulation and link to integrability

V) Extras
- Exact sols. in other models : Hubbard, spin-1 chains ...
- quasi-local charges

I) Quantum operations

1) Combined unitary evolution of system + bath/environment



$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$

U_{SE}

Only 1 assumption here:

No initial correlations in the initial state

$$\rho(0) = \rho_S \otimes \rho_E \qquad \rho(t) = U_{SE}\, \rho(0)\, U_{SE}^\dagger$$

$$\rho(t) = \mathrm{Tr}_E\, \rho(t) = \mathrm{Tr}_E \{ U_{SE} (\rho_S \otimes \rho_E) U_{SE}^\dagger \}$$

This can be understood as a linear machine

$$\rho_S \longrightarrow \boxed{\hat{\mathcal{M}}} \longrightarrow \rho(t) = \rho_S'$$

$\rho_S' = \mathcal{M}(\rho_S)$ acting on $\underline{\mathrm{End}(\mathcal{H}_S)}$

Which has the following properties

a) trace preservation

P.: $\underline{\underline{\mathrm{tr}\, \rho_S'}} = \mathrm{tr}_S\, \mathrm{tr}_E \left(U_{SE} (\rho_S \otimes \rho_E) U_{SE}^\dagger \right)$

$= \mathrm{tr}\left(U_{SE} (\rho_S \otimes \rho_E) U_{SE}^\dagger \right) =$

$= \mathrm{tr}\, \rho_S \otimes \rho_E = \mathrm{tr}\rho_S \otimes \mathrm{tr}\rho_E = \underline{\underline{\mathrm{tr}\, \rho_S}}\ (=1)$

b) Hermiticity preservation (<u>Excercise</u>!)

c) It is positive, meaning that $\rho_S \geq 0 \Rightarrow \rho_S' \geq 0$

(<u>Excercise</u>!)

d) It is completely positive, meaning that if
$\mathcal{H}_A$ is some finite dimensional ancilla, then

$$\mathrm{Id}_A \otimes \hat{\mathcal{M}} \text{ is positive over } \mathrm{End}(\mathcal{H}_A \otimes \mathcal{H}_S) \text{ for any } A$$

Remark: There are linear maps which are positive but not completely positive, i.e. obey (a–c) but not (d).

Example: transposition

$$\hat{\mathcal{J}}(\rho) = \rho^T$$

$\mathrm{Id}_A \otimes \hat{\mathcal{J}}$ is a partial transposition, some entangled states can be detected by non-positivity of

$$(\mathrm{Id}_A \otimes \hat{\mathcal{J}})(\rho) \ngeq 0$$

Excercise: construct an example!

Stinespring's theorem: any CPTPM can be written in a form

$$\mathcal{M}(\rho) = \mathrm{tr}_E\, U(\rho \otimes \rho_E) U^\dagger$$

for some U and ρ_E (which are highly nonunique)

2) Kraus representation

Let us assume $\rho_E = |0\rangle\langle 0|$ where $|0\rangle$ is some state in $\mathcal{H}_E$

Let $\{|k\rangle;\ k = 0, 1, \dots N_E - 1\}$ ($N_E = \dim \mathcal{H}_E$) be an ON basis of $\mathcal{H}_E$

Then:

$$\rho' = \mathrm{Tr}_E\, U(\rho \otimes |0\rangle\langle 0|) U^\dagger = \sum_{k=0}^{N_E-1} {}_E\langle k| U |0\rangle_E\, \rho_S\, {}_E\langle 0| U^\dagger |k\rangle_E = \sum_k U_{k0}\, \rho\, U_{k0}^\dagger$$

Now write

$$U = \sum_{k,l=0}^{N_E-1} U_{kl}\, |k\rangle\langle l|$$

$$U_{kl} \in \mathrm{End}(\mathcal{H}_S)$$

Define $K_k = U_{k0}$ and call them Kraus operators

They satisfy $\sum_k K_k^+ K_k = \mathbb{1}$ since U is unitary

Kraus theorem: Any CPTPM $\hat{M}$ can be represented as

$$\rho' = \hat{M}(\rho) = \sum_{k=0}^{N^2-1} K_k \rho K_k^+$$

where $\sum_k K_k^+ K_k = \mathbb{1}$ and $N = N_S = \dim \mathcal{H}_S$

Excercise: prove a–d)

Remark: Kraus representation is highly non-unique

$$K_k = \sum_{l=0}^{N^2-1} u_{kl} K_l' \quad \text{where } u \in U(N^2)$$

$$\hat{M}(\rho) = \sum_k \sum_l \sum_{l'} u_{kl} K_l' \rho K_{l'}'^+ \bar{u}_{kl'}$$

$$= \sum_l \sum_{l'} \underbrace{\left(\sum_k u_{kl} \bar{u}_{kl'}\right)}_{\delta_{ll'}} K_l' \rho K_{l'}'^+ = \sum_l K_l' \rho K_l'^+$$

and again $\sum_l K_l'^+ K_l' = \mathbb{1}$ (Excercise)

3) Infinitesimal generators of Kraus maps

$$U_{SE}(\delta t) ;$$

$$\Rightarrow M(\delta t) \qquad \rho(t+\delta t) = \hat{M}_t(\delta t)\, \rho(t) = \rho(t) + \dot{\rho}(t)\, \delta t + O(\delta t^2)$$

The Kraus terms which can generate linear corrections in δt to identity should be of one of the two forms:

i) $K_k = \sqrt{\delta t}\, L_k + O(\delta t)$ $\quad k \geq k_0$ $\qquad \sum_{k<k_0} \alpha_k = 1$

ii) $K_k = \alpha_k \mathbb{1} + \delta t(-iH_k + M_k) + O(\delta t^2)$, $k < k_0$, H_k, M_k Herm.

The first part yields:

$$\mathcal{M}_1(\rho) = \delta t \sum_{k=k_0}^{N^2-1} L_k \rho L_k^\dagger + O(\delta t^2)$$

$$H = \sum_{k=0}^{k_0-1} \alpha_k H_k$$

$$M = \sum_{k=0}^{k_0-1} \alpha_k M_k$$

$$\mathcal{M}_2(\rho) = \rho + \delta t\left(-i[H,\rho] + \{M,\rho\}\right) + O(\delta t^2)$$

So, I can choose $k_0 = 1$ for the most general situation

The condition

$$\sum_{k=0}^{N^2-1} K_k^\dagger K_k = \mathbb{1}$$

results in:

$$\mathbb{1} + \delta t\left(\left(-i\cancel{H} + i\cancel{H} + 2M\right) + \sum_{k=1}^{N^2-1} L_k^\dagger L_k\right) = \mathbb{1}$$

$$\Rightarrow \boxed{M = -\frac{1}{2}\sum_{k=1}^{N^2-1} L_k^\dagger L_k}$$

Finally we arrive at the Lindblad equation

$$\boxed{\frac{d\rho}{dt} = -i[H,\rho] + \sum_{k=1}^{N^2-1}\left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}\right)}$$

Lindblad, GKS (1976)

Remarks:

In general L_k and H can depend on time,

and remember, since we assumed factorizability of initial state,

this eq. is only compatible with Born-Markov approximation.

This means that

$$\underset{\substack{\text{typical system}\\ \text{relaxation timescale}}}{\tau_S} \gg \delta t \gg \underset{\substack{\text{typical environment}\\ \text{relaxation timescale}}}{\tau_E}$$

II) "More" or "less" serious derivation of QME

We start with the generic Hamiltonian

$$H = H_0 + \lambda V = H_S + H_E + \lambda V \quad \text{and assume} \quad V = \sum_k S_k \otimes E_k$$

$$S_k \in \mathrm{End}(\mathcal{H}_S)$$
$$E_k \in \mathrm{End}(\mathcal{H}_E)$$

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho]$$

$\Rightarrow$ Interaction picture

$$\rho_I(t) = e^{\frac{i}{\hbar}(H_S + H_E)t} \rho(t) e^{-\frac{i}{\hbar}(H_S + H_E)t}$$

$$\Rightarrow \frac{d\rho_I}{dt} = \frac{i}{\hbar} e^{\frac{i}{\hbar}H_0 t}\left([H_0, \rho(t)] - [H, \rho(t)]\right) e^{-\frac{i}{\hbar}H_0 t}$$
$$= -\frac{i}{\hbar}\lambda [V_I(t), \rho_I(t)]$$

Formal solution:

$$\rho_I(t) = \rho_I(0) - \frac{i}{\hbar}\lambda \int_0^t dt' [V_I(t'), \rho_I(t')]$$

Expand to second order in λ (assumption I)

$$\rho_I(t) = \rho_I(0) - \frac{i}{\hbar}\lambda \int_0^t dt' [V_I(t'), \rho_I(0)] - \frac{\lambda^2}{\hbar^2} \int_0^t dt' \int_0^{t'} dt'' [V_I(t'), [V(t''), \rho_I(t'')]]$$

$\rho_S(t) = \mathrm{tr}_E\, \rho_I(t)$ and assume (assumption II) $\rho_I(t) = \rho_S(t) \otimes \rho_E$ and $\rho_E(t) = \rho_E$ (say $\rho_E = \frac{1}{Z} e^{-\beta H_E}$)

and $\mathrm{tr}_E\, \rho_E E_k = 0$

(without loss of generality!)

Taking derivative $\frac{d}{dt}$ and assume system dynamics to be slow $\rho_S(t'') \cong \rho_S(t)$ (assumption III)

$$\boxed{\frac{d\rho_S(t)}{dt} = -\frac{\lambda^2}{\hbar^2} \int_0^t d\tau\, \mathrm{tr}_E \left[V_I(t), [V_I(\tau), \rho_S(t) \otimes \rho_E]\right]}$$

Redfield equation

From Redfield to Lindblad?

Additional assumption of the so-called "rotating-wave" approximation

Write

$$S_k(t) = e^{iH_S t} S_k e^{-iH_S t} \qquad E_k(t) = e^{iH_E t} E_k e^{-iH_E t}$$

$$S_k(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \tilde{S}_k(\omega) \qquad G_{jk}(t-\tau) = tr_E\left(\rho_E\, E_j(t)\, E_k(\tau)\right)$$

We have double integrals of the form $\quad \tilde{G}_{jk}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, G_{jk}(t)$

$$\int d\omega\, e^{-i\omega t} \int d\omega' e^{-i\omega'\tau}$$ Only keep terms <u>with $\omega + \omega' = 0$</u>

$\Rightarrow$ We arrive at:

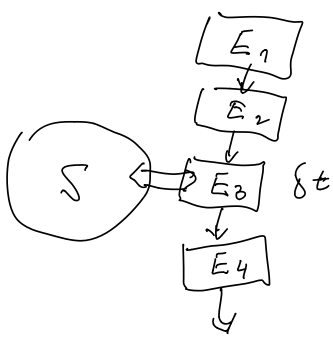
$$\frac{d\rho_S}{dt} = -\frac{\lambda^2}{\hbar^2} \int d\omega \sum_{jk} \tilde{G}_{jk}(\omega)\Big[\tilde{S}_j(\omega)\tilde{S}_k(-\omega)\rho_S(t) + \rho_S(t)\tilde{S}_j(\omega)S_k(-\omega) - 2S_k(-\omega)\rho_S(t)S_j(\omega)\Big]$$

Which is essentially the Lindblad equation (after diagonalizing $\tilde{G}_{jk}$)

<u>Excercise</u>: derive in detail

Alternative (non-perturbative) justifications of markovian QME:

Repeated interactions protocol



Quantum trajectories - stochastic wave-functions

$L_k \rightarrow$ jump operators

$$H_{ef} = H - i \sum_k \tfrac{1}{2} L_k^{\dagger} L_k$$

$$\frac{d\rho}{dt} = -i\Big(H_{ef}\rho - \rho H_{ef}^{\dagger} + \sum_k L_k \rho L_k^{\dagger}\Big)$$

1) $\dfrac{d|\psi(t)\rangle}{dt} = -\dfrac{i}{\hbar} H_{ef}|\psi(t)\rangle$

2) Poissonian clicks: in time δt, with probability

$$p_k = \delta t \, \| L_k |\psi(t)\rangle \|^2 = \delta t \langle \psi(t) | L_k^\dagger L_k | \psi(t)\rangle$$

the state jumps as $|\psi(t)\rangle \longrightarrow \dfrac{L_k |\psi(t)\rangle}{\| L_k |\psi(t)\rangle \|}$

$$\rho(t) = \mathbb{E}\left(|\psi(t)\rangle\langle\psi(t)| \right)$$

III Vectorization and superoperator formalism

"real" basis functions

$$\mathrm{vec}\,(|\psi\rangle\langle\varphi|) := |\psi\rangle|\varphi\rangle \qquad \mathrm{vec} : \mathrm{End}(\mathcal{H}) \to \mathcal{H} \otimes \mathcal{H}$$

$$\mathrm{vec}(A|\psi\rangle\langle\varphi|B) = (A \otimes B^T)\, \mathrm{vec}(|\psi\rangle\langle\varphi|)$$

but define vec as linear (not antilinear)

$$\mathrm{vec}(A\rho B) = (A \otimes B^t)\, \mathrm{vec}\rho$$

writing $\mathrm{vec}\rho = |\rho\rangle$, the Lindblad eq. attains Schrödinger lookalike form: ($\hbar := 1$)

$$\boxed{\begin{aligned} &\frac{d}{dt}|\rho\rangle = \hat{\mathcal{L}}|\rho\rangle \\ &\hat{\mathcal{L}} = -i(H\otimes \mathbb{1} - \mathbb{1}\otimes H) + \sum_k \left(L_k \otimes \bar{L}_k - \tfrac{1}{2} L_k^\dagger L_k \otimes \mathbb{1} - \tfrac{1}{2} \mathbb{1} \otimes \bar{L}_k^\dagger \bar{L}_k \right) \end{aligned}}$$

(Non-equilibrium) steady state (NESS) density matrix ρ_∞

$$\hat{\mathcal{L}}|\rho_\infty\rangle = 0$$

Always $\exists$, i.e. $0 \in \mathrm{spectrum}(\hat{\mathcal{L}})$, since $\langle \mathbb{1}|\hat{\mathcal{L}} = 0$
or $\hat{\mathcal{L}}^\dagger|\mathbb{1}\rangle = 0$

$$\hat{\mathcal{L}}^\dagger = i(H\otimes\mathbb{1} - \mathbb{1}\otimes H) + \sum_k \left(L_k^\dagger \otimes \bar{L}_k^\dagger - \tfrac{1}{2} L_k^\dagger L_k \otimes \mathbb{1} - \tfrac{1}{2}\mathbb{1}\otimes \bar{L}_k^\dagger \bar{L}_k \right)$$

$$\hat{\mathcal{L}}^\dagger|\mathbb{1}\rangle = \mathrm{vec}\left(iH - iH + \sum_k L_k^\dagger L_k - \tfrac{1}{2} L_k^\dagger L_k - \tfrac{1}{2} L_k^\dagger L_k \right) = \emptyset$$

This is also a consequence of trace preservation

$$0 = \frac{d}{dt}\,\mathrm{tr}\,\rho = \langle \mathbb{1}|\hat{\mathcal{L}}|\rho\rangle, \ \forall \rho$$

$$\mathrm{tr}\,\rho = \sum_j \langle j|\rho|j\rangle = \sum_j (\langle j|\otimes\langle j|)\,|\rho\rangle = \langle \mathbb{1}|\rho\rangle$$

Uniqueness of NESS $\Leftarrow$ Thm. (Evans 1977):

ρ_∞ is unique iff $\{H;\, L_k,\, L_k^\dagger\}$ generate the entire matrix algebra $\mathrm{End}(\mathcal{H}_s)$

An example: XXZ spin chain with boundary driving

$$H = \sum_{j=1}^{n-1} 2\sigma_j^+\sigma_{j+1}^- + 2\sigma_j^-\sigma_{j+1}^+ + \Delta\sigma_j^z\sigma_{j+1}^z$$

$$L_1 = \sqrt{\gamma_L^+}\,\sigma_1^+,\quad L_2 = \sqrt{\gamma_L^-}\,\sigma_1^-$$
$$L_3 = \sqrt{\gamma_R^+}\,\sigma_n^+,\quad L_4 = \sqrt{\gamma_R^-}\,\sigma_n^-$$

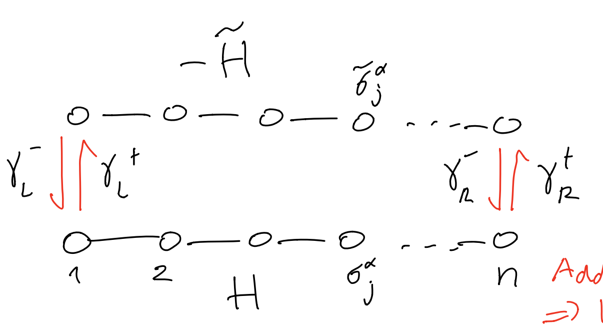
Define $\sigma_j^\alpha \equiv \sigma_j^\alpha \otimes \mathbb{1}$

$$\tilde{\sigma}_j^\alpha = \mathbb{1}\otimes P\sigma_j^\alpha P^{-1} \qquad P = \prod_j \sigma_j^x = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}^{\otimes n}$$

spin-flip

$$P\sigma_j^\pm P^{-1} = \sigma_j^\mp$$
$$P\sigma_j^z P^{-1} = -\sigma_j^z$$

$$\hat{\mathcal{L}} = -iH + i\tilde{H} - \frac{\gamma_L^+ - \gamma_L^-}{4}(\sigma_1^z - \tilde{\sigma}_1^z) + \gamma_L^-\,\sigma_1^+\tilde{\sigma}_1^- + \gamma_L^+\,\sigma_1^-\tilde{\sigma}_1^+$$
$$- \gamma_R^+ - \gamma_R(\sigma_n^z - \tilde{\sigma}_n^z) + \gamma_R^-\,\sigma_n^+\tilde{\sigma}_n^- + \gamma_R^+\,\sigma_n^-\tilde{\sigma}_n^+$$



Non hermitian interacting hopping problem on the ring of 2n sites

Additional dephasing jump op. $L_j = \sqrt{\gamma}\,\sigma_j^z$
$\Rightarrow$ Hubbard like hamiltonian with $U = i\gamma$

Operator Fock space approach ("Third quantization")

For simplicity we assume only, say n, fermionic d.o.f. represented by $2n$ Majorana operators

$$w_j \; ; \; j = 1 \ldots 2n$$

$$\{w_j, w_k\} = 2\delta_{j,k}$$

Examples: i) Physical fermions : $c_j = w_{2j-1} + i w_{2j}$
$c_j^{+} = w_{2j-1} - i w_{2j}$

ii) spins $\frac{1}{2}$:

$$\sigma_j^x = \Big(\prod_{l \le 2(j-1)} w_l\Big) w_{2j-1}$$

$$\sigma_j^y = \Big(\prod_{l \le 2(j-1)} w_l\Big) w_{2j}$$

$$\sigma_j^z = -i\, w_{2j-1} w_{2j}$$

A set of 2^{2n} operators

$$\mathcal{O}_{\alpha_1 \alpha_2 \ldots \alpha_{2n}} := 2^{-\frac{n}{2}} w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_{2n}^{\alpha_{2n}} \qquad \alpha_j \in \{0, 1\}$$

spans the entire $K = \mathrm{End}(\mathcal{H}_S)$ $\quad \mathcal{H}_S = \mathbb{C}^n$ and is ON w.r.t Hilbert-Schmidt inner product

$$\langle x | y \rangle = \mathrm{tr}\, x^{+} y$$

Define a set of $2 \times 2n$ superoperators (operators over K)

$$\hat{c}_j |\mathcal{O}_{\underline{\alpha}}\rangle = \alpha_j |w_j \mathcal{O}_{\underline{\alpha}}\rangle$$

$$\hat{c}_j^{+} |\mathcal{O}_{\underline{\alpha}}\rangle = (1-\alpha_j) |w_j \mathcal{O}_{\underline{\alpha}}\rangle \qquad \text{Excercise: derive expression for } \hat{c}_j^{+}$$

$\hat{c}_j, \hat{c}_j^{+}$ obey CAR:

$$\left.\begin{aligned} \{\hat{c}_j, \hat{c}_k\} &= 0 \\ \{\hat{c}_j, \hat{c}_k^{+}\} &= \delta_{j,k} \end{aligned}\right\} \quad \text{Excercise: prove CAR!}$$

Consider now a general quadratic system

$$H = \underline{w} \cdot \mathcal{H}\, \underline{w} \qquad \mathcal{H} \in i\mathbb{R}^{2n\times 2n}$$

$$L_k = \underline{\ell}_k \cdot \underline{w}$$

Task:

Write down the Lindbladian over $\mathcal{K}$

We start by def.:

$$\hat{w}_j^L |x\rangle = |w_j x\rangle$$

$$\hat{w}_j^R |x\rangle = |x w_j\rangle$$

$$\hat{w}_j^L = \hat{c}_j + \hat{c}_j^\dagger$$

$$\hat{w}_j^R = \hat{\mathcal{P}}(\hat{c}_j - \hat{c}_j^\dagger) = -(\hat{c}_j - \hat{c}_j^\dagger)\hat{\mathcal{P}} \quad (*)$$

where $\hat{\mathcal{P}} = \exp(i\pi \hat{\mathcal{N}})$

$$\hat{\mathcal{N}} = \sum_{j=1}^{2n} \hat{c}_j^\dagger \hat{c}_j \qquad \hat{\mathcal{P}}^2 = \mathbb{1}$$

To prove $(*)$:

$$\hat{c}_j |P_{\underline{\alpha}}\rangle = \alpha_j |w_j P_{\underline{\alpha}}\rangle = \alpha_j \hat{\mathcal{P}} |P_{\underline{\alpha}} w_j\rangle = -\alpha_j \hat{w}_j^R \hat{\mathcal{P}} |P_{\underline{\alpha}}\rangle$$

$$\hat{c}_j^\dagger |P_{\underline{\alpha}}\rangle = (1-\alpha_j)|w_j P_{\underline{\alpha}}\rangle = +(1-\alpha_j)\hat{\mathcal{P}}|P_{\underline{\alpha}} w_j\rangle = (1-\alpha_j)\hat{w}_j^R \hat{\mathcal{P}} |P_{\underline{\alpha}}\rangle$$

$$(\hat{c}_j - \hat{c}_j^\dagger)|P_{\underline{\alpha}}\rangle = -\hat{w}_j^R \hat{\mathcal{P}} |P_{\underline{\alpha}}\rangle$$

See note $*1$) at the bottom

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}_U + \hat{\mathcal{L}}_D$$

Unitary part:

$$\hat{\mathcal{L}}_U = -i\,\mathrm{ad} H = -i(\hat{\underline{w}}^L \cdot \mathcal{H}\, \hat{\underline{w}}^L - \mathcal{H}\, \hat{\underline{w}}^R \cdot \hat{\underline{w}}^R)$$

$$= -4i\, \hat{\underline{c}}^\dagger \cdot \mathcal{H}\, \hat{\underline{c}}$$

Dissipative part:

$$\hat{\mathcal{L}}_D \rho = \sum_k \left(2L_k^\dagger \rho L_k - \{L_k^\dagger L_k, \rho\}\right) = \sum_{j,j'} \overbrace{\left(\sum_k \ell_{kj} \bar{\ell}_{kj'}\right)}^{M_{jj'}} \hat{\mathcal{L}}_{jj'} \rho$$

$$\hat{\mathcal{L}}_{jj'} = 2\hat{w}_j^L \hat{w}_{j'}^R - \hat{w}_j^L \hat{w}_{j'}^L - \hat{w}_j^R \hat{w}_{j'}^R$$

Excercise: Derive $\hat{\mathcal{L}}_{jj'} = \frac{\mathbb{1}+\hat{\mathcal{P}}}{2}\left(4\hat{c}_j^{+}\hat{c}_{j'}^{+} - 2\hat{c}_j\hat{c}_{j'}^{+} - 2\hat{c}_{j'}^{+}\hat{c}_j\right) + \frac{\mathbb{1}-\hat{\mathcal{P}}}{2}\left(4\hat{c}_j\hat{c}_{j'} - 2\hat{c}_j\hat{c}_{j'}^{+} - 2\hat{c}_{j'}\hat{c}_j^{+}\right)$

$$\Rightarrow \hat{\mathcal{L}}_D = -2\hat{\underline{c}}^{+}\cdot(M-M^T)\hat{\underline{c}}^{+} - 2\hat{c}_j^{+}\cdot(M+M^T)\hat{\underline{c}}$$

Defining 4n Majorana operators:

on subspace

$$\mathcal{K}_+ = \frac{\mathbb{1}+\hat{\mathcal{P}}}{2}K$$

$$\hat{a}_{2j-1} = \frac{\hat{c}_j + \hat{c}_j^{+}}{\sqrt{2}} \qquad \hat{a}_{2j} = \frac{i(\hat{c}_j - \hat{c}_j^{+})}{\sqrt{2}}$$

$$\{\hat{a}_r, \hat{a}_s\} = 4\delta_{r,s} \qquad r,s = 1..4n$$

the Liouvillean can be expressed as a quadratic form:

$$\hat{\mathcal{L}} = \hat{\underline{a}}\cdot A\,\hat{\underline{a}} - A_0\mathbb{1} \qquad A_0 = 2\,\mathrm{tr}\,M$$

$$A = -2i\mathcal{H}\otimes\mathbb{1}_2 - 2M_r\otimes\sigma^y - 2M_i\otimes(\sigma^x - i\sigma^z)$$

$$M_r = \tfrac{1}{2}(M+\bar{M}) = M_r^T$$
$$M_i = \tfrac{1}{2}(M-\bar{M}) = -M_i^T$$

Writing $J = \mathbb{1}_{2n}\otimes\sigma^x$

$$\Rightarrow \bar{A} = JAJ$$

Non-hermitian Bogolyubov transformation

$$A = V^T B V \qquad VV^T = J \qquad B = \begin{pmatrix} 0 & \beta_1 & & & \\ -\beta_1 & 0 & & & \\ & & 0 & \beta_2 & \\ & & -\beta_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$A v_{2j-1} = \beta_j v_{2j-1} \qquad v_{2j}\cdot v_{2j'} = 0$$
$$A v_{2j} = -\beta_j v_{2j} \qquad v_{2j-1}\cdot v_{2j'} = 0$$
$$\mathrm{Re}\,\beta_j \geq 0 \qquad v_{2j-1}\cdot v_{2j'} = \delta_{jj'} \qquad V = \begin{pmatrix} v_1 \\ \hline v_2 \\ \hline \vdots \\ v_{4n} \end{pmatrix}$$

Excercise: Show that β_j are eigenvalues of $2n\times 2n$ matrix

$$X = -2iH + 2M_r$$

Normal master modes

nonhermitian CAR

$$\hat{\underline{b}} = (\hat{b}_1, \hat{b}_1', \ldots, \hat{b}_{2n}, \hat{b}_{2n}') := V\hat{\underline{a}}$$

$$\hat{b}_j = v_{2j-1}\cdot\hat{\underline{a}} \qquad \{\hat{b}_j, \hat{b}_{j'}\} = \{\hat{b}_j', \hat{b}_{j'}'\} = 0$$
$$\hat{b}_j' = v_{2j}\cdot\hat{\underline{a}} \qquad \{\hat{b}_j, \hat{b}_{j'}'\} = \delta_{jj'}$$

$$\hat{\mathcal{L}} = \hat{\underline{a}}\cdot V^T B V \hat{\underline{a}} - A_0 = \hat{\underline{b}} \cdot B \hat{\underline{b}} - A_0 \qquad A_0 = \sum_{j=1}^{2n} \beta_j$$

$$= -\sum_{j=1}^{2n} \beta_j (\hat{b}_j' \hat{b}_j - \hat{b}_j \hat{b}_j') - A_0$$

$$\boxed{\hat{\mathcal{L}} = -2 \sum_{j=1}^{2n} \beta_j \hat{b}_j' \hat{b}_j}$$

Normal master mode decomposition

$\hat{n}_j = \hat{b}_j' \hat{b}_j$ are mutually commuting ops. with eigenvalues $\{0,1\}$

NESS is a Fock vacuum, determined by

$$\hat{b}_j |NESS\rangle = 0$$

$$\langle \mathbb{1} | \hat{b}_j' = 0$$

And the full spectral decomposition of Liouvillean dynamics reads:

$$e^{t\hat{\mathcal{L}}} = \sum_{\underline{\nu} \in \{0,1\}^{2n}} \exp\left(-t \sum \nu_j \beta_j\right) |\underline{\nu}\rangle\langle\underline{\nu}|$$

$$|\underline{\nu}\rangle = (\hat{b}_1')^{\nu_1} \cdots (\hat{b}')^{\nu_{2n}} |NESS\rangle$$

$$\langle\underline{\nu}| = \langle\mathbb{1}| (\hat{b})^{\nu_1} \cdots (\hat{b})^{\nu_{2n}}$$

<u>Excercise:</u> Work out an example of 2-level quantum system $(n=1)$

$$w_1 = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$w_2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$H = -ih\, w_1 w_2 = \begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix}$$

$$\underline{l}_1 = \gamma^+ (1, i) \Rightarrow L_1 = \sqrt{\gamma^+}\, \sigma^+$$

$$\underline{l}_2 = \gamma^- (1, -i) \Rightarrow L_2 = \sqrt{\gamma^-}\, \sigma^-$$

Show that $\quad \operatorname{tr} \rho_\infty c^+ c = \frac{1}{2} - \frac{i}{2} \operatorname{tr} \rho_\infty w_1 w_2$

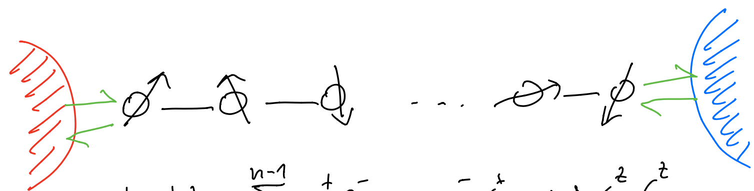
$$= \frac{1}{2} - \frac{i}{2} \langle \mathbb{1} | \hat{c}_1^+ \hat{c}_2^+ | NESS\rangle = \frac{\gamma^+}{\gamma^+ + \gamma^-}$$

$$\beta_{1,2} = \gamma^+ + \gamma^- \pm ih$$

IV) Boundary driven interacting open systems

— exact solutions in terms of matrix product ansatz —

Canonical example: BD XXZ chain:



$L_1 = \sqrt{\frac{\varepsilon}{2}(1+\mu)}\,\sigma_1^+$ $\quad H = \sum_{j=1}^{n-1} 2\sigma_j^+\sigma_{j+1}^- + 2\sigma_j^-\sigma_{j+1}^+ + \Delta\sigma_j^z\sigma_{j+1}^z$ $\quad L_3 = \sqrt{\frac{\varepsilon}{2}(1-\mu)}\,\sigma_n^+$

$L_2 = \sqrt{\frac{\varepsilon}{2}(1-\mu)}\,\sigma_1^-$ $\quad L_4 = \sqrt{\frac{\varepsilon}{2}(1+\mu)}\,\sigma_n^-$

ε – coupling strength

μ – bias / driving strength

Exact solutions possible for

i) – second order perturbative expansion in ε (for any μ)

ii) – nonperturbative, for any ε (for maximum driving $\mu = \pm 1$)

Steady state LE: (take $\mu = 1$; pure source-sink driving)

$$i[H,\rho_\infty] = \varepsilon\,\hat{\mathcal{D}}_{\sigma_1^+}(\rho_\infty) + \varepsilon\,\hat{\mathcal{D}}_{\sigma_n^-}(\rho_\infty) \qquad \text{(SSLE)}$$

$$\hat{\mathcal{D}}_L(\rho) := 2L\rho L^+ - \{L^+L,\rho\}$$

Step 1: Cholesky factorization of NESS density matrix

$$\rho_\infty = \frac{\Omega_n\Omega_n^+}{\mathrm{tr}(\Omega_n\Omega_n^+)} \qquad \Omega_n \text{ 'amplitude operator'}$$

Lemma: ρ_∞ solves SSLE iff: Ω_n satisfies a recurrence:

$$[H,\Omega_n] = -i\varepsilon(\sigma^z\otimes\Omega_{n-1} - \Omega_{n-1}\otimes\sigma^z) \qquad (*)$$

and is of upper triangular form

$$\Omega_n = \sigma^0\otimes\Omega_{n-1} + \sigma^+\otimes\Omega_{n-1}^{\wedge} = \Omega_{n-1}\otimes\sigma^0 + \Omega_{n-1}^-\otimes\sigma^-$$

Proof: Excercise

Step 2: MPA for amplitude operator

$$\Omega_n = \sum_{\alpha_1 \ldots \alpha_n \in \{+,-,0\}} \langle 0| A_{\alpha_1} A_{\alpha_2} \cdots A_{\alpha_n} |0\rangle \, \sigma^{\alpha_1} \otimes \sigma^{\alpha_2} \cdots \otimes \sigma^{\alpha_n}$$

Note that $\alpha = z$ is missing!

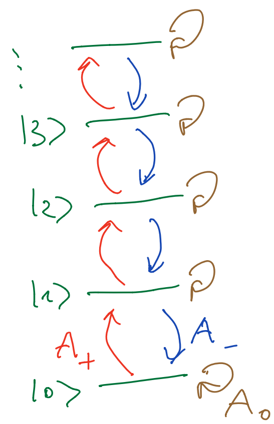
$[H, \Omega_n]$ contains a single σ^z operator! ("defect")

Organizing terms around a defect in $E_q(x)$, we obtain an algebra for A_0, A_+, A_- which we can solve in terms of tridiagonal ansatz:

$$A_0 = \sum_{k=0}^{\infty} a_k \, |k\rangle\langle k|$$

$$A_+ = \sum_{k=0}^{\infty} b_k \, |k\rangle\langle k+1|$$

$$A_- = \sum_{k=0}^{\infty} |k+1\rangle\langle k|$$



$\Rightarrow$ recurrence for a_k, b_k

$$a_{k+1} - 2\Delta a_k + a_{k-1} = 0$$

$$b_{k+1} - b_k = 2 a_{k+1} (\Delta a_{k+1} - a_k)$$

Writing $\Delta = \cos\eta$

we have a solution

$$\boxed{a_k = \cos k\eta + \frac{i\varepsilon}{2} \frac{\sin(k\eta)}{\sin\eta}}$$

$$\boxed{b_k = \sin(k+1)\eta \left(\frac{i\varepsilon}{\sin\eta} \cos k\eta - \left(1 + \frac{\varepsilon^2}{4\sin^2\eta}\right) \sin k\eta \right)}$$

Lax representation of NESS

Consider $U_q(sl_2)$ Lax operator

$$L(\varphi, s) = \begin{pmatrix} \sin(\varphi + \eta S_s^z) & (\sin\eta) S_s^- \\ (\sin\eta) S_s^+ & \sin(\varphi - \eta S_s^z) \end{pmatrix}$$

obeying YBE:

$$\check{R}_{1,2}(\varphi_1 - \varphi_2) L_1(\varphi_1) L_2(\varphi_2) = L_1(\varphi_2) L_2(\varphi_1) \check{R}_{1,2}(\varphi_1 - \varphi_2)$$

for a generic (complex spin s) highest weight irrep

$$S_s^z = \sum_{k=0}^{\infty} (s-k) |k\rangle\langle k|$$

$$S_s^+ = \sum_{k=0}^{\infty} \frac{\sin(k+1)\eta}{\sin\eta} |k\rangle\langle k+1|$$

$$S_s^- = \sum_{k=0}^{\infty} \frac{\sin(2s-k)\eta}{\sin\eta} |k+1\rangle\langle k|$$

Since $\partial_\varphi \check{R}(\varphi)\big|_{\varphi=0} = \frac{1}{2}(h^{XXZ} + \cos\eta\, \mathbb{1})$ $\quad h^{XXZ} = 2\sigma^+ \otimes \sigma^- + 2\sigma^- \otimes \sigma^+ + \Delta \sigma^z \otimes \sigma^z$

$$\Rightarrow \quad [h^{XXZ}_{1,2}, L_1(\varphi) L_2(\varphi)] = L_1'(\varphi) L_2(\varphi) - L_1(\varphi) L_2'(\varphi)$$

$$\Rightarrow \quad [H, \langle 0| L_1(\varphi) L_2(\varphi) \cdots L_n(\varphi) |0\rangle] = \langle 0| L_1'(\varphi) L_2(\varphi) \cdots L_n(\varphi) |0\rangle - \langle 0| L_1(\varphi) \cdots L_{n-1}(\varphi) L_n'(\varphi) |0\rangle$$

$\Omega^T(\varphi, s)$ fulfills The Lemma iff $\langle 0| L'(\varphi) = -i\varepsilon\, \sigma^z \langle 0|$

$\qquad L'(\varphi) |0\rangle = i\varepsilon\, \sigma^z |0\rangle$

$$\Leftrightarrow \quad \boxed{\varphi = \frac{\pi}{2} \qquad \tan\eta s = \frac{i\varepsilon}{2\sin\eta}}$$

Conclusions: What can we do with this NESS?

a) Compute local observables

$$\langle a_j \rangle = \operatorname{tr} \rho_\infty a = \frac{\operatorname{tr} \Omega_n \Omega_n^+ a_j}{\operatorname{tr} \Omega_n \Omega_n^+}$$

using an auxilliary transfer matrix technique

$$\langle a \rangle = \frac{\langle\langle 0,0| \mathbb{T}^{j-1} \mathbb{A} \mathbb{T}^{n-j} |0,0\rangle\rangle}{\langle\langle 0,0| \mathbb{T}^n |00\rangle\rangle}$$

b) Construct new, quasi-local conservation laws of the model.
Example, take a perturbative limit $\varepsilon \to 0$ and consider $O(\varepsilon)$ term of The Lemma:

$$\Omega_n = \mathbb{1} + i\varepsilon Z + O(\varepsilon^2)$$

$$\boxed{[H, Z] = -\sigma_1^z + \sigma_n^z}$$

c) Attack other models:

- Spin 1 or higher spin chains
- Hubbard model

...

More in the Review:
J. Phys. A 48 (2015) 373001

*1 Note

$$\hat{P}\,|P_{\underline{\alpha}}\rangle = (-1)^{|\alpha|}\,|P_{\underline{\alpha}}\rangle$$

$$\hat{c}_j^{+}\,|P_{\underline{\alpha}}\rangle = (1-\alpha_j)\,|w_j P_{\underline{\alpha}}\rangle = (-1)^{|\alpha|-1}(1-\alpha_j)\,|P_{\underline{\alpha}} w_j\rangle$$

$$\hat{c}_j\,|P_{\underline{\alpha}}\rangle = \alpha_j\,|w_j P_{\underline{\alpha}}\rangle = (-1)^{|\alpha|}\alpha_j\,|P_{\underline{\alpha}} w_j\rangle$$

$$(\hat{c}_j^{+}-\hat{c}_j)\,|P_{\underline{\alpha}}\rangle = -(-1)^{|\alpha|}\,\hat{w}_j^{R}\,|P_{\underline{\alpha}}\rangle$$

$$= -\hat{w}_j^{R}\,\hat{P}\,|P_{\underline{\alpha}}\rangle$$

$$\hat{w}_j^{R}\,\hat{P} = \hat{c}_j^{+}-\hat{c}_j$$

$$\hat{w}_j^{R} = (\hat{c}_j^{+}-\hat{c}_j)\,\hat{P}$$