Topological matter in cold atoms systems - Haldane phase and topological superconductivity

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The scope of these lectures is to illustrate the potential of cold atoms systems to investigate paradigmatic phenomena in topological matter. The focus will be on two examples, the Haldane phase and topological superconductivity. Beyond theoretical considerations, state of the art quantum engineering and probing methods will be discussed.

The lectures are divided into three parts. In the first one, we will review some generic features of topological phases in direct continuations with the previous series of lectures on quantum Hall effects, and some basic aspects of cold atoms experiments. In the second part, we will discuss the origin of topological phases in quantum spin chains, first from a theoretical perspective, and then by considering direct realization in cold atom systems. In the last part, we will discuss topological superconductivity in one-dimensional systems, in particular, Kitaev's wire and generalization to number conserving settings.

The *leitmotiv* of this short course is to emphasize general concepts via simple, paradigmatic examples. We will not discuss more sophisticated tools to characterize states of matter, such as the Altland-Zirnbauer classification of free theories and their generalizations to interacting systems, but rather point out connection to the models presented here. On the cold atom side, references to technical features will be kept to the minimum.

Several references will be given over the course of the lectures. Here are three general suggested readings:

- E. Fradkin, Field theories of condensed matter systems;
- B. A. Bernevig, Topological insulators and topological superconductors;
- J. Dalibard et al., Rev. Mod. Phys. 83, 1523 (2011)

1 Introduction to topological matter in cold atom systems

1.1 Some features of topological phases

Before discussing the specific examples below, it is worth reminding some generic features of topological phases (here, to be broadly intended as both symmetry-protected and true topological order):

- 1. **non-local order parameters**: in contrast to symmetry broken, paramagnetic, and critical phases, topological order is characterized by non-local order parameters i.e., correlation functions which are able to unambiguously distinguish such phase. A typical example here are Polyakov loops in lattice gauge theories (which we do not cover here). We will see how this applies to one-dimensional systems in the case of the Haldane chain;
- 2. **spectral degeneracies**: in addition to the presence of a bulk gap (we will however also briefly discuss how topological states can coexist with gapless bulk degrees of freedom), the spectrum of the Hamiltonian operator changes when changing the topology of the system (as already seen in the QH case). This is an hallmark feature of topology and is sometimes connected with
- 3. **existence of gapless boundary modes**, also seen in the QH case. We will see how this concept applies in both examples below;
- 4. **specific and universal entanglement properties**: this is a direct connection between the non-local correlations intrinsic of topological order, and entanglement. This connection is only partly explored: we will discuss few applications in the context of entanglement spectra;
- 5. **existence of anyonic excitations**: often times, topological matter hosts excitations with exotic statistics. Those are particularly useful in the context of fault tolerant quantum computation. We will see one example Majorana zero modes, realizing Ising anyons in the context of topological superconductivity.

This list is useful for two main reasons: first, we will re-call several of these concepts in the examples; second, it makes crystal clear that the challenges in realizing topological matter are not only at the level of finding the proper system supporting topological order, but actually, probing the latter might be equally (if not more) difficult!

There is another important element which is missing in the list, that is, resilience to certain classes of local perturbations. This is directly related to 1 and 2 above. If time allows, we will discuss this point as well in the last lecture.

1.2 Realizing and probing topological phases in cold atom systems: challenges

The simple fact that the first observation of topologically ordered phases took place in the early 80s evidences how experimentally accessing topological matter presents qualitatively new challenges with respect to (Ginzbug-Landau) long-range order, such as antiferromagnetism. Within the cold atom context, the challenges are both in terms of realizing a Hamiltonian dynamics that leads to states displaying topological order, and of probing the characteristics features above. Over the course of the lectures, we will touch both aspects.

2 The Haldane phase - a paradigmatic example of symmetryprotected topological phase

Goal of this section:

to show that the 1D S-1 Heisenberg model supports a **gapped phase**, which we call **Haldane phase**. The latter is characterized by **non-local order parameters**, the existence of **boundary zero-energy modes**, and a **degenerate entanglement spectrum**.

2.1 The spin-s 1D Heisenberg model: Hamiltonian and symmetries

We are interested in the dynamics of the one-dimensional Heisenberg model, defined by the Hamiltonian:

$$H_{HS} = J \sum_{i=1}^{L} \vec{S}_{i} \vec{S}_{i+1} = J \sum_{i=1}^{L} \sum_{\alpha = x, y, z} S_{i}^{\alpha} S_{i+1}^{\alpha}$$
(1)

where S_i^{α} is the spin-*s* operator along the α direction at the site *i*, *J* is the coupling strength, and we have assumed periodic boundary conditions, $S_{I+k}^{\alpha} \equiv S_k^{\alpha}$.

The model is invariant under SU(2) transformation, and thus has SU(2) symmetry.

Exercise 1

verify the SU(2) symmetry of the model.

We really do not need much more information before stating the first result on the model.

2.2 The Haldane 'conjecture'

The Haldane conjecture states that the spectrum of H_{HS} is gapless for half-integer *s*, and gapped for integer *s*.

This 'conjecture' (actually, it was proven by Haldane within certain approximations, see below) represents the starting point to analyze the Haldane phase, so we start by proving it. The proof relies on an approximate mapping between H_{HS} and an O(3) non-linear sigma model (NLSM), which at the time of Haldane's paper was already quite well understood in the absence of a topological term.

2.2.1 Mapping to an O(3) non-linear σ -model

We start by replacing the original spin operators \vec{S}_i with a pair of slowly varying field (slowly with respect to the lattice spacing *a*):

$$\vec{S}_i = S_i(\pm s\vec{\varphi}(x) + \vec{\ell}(x)) \tag{2}$$

where the first field represents a staggered magnetization ($\varphi_i^z = +1$ for a classical antiferromagnet along the *z* direction), and \vec{l} represent fluctuations on the top of this order. The justification of this representation is that the classical ground state of the model displays long-range order. We will see later on in which respect this allows simple approximations to carried out.

We now impose that these low-lying fields are defined on the dual lattice of sites - and will check the consequences of this choice later on. In particular, we defined them on the sites (2i + 1/2), so that:

$$\vec{\varphi}(2i+1/2) = \frac{\vec{S}_{2i+1} - \vec{S}_{2i}}{2s}, \qquad \vec{\ell}(2i+1/2) = \frac{\vec{S}_{2i+1} + \vec{S}_{2i}}{2a_0}$$
(3)

The choice of a two-site magnetic unit cell is again inspired by the classical solution; the latter is recovered by suppressing the fluctuation field ℓ . Here, the lattice spacing is $a_0 = 2a$.

The pair of fields we introduce satisfy some noteworthy properties:

$$\vec{\varphi}_{2i+1/2} \cdot \vec{\ell}_{2i+1/2} = 0, \qquad \varphi_{2i+1/2}^2 = 1 + \frac{1}{s} - a_0^2 \ell_{2i+1/2}^2 / s^2$$
 (4)

and the following commutation relations:

$$[\varphi_{2i+1/2}^{a}, \varphi_{2j+1/2}^{b}] = a_0 i \epsilon^{abc} \delta_{ij} \ell_{2i+1/2}^{c} / 2s^2, \quad [\ell_{2i+1/2}^{a}, \ell_{2j+1/2}^{b}] = i \epsilon^{abc} \delta_{ij} \ell_{2i+1/2}^{c} / 2a_0$$
(5)

$$[\ell_{2i+1/2}^{a}, \varphi_{2j+1/2}^{b}] = i\epsilon^{abc}\delta_{ij}\varphi_{x}^{c}/2a_{0}$$
(6)

Exercise 2

We now take the continuum limit, by replacing these unit cell operators with slowly varying fields:

$$\varphi_{2i+1/2}^{a} \to \varphi^{a}(x), \ell_{2i+1/2}^{a} \to \ell^{a}(x),$$
(7)

with $x = (2i + 1/2)a_0$, and then continuum delta function defined as $\frac{\delta_{ij}}{2a_0} \rightarrow \delta(x - y)$. After this limit is taken, the commutation relations read:

$$[\varphi^a(x),\varphi^b(y)] = 0, \quad [\ell^a(x),\ell^b(y)] = i\epsilon^{abc}\ell^c(x)\delta(x-y), \quad [\ell^a(x),\varphi^b(y)] = i\epsilon^{abc}\varphi^c(x)\delta(x-y)$$
(8)

which shall be interpreted as follows: the smooth variations ℓ satisfy SO(3) algebra, while their action on the orientation field is just to rotate it. In addition, we retain $n(x) \cdot m(x) = 0$.

We have now all the pieces to carry out the continuum limit of the Hamiltonian operator. First, we decompose it into two parts (due to our violation of the lattice parity symmetry, this is convenient), $H = J \sum_{i=1}^{L/2} \vec{S}_{2i} \cdot (\vec{S}_{2i+1} + \vec{S}_{2i-1})$, then we express each term as a function of the field operators:

$$\vec{S}_{2n}\vec{S}_{2n+1} = (s\varphi + a_0\ell)(a_0\ell - s\varphi) = a_0^2\ell^2 - s^2\varphi^2 = -s^2 + 2a_0^2\ell^2$$
(9)

where we used Eq. (4) and neglected 1/s corrections, and

$$\vec{S}_{2n}\vec{S}_{2n-1} = \dots \tag{10}$$

$$= 2a_0^2\ell^2 + 2a_0s^2\varphi \cdot (\varphi' - a_0\varphi'') + 2sa_0^2(\ell\varphi' - \ell'\varphi) + \dots =$$
(11)

which is obtained by expanding properly the field φ , keeping its first derivatives (since $\ell = \partial_t \varphi$, we neglet its derivatives are they're higher order. This can also be seen by looking at the lattice spacing contributions before taking the continuum limit), and considering $\varphi \cdot \varphi' = 0$, $-\varphi' \ell = \varphi \ell'$

2.2.2 Hamiltonian in the continuum

The total Hamiltonian in the continuum limit thus reads (remember: $\sum_i \rightarrow \int dx/a_0$):

$$H_{HS} = \frac{J}{2a_0} \int dx \left[+4a_0^2 \ell^2 - 2a_0^2 s^2 \varphi \cdot \varphi'' + 2sa_0^2 (\ell \varphi' + \varphi' \ell) \right] = \frac{J}{2a_0} \int dx \left[4a_0^2 \ell^2 + 2a_0^2 s^2 (\varphi')^2 + 2sa_0^2 (\ell \varphi' + \varphi' \ell) \right] + \dots$$
(13)

and then, after defining the sound velocity v (actually, it is the same as the spin wave one) and the θ angle

$$v = 2sJa_0, \quad \theta = 2\pi s, \quad g = 2/s \tag{14}$$

the final Hamiltonian reads

$$H_{HS} = \frac{v}{2} \int dx \left[g(\ell - \frac{\theta}{4\pi} \varphi')^2 + \frac{(\varphi')^2}{g^2} \right]$$
(15)

which, when completed with the constraint $\varphi^2 = 1$, corresponds to the (1+1)-d non-linear σ model. For this model, it is known that the spectrum is massive for generic values of θ (in fact, this is a typical occurrence of dynamical mass generation via instantons), while it is gapless for $\theta = \pi$. This proves Haldane's 'conjecture' ¹. The topological origin of the θ -term is directly manifest in the Lagrangian formulation, where this is related to the Pontryagin index - we will see this in the next subsection.

It is worth noting that, at the time of Haldane's original work, the $\theta = 0$ case was well studied, and the presence of a mass gap was proven. However, the gaplessness of the $\theta = \pi$ case was not yet established, and Haldane provided arguments in support if that (it was later proven by Read and ...). This somehow was going into the opposite direction with respect to what was known on the lattice, where the S=1/2 case was known to be gapless from Bethe's solution, and S>1/2 exactly soluble chains were already known to be gapless upon the addition of further couplings (Takhtajan and Babujian).

2.2.3 Lagrangian formulation of the NLSM and topological origin of the θ -term

It is possible to verify that the Hamiltonian below corresponds to the following action in Euclidean space-time:

$$S = S_{NLSM} + i\theta Q, \quad S_{NLSM} = \int dx d\tau (\partial_{\mu}\varphi)^2, \quad Q = \int dx d\tau \frac{\epsilon_{\mu\nu}}{8\pi} \varphi(x) \cdot (\partial_{\mu}\varphi(x) \times \partial_{\nu}\varphi(x))$$
(16)

It is already apparent that the last term does not have a clear-cut interpretation in the twodimensional classical statistical mechanics context. However, the important aspect here is that Q is 1) a conserved quantity, and 2) has topological origin, and is an integer.

2.2.4 Numerical experiments

Numerical evidence in support of the Haldane conjecture for

¹The mass gap of the NLSM would be scaling as Js^2e^{-s} for $s \gg 1$ - a signature of non-perturbative dynamical mass generation.

2.3 String order parameters and Kennedy-Tasaki transformation

So far, we have only provided evidence for the presence of a finite mass gap.

3 Haldane phase in synthetic quantum systems

So far, several properties of the Haldane phase have been already experimentally verified in quasi-1D magnetic systems - in particular, the finite gap was spectroscopically confirmed. Synthetic quantum systems are complementary to solid state ones in the sense that a different set up probing tools is available, and as such, other key feature of topological order - not accessible in magnetic systems - can be probed. What we have in mind here are especially non-local order parameters and entanglement features.

The first challenge on this way is to find system dynamics which (approximately) reproduce the behavior of the 1D s-1 Heisenberg model, or, even better, of the Bilinear-biquadratic version.

4 One-dimensional topological superconductors - the Kitaev model

5 Syllabus

- 1. Second part [Lect. 4-5]: non-local order in one-dimensional spin liquids. On the nature of the hidden order, a very good reference is the long article by Kennedy and Tasaki [?] (pay attention to few typos in Sec. 2). A good reference on antiferromagnetic spin chains is the review article by Affleck [?], a bit old but excellent in terms of field theory insights. Some good references for the O(3) non-linear σ -model are contained in Ref. [?].
 - One-dimensional spin-liquids hidden order
 - the Haldane chain: mapping to a non-linear sigma model and topological angles;
 - string order, hidden symmetries, and Kennedy-Tasaki transformation;
 - AKLT ground state and edge states.
- 2. Fourth part [Lect. 8-9]: topological insulators and non-Abelian anyons. The best reference on the Kitaev model is Kitaev's original article [??]. A nice review by Ludwig provides an relatively up-to-date account on the classification of non-interacting fermions [?], which an especially clear part on Anderson localization.
 - Kitaev model: phase diagram via exact solution. Relation to the Ising model and role of the boundary conditions.
 - Majorana edge modes.
 - beyond Majorana: parafermions and clock chains.
 - (if time allows) towards the Altland-Zirnbauer classification of non-interacting topological matter

6 Other info

Exam. – Some of you might be interested in carrying out an exam, valid for certain phDs in Italy. Please, contact me before the end of the lectures to set this up. The exam will consist in a carrying out an exercise, either analytical or numerical, and producing a written report discussing the solution.