Cold atoms, Random Matrix Theory and the KPZ equation

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P. Le Doussal (LPT, ENS, Paris, France)
A. Rosso, G. Schehr, C. Texier (LPTMS, Université Paris-Sud, Orsay, France)
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- Phys. Rev. A 91, 012303 (2015)
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- J. Stat. Mech. P063301 (2017)
- Phys. Rev. Lett. 119, 130601 (2017)
- Europhys. Lett. 120, 10006 (2017)
- arXiv: 1711.07770, to appear in Sci. Post. (2018)
- arXiv: 1801.02680

Plan

- N spinless Fermions in a 1-d harmonic trap at $T = 0$

  $\Rightarrow$ GUE random matrices

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- Summary and Conclusion

Ultracold atoms

- Recent great progress in the experimental manipulation of cold atoms

  $\Rightarrow$ to investigate the interplay between quantum and statistical behaviors in many-body systems at low temperatures

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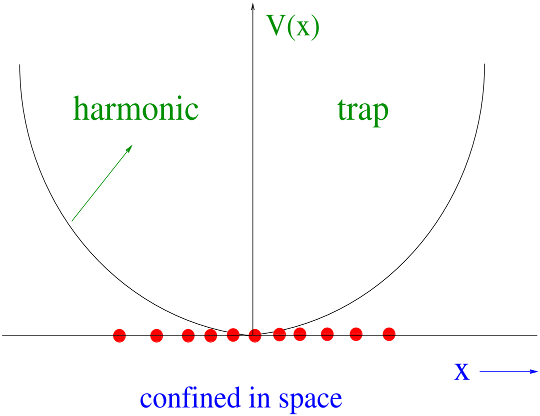
Bosons: Bose-Einstein condensation

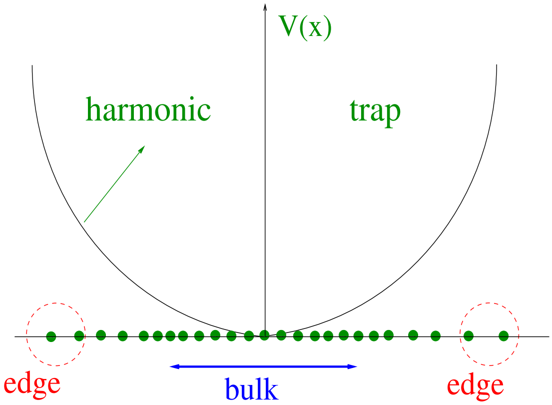
Fermions: Pauli exclusion principle $\Rightarrow$ rich quantum many-body physics

Ultracold atoms in a confining potential

A common feature of these experiments ⇒ presence of a confining potential that traps the particles within a limited spatial region



Ultracold atoms in a trap → edge physics



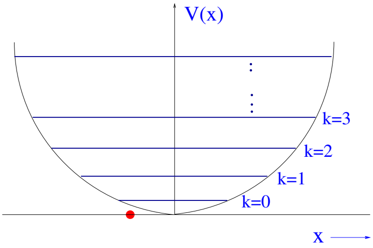
- bulk: traditional many-body physics (translationally invariant system)
- edge: new physics induced by confinement ⇒ universal edge properties

$T = 0$ free fermions in a 1-d harmonic trap

&

Random matrix theory (RMT)

A single Fermion in a harmonic trap



A single quantum particle in a harmonic potential: $V(x) = \frac{1}{2} m\omega^2 x^2$

Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi_k}{dx^2} + \frac{1}{2} m\omega^2 x^2 \varphi_k(x) = \epsilon_k \varphi_k(x)$$

with $\varphi_k(x \to \pm\infty) = 0$

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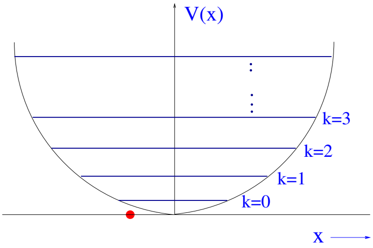

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single particle eigenfunctions: $\varphi_k(x) = \left[\frac{\alpha}{\sqrt{\pi}\, 2^k\, k!}\right]^{1/2} e^{-\alpha^2 x^2/2} H_k(\alpha\, x)$

with energy levels: $\epsilon_k = (k + 1/2)\,\hbar\,\omega \quad k = 0, 1, 2, 3 \ldots$

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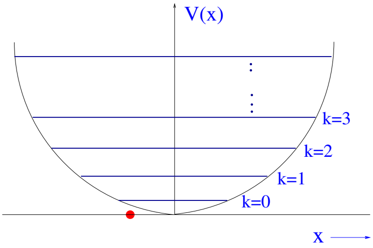

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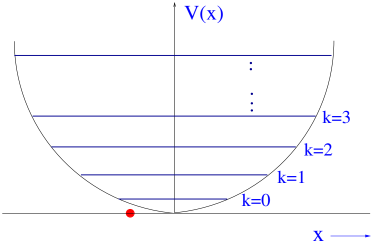

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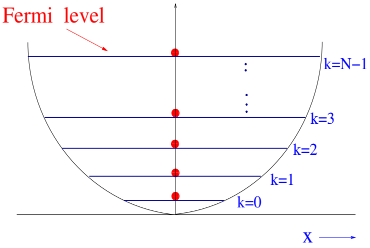
$H_k(x) \to$ Hermite polynomials

For example, $H_0(x) = 1$, $H_1(x) = 2\, x$, $H_2(x) = 4\, x^2 - 2$, etc.

N spinless Fermions in a harmonic trap: T=0



ground state many-body wavefunction → Slater determinant

$\Psi_0(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det[\varphi_i(x_j)]$

with $0 \leq i \leq (N-1)$, $1 \leq j \leq N$

ground state energy: $E_0 = \hbar\omega\, N^2/2$

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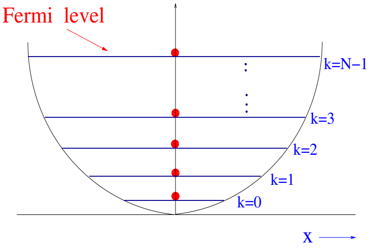

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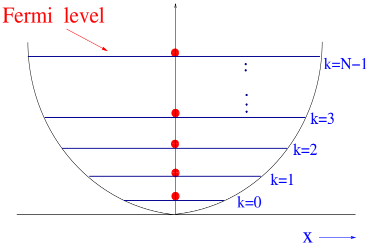

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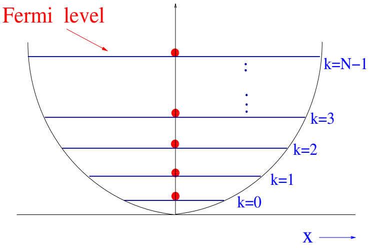

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The determinant $\det_{1\le i,j\le N}[H_i(\alpha\, x_j)] \Rightarrow$ can be explicitly evaluated

Vandermonde determinant

Example: $N = 3$: $H_0(x) = 1,\ H_1(x) = 2\,x,\ H_2(x) = 4\,x^2 - 2$

$$\det \begin{pmatrix} H_0(x_1) & H_0(x_2) & H_0(x_3) \\ H_1(x_1) & H_1(x_2) & H_1(x_3) \\ H_2(x_1) & H_2(x_2) & H_2(x_3) \end{pmatrix}$$

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$$= 8 \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}$$

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$$= 8\,(x_1 - x_2)\,(x_2 - x_3)\,(x_3 - x_1)$$

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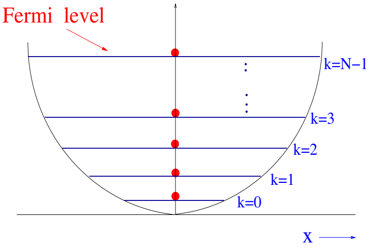

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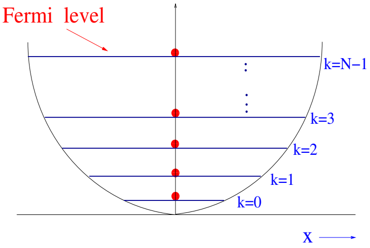

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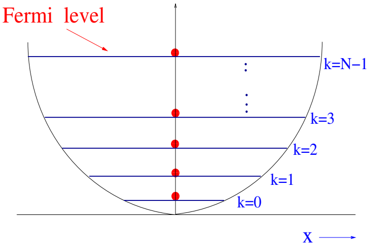

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$$\Longrightarrow \quad \boxed{|\Psi_0(\{x_i\})|^2 = \frac{1}{Z_N} e^{-\alpha^2 \sum_{i=1}^N x_i^2} \prod_{j<k} (x_j - x_k)^2}$$

Eigenvalues of Gaussian random matrix

$J_{ij} \Rightarrow$ complex, hermitian $N \times N$ Gaussian random matrix

$$J = \begin{pmatrix} J_{11} & J_{12} & \dots & J_{1N} \\ J_{12} & J_{22} & \dots & J_{2N} \\ \dots & \dots & \dots & \dots \\ J_{1N} & J_{2N} & \dots & J_{NN} \end{pmatrix}$$

$$\mathrm{Prob.}[J] \propto \exp\left[-\sum_{i,j} |J_{ij}|^2\right]$$

$$= \exp\left[-\mathrm{Tr}\left(J^\dagger J\right)\right]$$

$\rightarrow$ invariant under rotation (GUE)

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N real eigenvalues: $\lambda_1, \lambda_2, \dots, \lambda_N$

Joint distribution of eigenvalues (GUE):

$$P(\lambda_1, \lambda_2, \dots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^2$$

Free fermions at T=0 $\equiv$ GUE eigenvalues

- Fermions: squared many-body wave function at $T = 0$ (quantum probability density)

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$\Rightarrow$ The positions of free fermions in a harmonic trap at $T = 0$ behave statistically as the eigenvalues of a GUE random matrix

$$\boxed{(\alpha\, x_1,\, \alpha\, x_2,\, \ldots, \alpha\, x_N) \equiv (\lambda_1, \lambda_2, \ldots, \lambda_N)}$$

Properties of fermions in a harmonic trap at T=0

Squared many-body wave function at $T = 0$ for fermions

$\Rightarrow$ quantum probability density

$$|\Psi_0(\{x_i\})|^2 = \frac{1}{Z_N} \exp\left[-\sum_{i=1}^{N} \alpha^2 x_i^2\right] \prod_{j<k} (x_j - x_k)^2 \text{ where } \alpha = \sqrt{m\omega/\hbar}$$

$\Rightarrow$ several spatial properties of free fermions in a harmonic trap at $T = 0$ can directly be obtained from the known results in random matrix theory (RMT)

Eisler '13, Marino, S.M., Schehr, Vivo, '14, Calabrese, Le Doussal, S.M., '15, ...

RMT predictions

for

$T = 0$ **properties of free fermions in** 1**-d**

Slater determinant and the Kernel

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$$\boxed{K_N(x, x') = \sum_{k=0}^{N-1} \varphi_k(x)\varphi_k(x')} \to \text{Kernel}$$

m-point correlation function

- m-point correlation function: $1 \le m \le N$

$$R_m(x_1, x_2, \ldots, x_m) = \frac{N!}{(N-m)!} \int dx_{m+1} \ldots dx_N \, |\Psi_0(x_1, \, x_2, \ldots, x_m, \, x_{m+1}, \ldots, x_N)|^2$$

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- $m = N$:

$$R_N(x_1,\, x_2, \ldots, x_N) = N! \; |\Psi_0(x_1,\, x_2, \ldots, x_N)|^2 = \det_{1 \leq i,j \leq N} [K_N(x_i, x_j)]$$

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$$R_m(x_1, x_2, \ldots, x_m) = \frac{N!}{(N-m)!} \int dx_{m+1} \ldots dx_N \, |\Psi_0(x_1,\, x_2, \ldots, x_m,\, x_{m+1}, \ldots, x_N)|^2$$

- $m = N$:

$$R_N(x_1,\, x_2, \ldots, x_N) = N! \; |\Psi_0(x_1,\, x_2, \ldots, x_N)|^2 = \det_{1 \leq i,j \leq N} [K_N(x_i, x_j)]$$

- $m = 1$: one-point function:

$$R_1(x) = N \int dx_2 \ldots dx_N \, |\Psi_0(x, x_2, \ldots, x_N)|^2$$

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$\Rightarrow$ Average density of fermions (normalized to 1):

$$\rho_N(x) = \frac{1}{N} \sum_{i=1}^{N} \langle \delta(x - x_i) \rangle = \frac{1}{N} R_1(x)$$

m- point correlation function: determinantal process

- Beautiful determinantal structure:

$$R_m(x_1, x_2, \ldots, x_m) = \frac{N!}{(N-m)!} \int dx_{m+1} \ldots dx_N \, |\Psi_0(x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_N)|^2$$

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$$\boxed{K_N(x, x') = \langle \Psi_0 | \hat{c}^\dagger(x) \, \hat{c}(x') | \Psi_0 \rangle = \sum_{k=0}^{N-1} \varphi_k(x) \varphi_k(x')}$$ $\Rightarrow$ central object

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- In particular, the average density:

$$\boxed{\rho_N(x) = \frac{1}{N} K_N(x,x) = \frac{1}{N}\sum_{k=0}^{N-1} |\varphi_k(x)|^2}$$

Average density of fermions at T=0

Average density of fermions ($T=0$): Wigner semi-circle law

$$\rho_N(x) = \frac{1}{N}\, \textstyle\sum_{i=1}^{N} \langle \delta(x-x_i)\rangle = \frac{1}{N} \displaystyle\sum_{k=0}^{N-1} |\varphi_k(x)|^2$$

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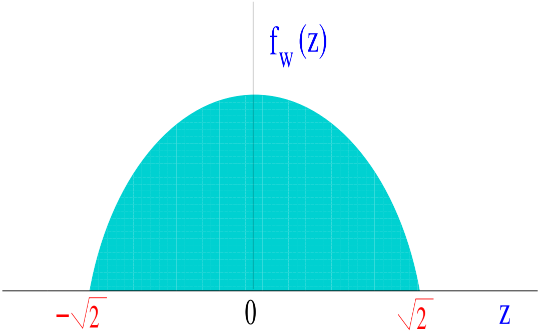
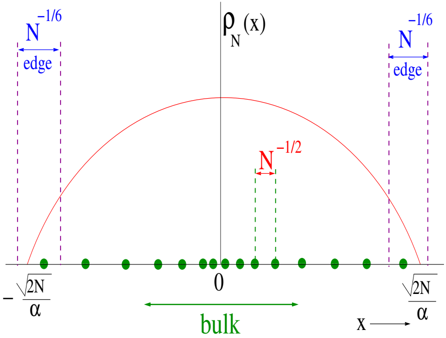
For $N >> 1$, $\rho_N(x) \to \frac{\alpha}{\sqrt{N}} f_W\left(\frac{\alpha\, x}{\sqrt{N}}\right)$, where $\boxed{f_W(z) = \frac{1}{\pi}\sqrt{2 - z^2}}$

(see also Local Density (or Thomas-Fermi) Approx. in the fermion literature)



Average density of fermions at T=0



- Average density: $\rho_N(x) \to \frac{\alpha^2}{\pi N}\sqrt{\frac{2N}{\alpha^2} - x^2}$

  where $\alpha = \sqrt{m\omega/\hbar}$ and the edge location: $r_{\rm edge} = \sqrt{2N}/\alpha$

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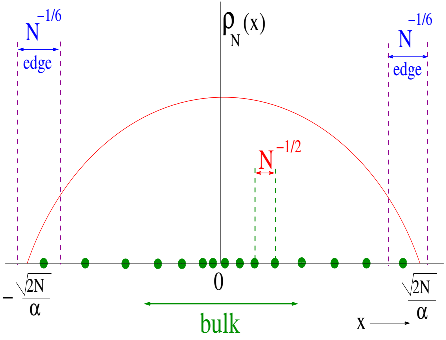

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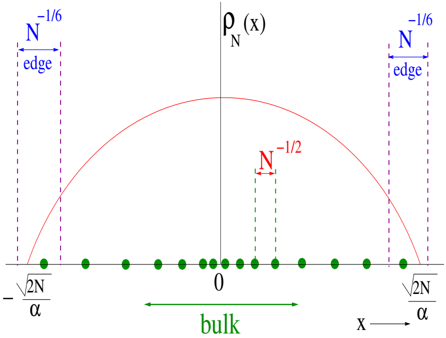

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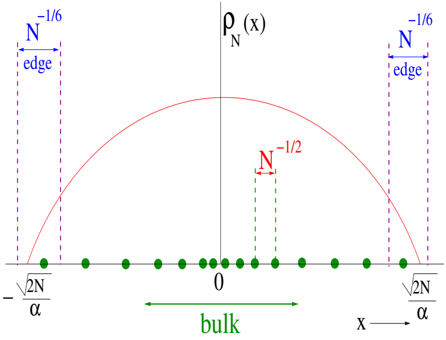

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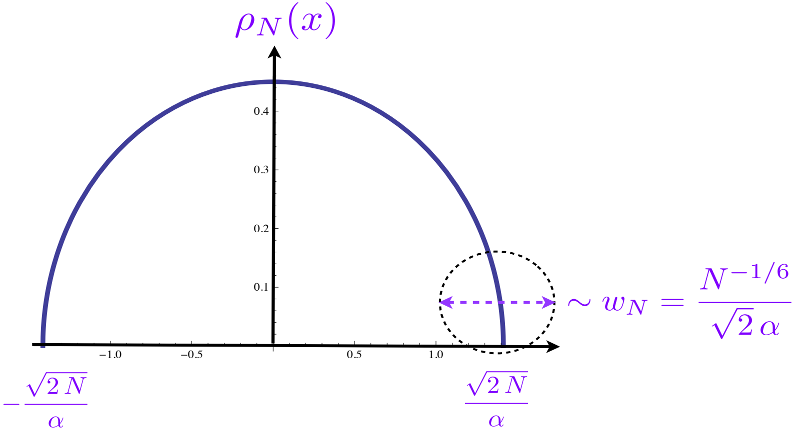
$$l_{\rm edge} >> l_{\rm bulk}$$

Edge density for finite N at T=0

Edge density of free fermions at $T = 0$: finite but large N



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Edge density of free fermions at $T = 0$ Bowick, Brezin '91/Forrester '93

$$\rho_N(x) \approx \frac{1}{N\, w_N}\, F_1\left(\frac{x-\sqrt{2N}/\alpha}{w_N}\right)$$

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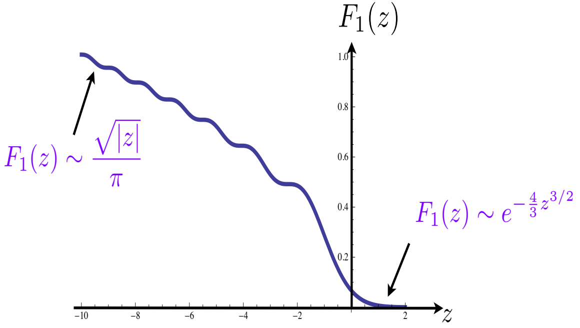
where $w_N = \frac{N^{-1/6}}{\alpha\,\sqrt{2}} \sim l_{\rm edge}$ and $F_1(z) = \left[\mathrm{Ai}'(z)\right]^2 - z\,[\mathrm{Ai}(z)]^2$



Limiting form of the Kernel at $T = 0$

- Bulk limit: when x and x' are far from the edge and

$$|x - x'| \sim \frac{1}{N\rho_N(x)} \equiv \text{interparticle distance}$$

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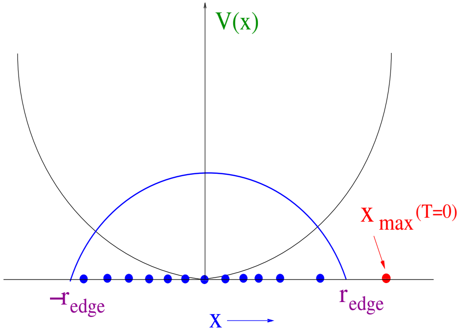
Position of the rightmost fermion at T=0



Connection to RMT $\Rightarrow$ $\boxed{x_{\max}(T=0) \equiv \frac{\lambda_{\max}}{\alpha}}$ where $\alpha = \sqrt{m\omega/\hbar}$

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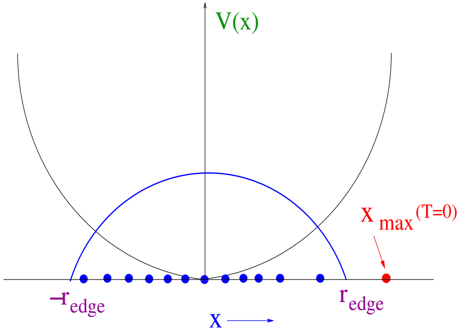

Connection to RMT $\Rightarrow$ $\boxed{x_{\rm max}(T=0) \equiv \frac{\lambda_{\rm max}}{\alpha}}$ where $\alpha = \sqrt{m\omega/\hbar}$

$\Rightarrow$ fluctuations of $x_{\rm max}(T=0)$ are governed by the Tracy-Widom distribution for GUE

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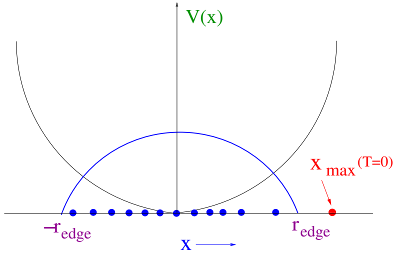

$$r_{\mathrm{edge}} = \frac{\sqrt{2N}}{\alpha}$$

$$w_N = \frac{N^{-1/6}}{\alpha\,\sqrt{2}}$$

$$\mathrm{Prob.}[x_{\mathrm{max}}(T=0) \le M] \approx \mathcal{F}_2\left(\frac{M - r_{\mathrm{edge}}}{w_N}\right)$$

$\mathcal{F}_2(s) \rightarrow$ GUE Tracy-Widom scaling function

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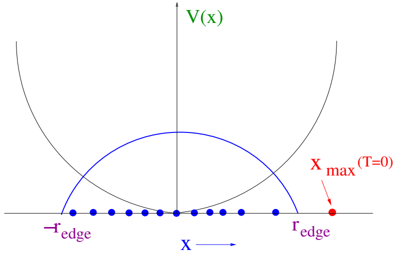

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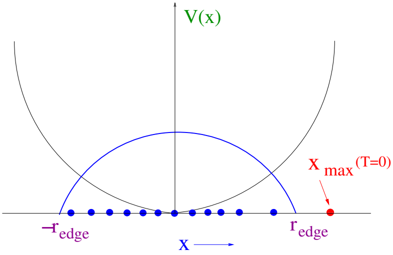

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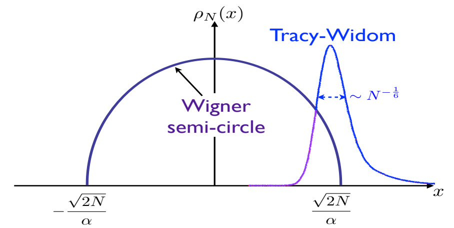
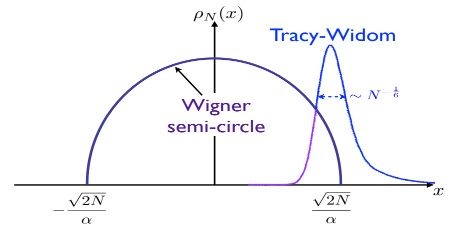
$P_s \rightarrow$ projector over the interval $[s, \infty]$

$\mathcal{K}_{\rm edge}(\mathbf{z}, \mathbf{z}') = \int_0^\infty du\, \mathrm{Ai}(\mathbf{z}+u)\, \mathrm{Ai}(\mathbf{z}'+u) \Rightarrow$ Airy-kernel

Tracy-Widom distribution



Tracy-Widom distribution



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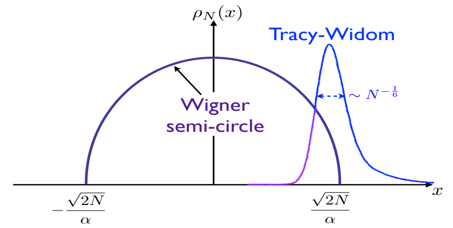
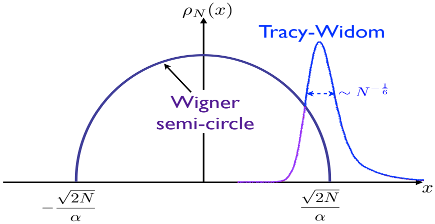


Asymptotics: $\mathcal{F}_2'(s) = f_2(s) \sim \exp\left[-\frac{1}{12}|s|^3\right] \quad \text{as} \quad s \to -\infty$

$$\sim \exp\left[-\frac{4}{3}\, s^{3/2}\right] \quad \text{as} \quad s \to \infty$$

Tracy-Widom distribution



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Tracy-Widom distribution $\rightarrow$ ubiquitous

directed polymer, random permutation, growth models–KPZ equation, sequence alignment, large N gauge theory, liquid crystals, spin glasses,...

Tracy-Widom distribution: Experiments

PRL **104**, 230601 (2010) PHYSICAL REVIEW LETTERS week ending 11 JUNE 2010

Universal Fluctuations of Growing Interfaces: Evidence in Turbulent Liquid Crystals

Kazumasa A. Takeuchi* and Masaki Sano
Department of Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

RAPID COMMUNICATIONS

PHYSICAL REVIEW E **85**, 020101(R) (2012)

Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers

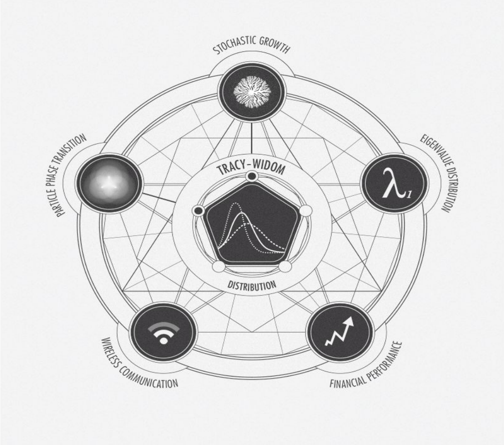
Moti Fridman, Rami Pugatch, Micha Nixon, Asher A. Friesem, and Nir Davidson*
Weizmann Institute of Science, Department of Physics of Complex Systems, Rehovot 76100, Israel

PHYSICAL REVIEW B **87**, 184509 (2013)

Universal scaling of the order-parameter distribution in strongly disordered superconductors

G. Lemarié,[1,2] A. Kamlapure,[3] D. Bucheli,[2] L. Benfatto,[2] J. Lorenzana,[2] G. Seibold,[4] S. C. Ganguli,[3] P. Raychaudhuri,[3] and C. Castellani[2]
[1]*Laboratoire de Physique Théorique UMR-5152, CNRS and Université de Toulouse, F-31062 France*
[2]*ISC-CNR and Department of Physics, Sapienza University of Rome, P.le A. Moro 2, 00185 Rome, Italy*
[3]*Tata Institute of Fundamental Research, Homi Bhabha Rd., Colaba, Mumbai 400005, India*
[4]*Institut Für Physik, BTU Cottbus, P.O. Box 101344, 03013 Cottbus, Germany*

Ubiquity of Tracy-Widom distribution



Olena Shmahalo/Quanta Magazine

"Equivalence Principle", M. Buchanan, Nature Phys. 10, 543 (2014)

"At the far ends of a new universal law", N. Wolchover, Quanta Magazine (October, 2014)

Free fermions at $T = 0 \rightarrow$ Tracy-Widom

One of the main conclusions:

free fermions in a harmonic trap at $T = 0$

$\Rightarrow$ **a physical realization of Tracy-Widom distribution**

Summary of $T = 0$ results

- free fermions in a harmonic trap at $T = 0$

  $\Rightarrow$ provides a physical realization of **GUE**

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- free fermions in a harmonic trap at $T = 0$

  $\Rightarrow$ provides a physical realization of **GUE**

- positions of fermions $\Rightarrow$ determinantal process with kernel $K_N(x, x')$

  - Bulk: Sine-kernel

$$\mathcal{K}_{\rm bulk}(z) = \frac{\sin(2z)}{\pi\, z}$$

  - Edge: Airy-kernel

$$\mathcal{K}_{\rm edge}(\mathbf{z}, \mathbf{z}') = \int_0^\infty du\, \mathrm{Ai}(\mathbf{z} + u)\, \mathrm{Ai}(\mathbf{z}' + u)$$

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- free fermions in a harmonic trap at $T = 0$

  $\Rightarrow$ provides a physical realization of **GUE**

- positions of fermions $\Rightarrow$ determinantal process with kernel $K_N(x, x')$

  - Bulk: Sine-kernel

$$\boxed{\mathcal{K}_{\rm bulk}(z) = \frac{\sin(2z)}{\pi\, z}}$$

  - Edge: Airy-kernel

$$\boxed{\mathcal{K}_{\rm edge}(\mathbf{z}, \mathbf{z}') = \int_0^\infty du\, \mathrm{Ai}(\mathbf{z} + u)\, \mathrm{Ai}(\mathbf{z}' + u)}$$

- Scaled kernels are universal, i.e., independent of the trapping potential

  $V(x) \sim |x|^p$ with a single minimum (and $p > 1$)

What happens at finite $T > 0$?

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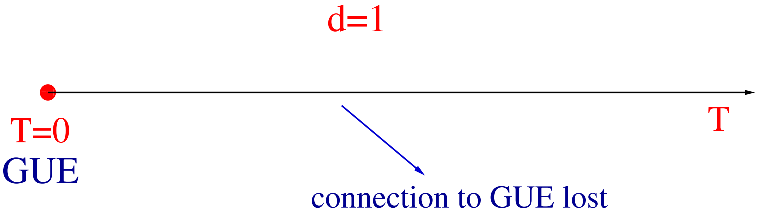
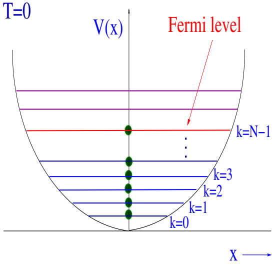
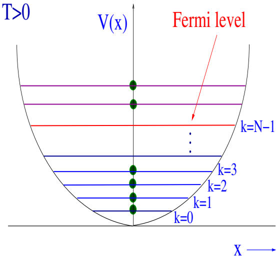


N free fermions in a harmonic trap at $T > 0$





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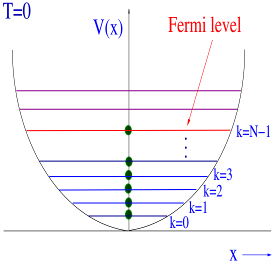

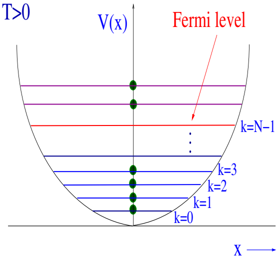

At finite $T > 0$: setting $\beta = 1/k_B T$ and $\epsilon_k = (k + 1/2)\,\hbar\omega$

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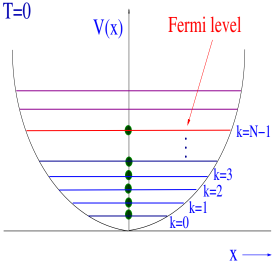

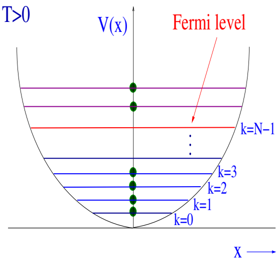

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Joint distribution of the positions of fermions

$$P\left(x_1,\, x_2, \ldots, x_N\right) = \frac{1}{N! Z_N(\beta)} \sum_{k_1 < k_2 < \cdots < k_N} \left[\det(\varphi_k(x_j))\right]^2 \, e^{-\beta\left(\epsilon_{k_1} + \epsilon_{k_2} + \ldots + \epsilon_{k_N}\right)}$$

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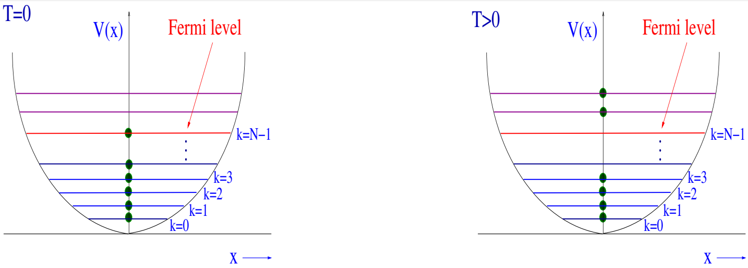

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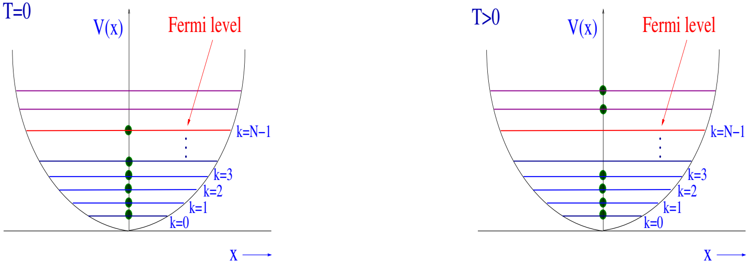

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Correlation function for $T > 0$

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Dean, Le Doussal, S.M., Schehr, '15

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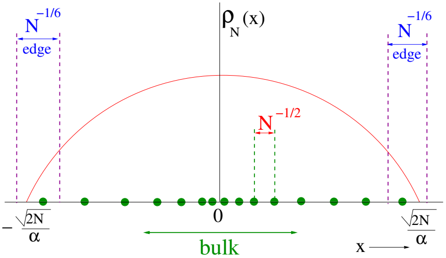
- same kernel also appears in a class of matrix models

  Moshe, Neuberger, Shapiro '94/Johansson '07, Johansson & Lambert, '15

Relevant scales at finite T



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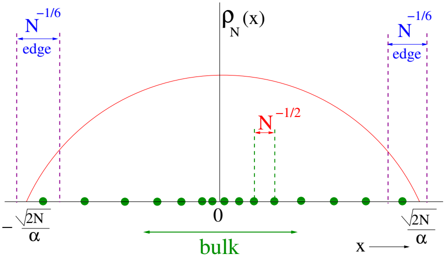

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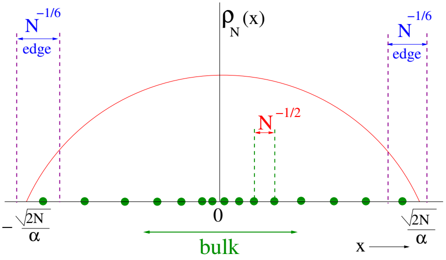

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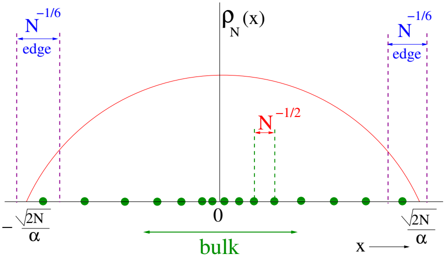

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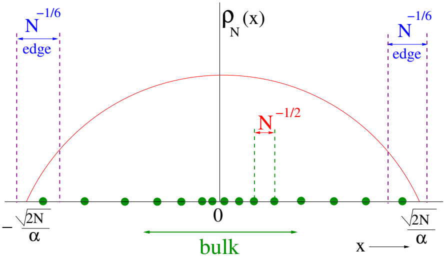

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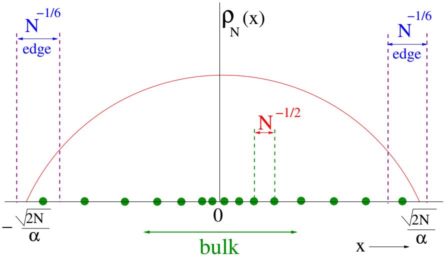

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Average bulk density at finite T

$$N\,\rho_N(x,T) = K_\mu(x,x) = \sum_{k=0}^{\infty} \frac{|\varphi_k(x)|^2}{1+e^{\beta\,(\epsilon_k-\mu)}}$$

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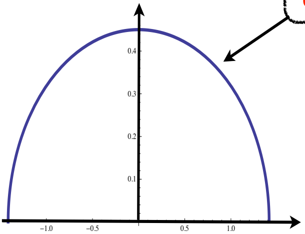
Two well understood limits:

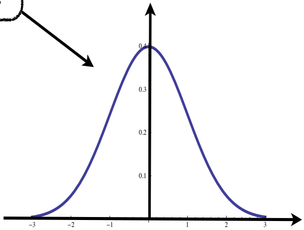
Interpolation at finite T ?

$T=0$



Wigner semi-circle

$T\to\infty$



Gibbs-Bltzmann

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Local Density (or Thomas-Fermi) Approx. in the literature on fermions

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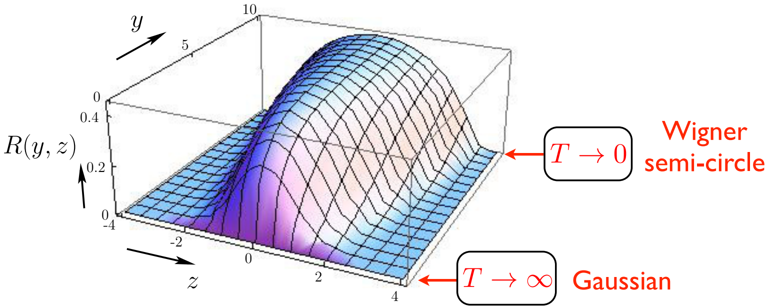

Edge density at finite T

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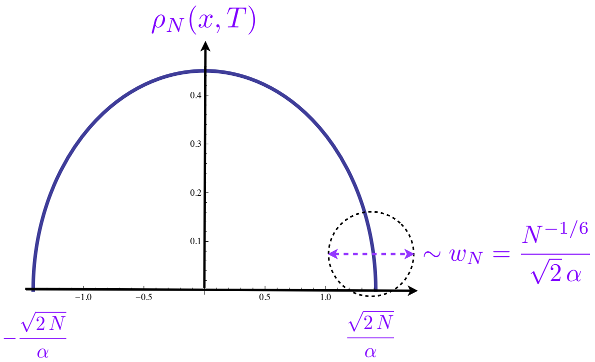

$\rho_N(x,T) \sim \frac{1}{N\,w_N}\,F_1\left(\frac{x - \sqrt{2N}/\alpha}{w_N}\right)$ where $$F_1(z) = \int_{-\infty}^{\infty} \frac{[\mathrm{Ai}(z+u)]^2}{1+e^{-\mathbf{b}\,u}}\,du$$

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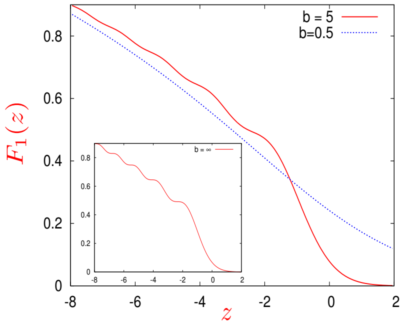

Asymptotic behaviors

$$F_1(z) \sim \begin{cases} \frac{\sqrt{|z|}}{\pi}, & z \to -\infty \\ \exp(-\mathbf{b}\,z), & z \to \infty \end{cases}$$

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Bulk kernel for N free fermions at $T > 0$

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Bulk kernel for N free fermions at $T > 0$

$$K_\mu(x,x') = \sum_{k=0}^{\infty} \frac{\varphi_k(x)\,\varphi_k(x')}{1+e^{\beta\,(\epsilon_k-\mu)}}, \quad \text{and} \quad N = \sum_{k=0}^{\infty} \frac{1}{1+e^{\beta\,(\epsilon_k-\mu)}}$$

- Bulk scaling limit: $N \to \infty, \quad T \sim N, \quad x \;\&\; x' \sim \sqrt{T}$

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$\Rightarrow$ generalisation of the Sine-kernel

In the context of matrix models, see Garcia-Garcia, Verbaarshot '03, Johansson '07

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  $V(x) \sim |x|^p$ Dean, Le Doussal, S.M., Schehr, '15

Edge kernel for N free fermions at $T > 0$

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  x and x' both close to the edge $r_{\rm edge} = \sqrt{2N}/\alpha$

  $K_\mu(x,x') \approx \frac{1}{w_N}\,\mathcal{K}_{\rm edge}\left(\frac{x-r_{\rm edge}}{w_N}, \frac{x'-r_{\rm edge}}{w_N}\right)$ with $\quad w_N = \frac{N^{-1/6}}{\alpha\,\sqrt{2}}$

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Dean, Le Doussal, S.M., Schehr, '15, see also Johansson '07

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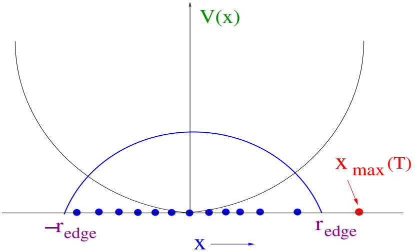
Position of the rightmost fermion at $T > 0$



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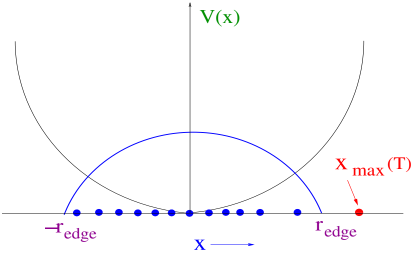

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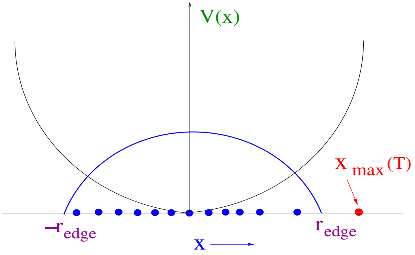

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$\Rightarrow$ finite T generalisation of the Tracy-Widom distribution

Dean, Le Doussal, S.M., Schehr, '15

Connection to the KPZ equation

Kardar-Parisi-Zhang (KPZ) equation in 1-d

- KPZ equation in $(1+1)$-dimensions in a curved geometry

(in dimensionless parameters) $\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \sqrt{2}\,\eta(x,t)$

$\eta(x,t) \to$ white noise $\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\,\delta(t-t')$

Kardar, Parisi, & Zhang '86

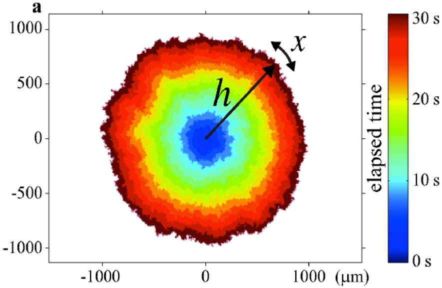

image from the liquid crystal experiment

Takeuchi et. al., Sci. Rep. '11

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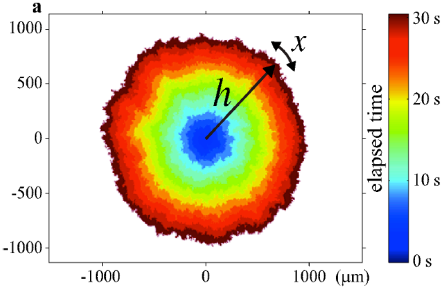

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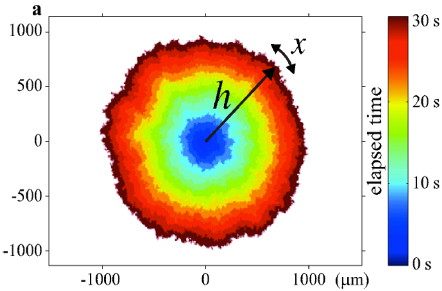

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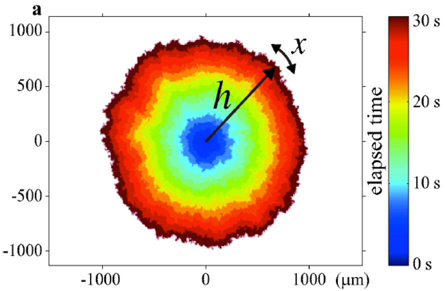

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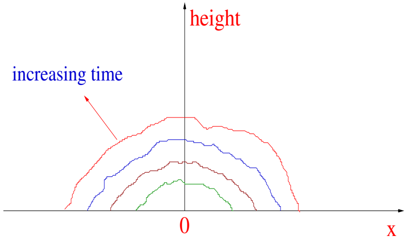
Sasamoto & Spohn '10/Calabrese, Le Doussal & Rosso '10/Dotsenko '10/ Amir, Corwin & Quastel '10

KPZ equation in curved geometry at finite time t

- KPZ growth in a curved geometry



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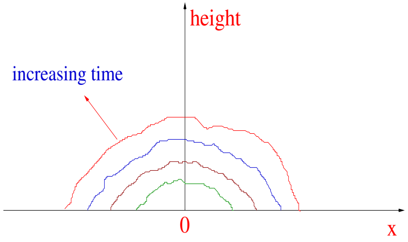

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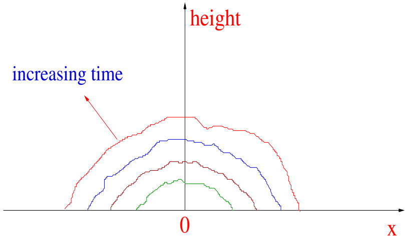

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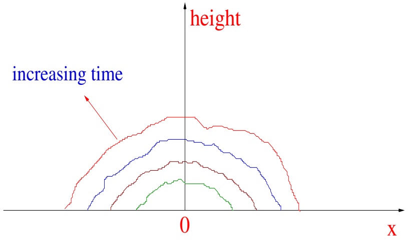

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Free fermions at finite T and KPZ at finite t

- Free fermions at finite T: fluctuations of $x_{\max}(T)$; $\mathbf{b} = \frac{\hbar\,\omega\,\mathbf{N}^{1/3}}{\mathbf{T}}$

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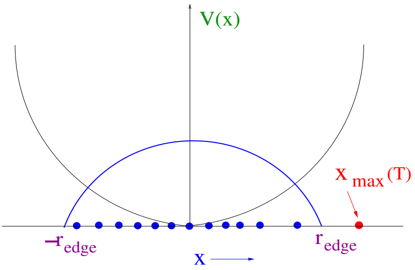
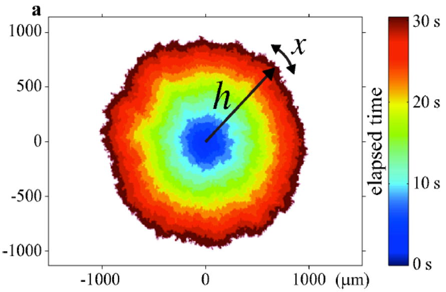
- two problems share the same kernel with the identification

$$\boxed{\frac{\hbar\,\omega\,\mathbf{N}^{1/3}}{\mathbf{T}} \Longleftrightarrow \mathbf{t}^{1/3}}$$

[Dean, Le Doussal, S.M., Schehr, '15]

Free fermions at finite T and KPZ at finite t





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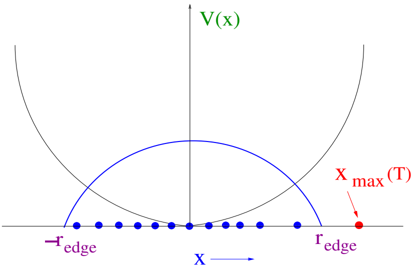

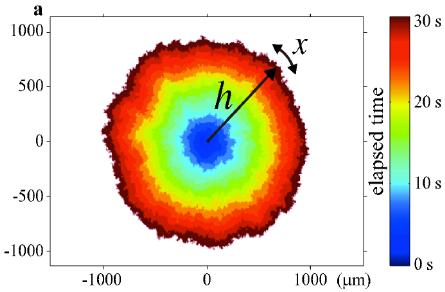

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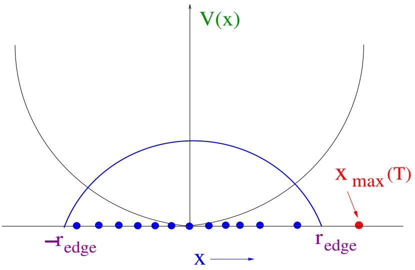
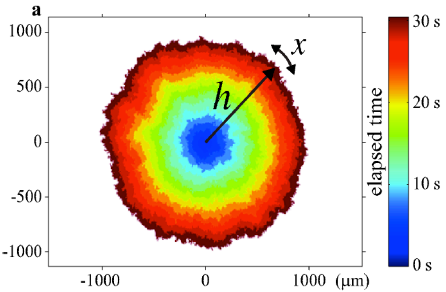
$$\lim_{N\to\infty}\left[\frac{x_{\max}(T) - r_{\rm edge}}{w_N}\right] \overset{\text{in law}}{=} \frac{h(0,t) + t/12 + G}{t^{1/3}}$$

Free fermions at finite T and KPZ at finite t





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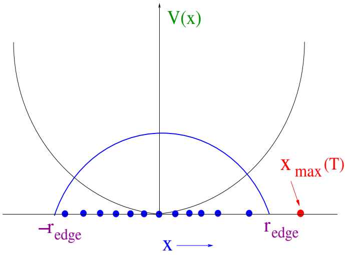
G $\Rightarrow$ Gumbel variable independent of $h(0,t)$: $\quad \mathrm{Prob.}[G \le z] = \exp\left[-e^{-z}\right]$

[Dean, Le Doussal, S.M., Schehr, PRL, 114, 110402 (2015); PRA, 94, 063622 (2016)]

Rightmost fermion in a harmonic trap at finite T



$$\boxed{\mathbf{t^{1/3}} \Longleftrightarrow \frac{\hbar\omega\, \mathbf{N^{1/3}}}{\mathbf{T}}}$$ implies

early time $\Longleftrightarrow$ finite temperature

(KPZ) (Fermions)

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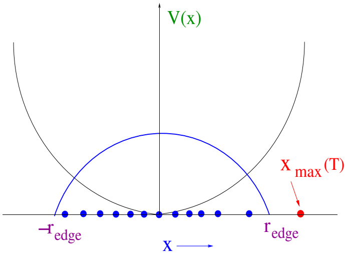

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setting $\alpha = \sqrt{m\omega/\hbar} \equiv 1$

Position distribution of the rightmost fermion at high T

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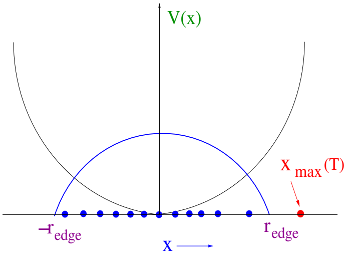

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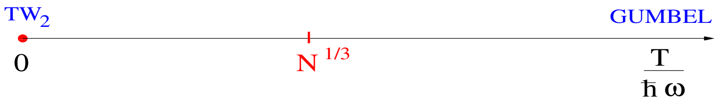
Position distribution of the rightmost fermion at high T

$$\boxed{\text{Prob.}\left(\frac{x_{\max}(T) - r_{\text{edge}}}{T/\sqrt{2N}} \le s\right) \sim \exp\left[\sqrt{\frac{T^3}{4\pi N}}\,\text{Li}_{5/2}(-e^{-s})\right]}$$

Le Doussal, S.M., Rosso & Schehr, PRL, 117, 070403 (2016)

Distribution of $x_{\max}(T)$ as T increases



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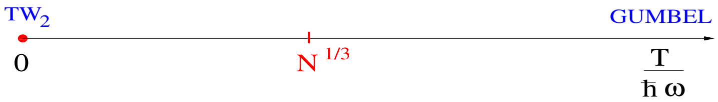

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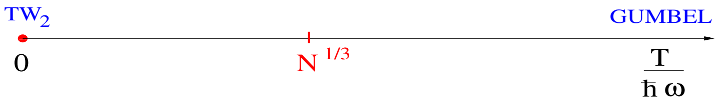

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- if $T/\hbar\omega >> N^{1/3}$: distribution of $x_{\max}(T) \longrightarrow$ Gumbel
- If $T/\hbar\omega = \left(\frac{N}{\mathbf{t}}\right)^{1/3}$: distribution of $x_{\max}(T) \longrightarrow$ parametrized by $\mathbf{t}$

  $\longrightarrow$ related to the height distribution of KPZ at finite 'time' $\mathbf{t}$

$\Longrightarrow$ finite T generalisation of Tracy-Widom

$\Longrightarrow$ the limit $T \to 0$ (or equivalently $\mathbf{t} \to \infty$ in KPZ) $\to$ TW GUE

[Dean, Le Doussal, S.M., Rosso, Schehr, '15-'17]

Generalisations to higher dimensions

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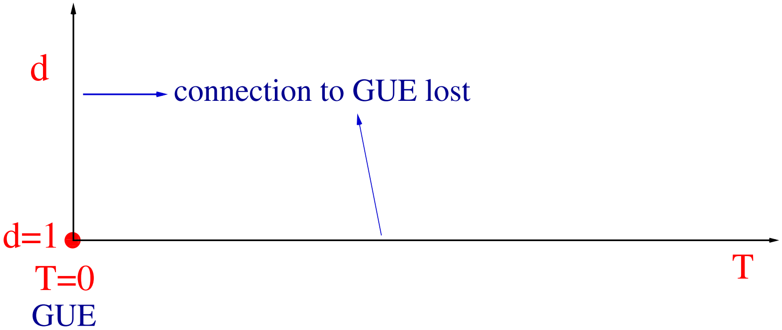

Free fermions in a d-dim. harmonic trap at $T = 0$

- Single particle Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2\,m}\left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2}\right) + \frac{1}{2}\,m\,\omega^2\,\left(x_1^2 + \ldots + x_d^2\right)$$

$$x_1^2 + \ldots + x_d^2 = r^2$$

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- Average global density ($T = 0$) for large N:

$$\rho_N(\vec{x}) \approx \frac{1}{N\,\Gamma(d/2+1)} \left(\frac{m}{2\pi\hbar^2}\right)^{d/2} \left[\mu - \frac{1}{2}\, m\,\omega^2\, r^2\right]^{d/2}$$

where $\mu \approx \hbar\omega\,[\Gamma(d+1)\,N]^{1/d}$ [Dean, Le Doussal, S.M., Schehr, '15]

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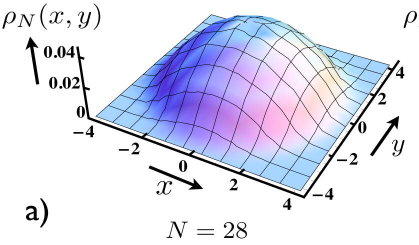

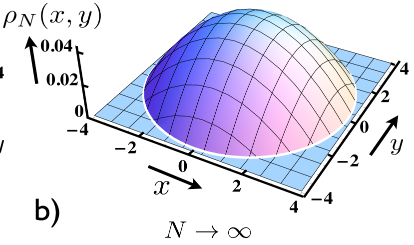

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$\rho_{\rm edge}(\vec{x}) \approx \frac{1}{N}\, \frac{1}{w_N^d}\, F_d\left(\frac{r - r_{\rm edge}}{w_N}\right)$ with $w_N = b_d\, N^{-1/6d}$

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$$F_d(z) = \frac{1}{\Gamma(d/2+1)\, 2^{4d/3}\, \pi^{d/2}} \int_0^\infty du\, u^{d/2}\, \mathrm{Ai}(u + 2^{2/3}\, z)$$

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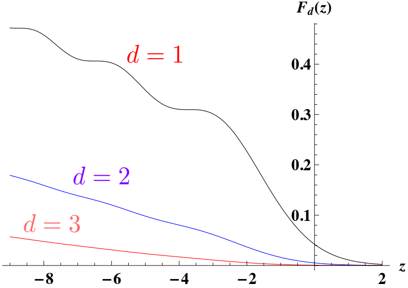

Recall that

$$F_1(z) = \left[\mathrm{Ai}'(z)\right]^2 - z\,\left[\mathrm{Ai}(z)\right]^2$$

$$F_d(z) \sim \begin{cases} \frac{(4\pi)^{-d/2}}{\Gamma(d/2+1)}\,|z|^{d/2}, & z \to -\infty \\ (8\pi)^{-\frac{d+1}{2}}\, z^{-\frac{d+3}{4}}\, e^{-\frac{4}{3}\, z^{3/2}}, & z \to \infty \end{cases}$$

Correlation kernel in d dimensions at $T = 0$

$$K_N(\vec{x}, \vec{x}') = \sum_{\vec{k}} \theta\left(E_F - \epsilon_{\vec{k}}\right)\, \varphi_{\vec{k}}(\vec{x})\, \varphi_{\vec{k}}(\vec{x}')$$

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$z_n = \frac{\vec{z}.\vec{r}_{\rm edge}}{r_{\rm edge}}$, $z_n' = \frac{\vec{z}'.\vec{r}_{\rm edge}}{r_{\rm edge}}$, and ${\rm Ai}_1(z) = \int_z^\infty {\rm Ai}(u)\, du$

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- Can one observe these new universal edge behavior in cold atom experiments?

Quantum gas microscope   M. Greiner et. al. PRL (2015)