Ljubljana 13/01/19
Bruno Bertini Uni-Lj

In these lectures we will be concerned with the study of

TRANSPORT IN QUANTUM MANY-BODY SYSTEMS OUT OF EQUILIBRIUM

- There is a non-trivial "flow" of conserved quantities in the system
- ⇒ Our goal is to <u>exactly determine</u> such flow

- We study the problem far away from equilibrium
- ⇒
  - no linear response
  - no time-dependent perturbation theory

Since we look for exact descriptions we consider an idealised situation

1. Study the problem in the thermodynamic limit (Volume $L \to +\infty$ with fixed densities)

   ⇒ forget all finite-size effects

2. Study stationary flows (We prepare the system in some initial configuration at time $t=0$ and consider the limit $t \to +\infty$)

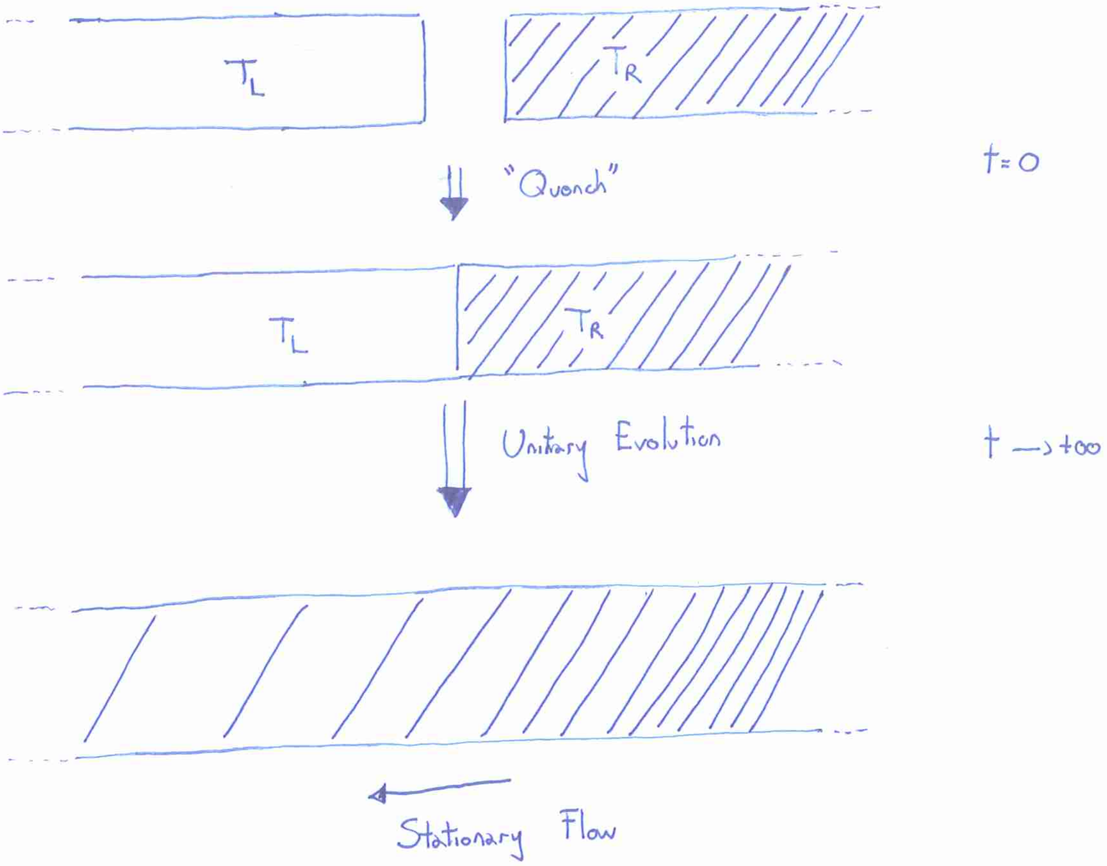
   ⇒ forget all the details of finite-time dynamics

3. Consider closed systems

   ⇒
   - a. No ad-hoc system-bath interaction, system and bath are of the same nature
   - b. Unitary dynamics

Note that this is not really a restriction: the open-system description can be recovered by "tracing out" a part of the system

In this setting we generate non-trivial transport by means of a "Bipartite Quench" protocol [see the reviews Doyon and Bernard '16 Vasseur and Moore '16]



Problem:

"Generic systems" do not support stationary flows!

Argument:

In generic systems one normally assumes

$$J_q \sim -\partial_x q$$ "Fourier Law"

↑ Current of the conserved charge $Q=\int dx\, q$ (Typically you can think of Q as the energy)

Combining this with the continuity equation

$$\partial_t q + \partial_x J_q = 0$$

we have

$$\partial_t q - \partial_x^2 q = 0 \qquad q(x,0) = q_L \overset{\text{Step function}}{\theta(-x)} + q_R\, \theta(x)$$

Everything is invariant under the scale transformation $q(x,t) \to q(x/\sqrt{a}, t/a)$

Assume the scaling form $q(x,t) = f(x/\sqrt{t})$

$$\Longrightarrow \quad \xi f'(\xi) + f''(\xi) = 0 \quad + \quad \begin{cases} f(-\infty) = q_L \\ f(+\infty) = q_R \end{cases} \quad \Longrightarrow \quad f(\xi) = \left(\frac{q_R - q_L}{\sqrt{\pi}} \right) \int_{-\infty}^{\xi} dy \, e^{-y^2} + q_L$$

but then

$$J_q \sim -\partial_x q = \frac{1}{\sqrt{t}} f'(x/\sqrt{t}) \xrightarrow[t \to +\infty]{} 0$$

(Same in higher dimensions)

In parity-invariant systems this is consistent with a relaxation to a local thermal state

$$\mathrm{Tr}[\hat{O}(x)\hat{\rho}(t)] \sim \mathrm{Tr}[\hat{O}(x)\hat{\rho}^{th}_{x,t}] \qquad \hat{\rho}^{th}_{x,t} = \frac{1}{Z} \exp[-\beta_{x,t} H]$$

Since currents of even charges are odd

$$\mathrm{Tr}[\hat{J}_e \hat{\rho}^{th}_{x,t}] = 0$$

This is not true for $\boxed{\text{integrable systems}}$

A special kind of systems (non-trivial in one dimension) with a macroscopic ($\propto L$) number of conservation laws with localised density

↳ "like" the Hamiltonian

In these systems

- Fourier's law does not hold
- Exotic non-equilibrium steady states (NESS) emerge

↳ $\neq \hat{\rho}^{th}$

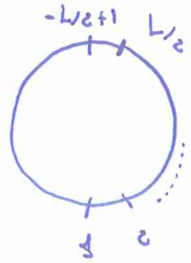
These are the systems we consider here

⟹ $\boxed{\text{TRANSPORT IN CLOSED INTEGRABLE QUANTUM MANY-BODY SYSTEMS OUT OF EQUILIBRIUM}}$

A bit more formally, we consider the following setting:

Consider a 1d discrete system of length L (even) with periodic boundary conditions

$$H = \sum_{j=-L/2+1}^{L/2} h_{j,j+1} \qquad\qquad h_{L/2,\,L/2+1} = h_{L/2,\,-L/2+1}$$



Hamiltonian density: acts non-trivially on the sites j and j+1 (local interactions)

H is integrable $\Rightarrow$ $[H, Q^{(n)}] = 0$ $\quad n = 1, \dots, \alpha L \qquad Q^{(2)} = H$

$$Q^{(n)} = \sum_{j=-L/2+1}^{L/2} q^{(n)}_{j,j+r}$$

"local" charge

Charge density: acts non-trivially from site j to site j+r

$r+1$ = "range" of the charge

Obs.

Typically (for interacting integrable models) one also needs to consider charges with density $\sim \sum_r q_{j,j+r}\, e^{-r}$. These are called quasi-local charges (see Ilievski, Medenjak, Prosen, and Zadnik '16)

The relation $[H, Q^n] = 0$ implies that there exist a local operator $J_j^{(n)}$ such that

$$\underset{-i\partial_t q_j^{(n)}}{\underset{\shortparallel}{[H, q_j^{(n)}]}} = -i\left(J_{j-1}^{(n)}(t) - J_j^{(n)}(t)\right) \qquad \text{(OCE)}$$

here for convenience we left implicit the range

This relation is called "Operatorial Continuity Equation" and the operator $J_j^{(n)}(t)$ is called "Current of the charge $Q^{(n)}$".

Bipartite Quench

A. Initialise the system in the state

We can add these if $H_{L/R}$ conserve the number of particles

$$\hat{\rho}_0 = \frac{1}{Z} \exp\left[-\beta_L (H_L - \mu_L N_L) - \beta_R (H_R - \mu_R N_R) \right]$$

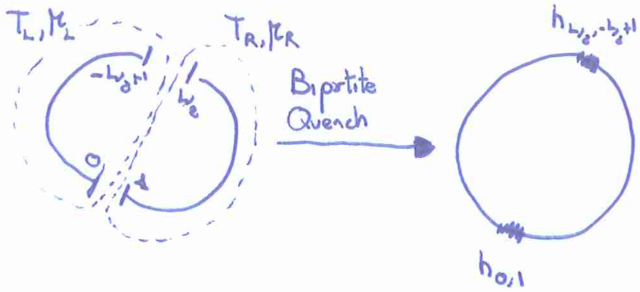
$$H_L = \sum_{j=-L/2+1}^{-1} h_{j,j+1} \qquad N_L = \sum_{j=-L/2+1}^{0} n_j$$

\# of particles at site j

$$H_R = \sum_{j=1}^{L/2-1} h_{j,j+1} \qquad N_R = \sum_{j=1}^{L/2} n_j$$

$$Z = \mathrm{tr}\left[\exp\left[-\beta_L (H_L - \mu_L N_L) - \beta_R (H_R - \mu_R N_R) \right] \right]$$

Pictorial representation



B. Evolve with $H = H_L + H_R + h_{0,1} + h_{L/2; -L/2+1}$

$$\Rightarrow \quad \hat{\rho}(t) = e^{-iHt} \hat{\rho}_0 e^{iHt}$$

C. Study

local observable: acts non-trivially in some finite neighbourhood of j. It can be
- local density of a conserved charge
- current
- more general stuff!

$$\langle O \rangle_\xi \equiv \lim_{\substack{t,j \to +\infty \\ j/t = \text{fixed} = \xi}} \lim_{\substack{L \to +\infty \\ \text{fixed densities}}} \langle O_j \rangle_{\hat{\rho}(t)}$$

Notation $\langle O \rangle_\rho \equiv \mathrm{Tr}[\rho O]$

- The order of limits matters!

First $\lim_{\substack{L \to +\infty \\ \text{fixed densities}}} \equiv \lim_{th}$ "Thermodynamic limit"

Then $\lim_{\substack{t,j \to +\infty \\ j/t = \text{fixed} = \xi \\ (\text{range of the operator = fixed})}} \equiv \lim_{sc,\xi}$ "Scaling limit"

"ray": for $\xi = 0$ we describe observables in the non-equilibrium steady state developing at the junction

To build up our physical intuition let us start by considering the simplest possible example of integrable model and compute $\langle O \rangle_{\xi}$ "blindly" for all local observables $\{O_j\}$.

We consider free fermions on a periodic lattice

$$H = -\frac{J}{2} \sum_{j=-L/2+1}^{L/2} c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j - h \sum_{j=-L/2+1}^{L/2} c_j^\dagger c_j \qquad c_{-L/2+k} = c_{L/2+k} \quad k=0,1 \quad \text{periodic b.c.}$$

$c_j^\dagger$ and c_j are canonical spinless fermionic operators

$$\{c_i^\dagger, c_j^\dagger\} = 0 = \{c_i, c_j\} \qquad (\text{anticommutator: } \{A,B\} = AB + BA)$$

$$\{c_i^\dagger, c_j\} = \delta_{ij}$$

Obs.

The Hamiltonian H can be expressed as that of a spin-$\frac{1}{2}$ chain by means of the following mapping

$$S_i^x = \prod_{j=1}^{i-1} (2c_j^\dagger c_j - 1) \cdot \left(\frac{c_i + c_i^\dagger}{2} \right)$$

$$S_i^y = \prod_{j=1}^{i-1} (2c_j^\dagger c_j - 1) \left(\frac{c_i - c_i^\dagger}{2i} \right) \qquad \text{"Jordan-Wigner" Transformation} \qquad (JW)$$

$$S_i^z = c_i^\dagger c_i - \frac{1}{2}$$

Exercise 1. Verify that

$$[S_i^\alpha, S_j^\beta] = i \delta_{ij} \varepsilon^{\alpha\beta\gamma} S_j^\gamma$$

$$(S_j^\alpha)^2 = \frac{1}{4}$$

i.e. they define a spin-$\frac{1}{2}$ representation of SU(2)

Inverting (JW) and plugging into H we have

$$H = J \sum_{j=-L/2+1}^{L/2-1} S_j^x S_{j+1}^x + S_j^y S_{j+1}^y - h \sum_{j=-L/2+1}^{L/2} \left(S_j^z + \frac{1}{2} \right) \qquad \text{XX model}$$

The mapping (JW), however, "spoils" the boundary conditions. Periodic b.c. for the fermions imply that the b.c. on the spins depend on the total spin along z (more precisely on the parity of
$\Rightarrow$ to avoid complications we work with the fermionic Hamiltonian $S^z_{tot} + L/2$)

A. "Diagonal Form" of H

H can be written as the sum of independent modes by means of a discrete Fourier transformation

$$C(k) \equiv \frac{1}{\sqrt{L}} \sum_{j=-L/2+1}^{L/2} e^{ikj} c_j \quad \underset{\text{verify it!}}{\Rightarrow} \quad c_j = \frac{1}{\sqrt{L}} \sum_{\{k_n\}} e^{-ik_n j} C(k_n) \qquad (*)$$

where we introduced $k_n \equiv \frac{2\pi}{L} \cdot n \quad n \in [-L/2+1, L/2] \cup \mathbb{Z}$

and $\sum_{\{k_n\}} f(k_n) \equiv \sum_{n=-L/2+1}^{L/2} f(\frac{2\pi}{L} \cdot n)$

Note that $C_{L/2+k} = e^{ik_n L} C_{-L/2+k} \underset{e^{ik_n L}=1}{=} C_{-L/2+k}$

Exercise 2:

Verify that

$\{C^+(k_n), C^+(k_m)\} = 0 = \{C(k_n), C(k_m)\}$

$\{C^+(k_n), C(k_m)\} = \delta_{n,m}$

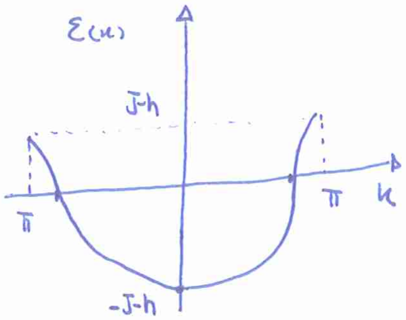
Plugging (*) into the definition of H we have

$$H = \sum_{\{k_n\}} \varepsilon(k_n) \underset{\substack{||| \\ \hat{n}_{k_n}}}{C^+_{k_n} C_{k_n}} \qquad \text{where} \quad \varepsilon(k) = -J\cos k - h$$

$\hat{n}_{k_n}$ "mode occupation" operator

$[\hat{n}_{k_n}, \hat{n}_{k_m}] = 0$

$(\hat{n}_{k_n})^2 = \hat{n}_{k_n} \Rightarrow$ Fermionic



B. Conserved Charges and Currents

For h ≠ 0 all the conserved charges with local densities can be expressed as linear combinations of mode occupation operators

For h=0 there are additional charges ⇒ we do not consider this case here

see. Fagotti '14

A complete set of local charges is given by $\{Q_m\}_{m=0,\dots,L}$ ⑧

Particle number

$$Q_0 : \sum_{\{k_r\}} \underset{\overset{\equiv}{q_0(k_r)}}{1}(\hat{n}_{k_r}-\tfrac{1}{2}) = \sum_{j=-L/2+1}^{L/2} (c_j^\dagger c_j - \tfrac{1}{2}) \equiv N - L/2$$

$$Q_{2n+1} = \sum_{\{k_r\}} \underset{\overset{\equiv}{q_{2n+1}(k_r)}}{\sqrt{2}\sin((n+1)k_r)}(\hat{n}_k-\tfrac{1}{2}) = \tfrac{i}{\sqrt{2}} \sum_{j=-L/2+1}^{L/2} \underbrace{c_j^\dagger c_{j+n} - c_{j+n}^\dagger c_j} \qquad n = 0, \dots, L/2-1$$

$$Q_{2n+2} = \sum_{\{k_r\}} \underset{\overset{\equiv}{q_{2n+2}(k_r)}}{\sqrt{2}\cos((n+1)k_r)}(\hat{n}_{k_n}-\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \sum_{j=-L/2+1}^{L/2} \underbrace{c_j^\dagger c_{j+n} + c_{j+n}^\dagger c_j} \qquad n = 0, \dots, L/2-1$$

localised densities $\equiv q_j^{(n)}$

In short

$$Q_n = \sum_{\{k_r\}} q_n(k_r)(\hat{n}_k - \tfrac{1}{2})$$

Obs.

Note that $\{\vec{q}_n\}_{n=0,\dots,L}$, where $\vec{q}_n = \begin{pmatrix} q_n(k_{-L/2+1}) \\ \vdots \\ q_n(k_{L/2}) \end{pmatrix}$, is an orthogonal basis of $\mathbb{C}^L$. Namely

$$\vec{q}_n \cdot \vec{q}_m = \sum_{\{k_r\}} q_n^*(k_r) q_m(k_r) = L\,\delta_{n,m}$$

$$\vec{q}_n \cdot \vec{v} = 0 \quad \forall n \iff \vec{v} \equiv 0$$

Exercise 3

Using the above Obs. prove that $\{Q_n\}$ are linearly independent because orthogonal with respect to the scalar product

$$(A,B) = \frac{1}{2^L} \underset{\overset{\uparrow}{\text{trace}}}{\mathrm{tr}}[A^\dagger B]$$

Moreover, they are complete. Namely if $Q^f = \sum_{\{k_r\}} f(k_r)(\hat{n}_{k_r} - \tfrac{1}{2})$ is such that

$$(Q^f, Q^{(n)}) = 0 \quad \forall n$$

$$\Rightarrow \quad Q^f = 0.$$

(up to operators having e.v. 0 in every translationally invariant state)

The currents are defined via

to fix the additive constant

$$i[H, q_j^{(n)}] = J_{j-1}^{(n)} - J_j^{(n)} \quad \oplus \quad \mathrm{tr}[J_j^{(n)}] = 0$$

We don't need their explicit expression here, see Fagotti '16 if you are interested in it.

C. Eigenstates

The eigenstates of H are constructed by introducing the vacuum state $|0\rangle$ such that

$$C(k_n)|0\rangle = 0 \quad \forall\ n$$

and noting that

$$[H, C^\dagger(k_n)] = \mathcal{E}(k_n)\, C^\dagger(k_n) \quad \Rightarrow \quad e^{iHt}\, C^\dagger(k_n)\, e^{-iHt} = e^{i\mathcal{E}(k_n)t}\, C^\dagger(k_n) \qquad (*)$$

So that one can easily verify that

$$|\Psi_{k_1 \cdots k_N}\rangle = C^\dagger(k_1) \cdots C^\dagger(k_N)\, |0\rangle \qquad \text{where} \quad k_j \equiv \frac{2\pi}{L} n_j \quad n_j \in [-L/2+1, L/2] \cap \mathbb{Z}$$

is an eigenstate of H with eigenvalue

$$E_{k_1 \cdots k_N} = \sum_{j=1}^{P} \mathcal{E}(k_j).$$

In terms of $\{c_j^\dagger\}$ we have

$$|\Psi_{k_1 \cdots k_N}\rangle = \sum_{j_1 < \cdots < j_N} \underset{\displaystyle \sum_{P \in S_N} (-1)^P e^{i \sum_{a=1}^{N} k_{P(a)} j_a}}{\underset{|||}{\det\left[e^{i k_a j_b} \right]_{a,b=1,\dots,N}}}\ c_{j_1}^\dagger \cdots c_{j_N}^\dagger |0\rangle$$

↑ permutations of N elements

Considering the "one-site shift" operator T (unitary), defined as

$$T\, c_j^\dagger\, T^\dagger = c_{j+1}^\dagger \qquad (\text{elementary translation on the lattice})$$

We have

$$T\, C_{k_n}^\dagger\, T^\dagger = e^{-ik_n}\, C_{k_n}^\dagger$$

$$T\, |\Psi_{k_1 \cdots k_N}\rangle = e^{-i \sum_{j=1}^{N} k_j}\, |\Psi_{k_1 \cdots k_N}\rangle$$

⇒ the state diagonalises the lattice momentum (defined as $T = e^{-iP}$) with eigenvalue
("quasi" momentum)

$$P_{k_1 \cdots k_N} = \left(\sum_{j=1}^{N} k_j \right) \bmod 2\pi$$

A complete basis of H is given by

$$B = \left\{ |\Psi_{k_1 \cdots k_N}\rangle \;/\; N = 0, \dots, L\,;\ k_j = \frac{2\pi}{L} n_j\,;\ n_j \in [-L/2+1, L/2] \cap \mathbb{Z}\,;\ n_{i+1} > n_i \quad \forall i \right\}$$

All this has a very simple physical interpretation

1. $c^\dagger_k$ "creates" a particle excitation of energy $\varepsilon(k)$ and quasi momentum k (c_k "destroys" it)

2. The eigenstates of H are collections of these particles

3. The particles are independent from each other: <u>no interaction</u>!

D. Thermodynamic Description

let us consider

$$\lim_{th} \langle \Psi_{k_1 \dots k_N} | c^\dagger_{i_1} c_{j_1} | \Psi_{k_1 \dots k_N} \rangle = \lim_{\substack{L,N \to +\infty \\ N/L \text{ fixed}}} \frac{1}{L} \sum_{j=1}^{N} e^{ik_j(i_1 - j_1)} = \lim_{\substack{N,L \to +\infty \\ N/L \text{ fixed}}} \sum_{j=1}^{N-1} \overset{\sim \mathcal{O}(1)}{\frac{1}{L(k_{j+1}-k_j)}} \, e^{ik_j(\overset{\sim \mathcal{O}(1):\ \text{local}}{i_1 - j_1})} \overset{\sim \frac{1}{L}}{(k_{j+1}-k_j)}$$

$$\parallel$$

$$\langle c^\dagger_{i_1} c_{j_1} \rangle_{\rho_{\{k\}}} \qquad \rho_{\{k\}} \rightarrow |\Psi_{k_1 \dots k_N}\rangle\langle \Psi_{k_1 \dots k_N}|$$

$$= \int_{-\pi}^{\pi} dk \, \rho(k) \, e^{ik(i_1 - j_1)} \qquad (*)$$

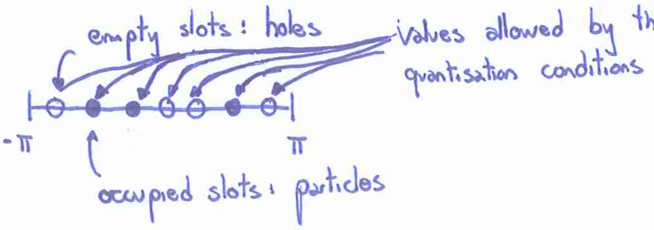
Where we introduced the "root density"

$$\rho(k_j) \equiv \lim_{L \to +\infty} \frac{1}{L(k_{j+1} - k_j)}$$

such that for large L

$$2\pi L \, \rho(k) \Delta k \approx \# \text{ of } \underset{\uparrow}{\{k_j\}} \text{ in } [k, k+\Delta k]$$

"Occupied momenta" or "particles"



Eq $(*)$ proves that $\langle c^\dagger_i c_j \rangle_{\rho_{\{k\}}}$, in the thermodynamic limit, depends only on the root density. Using that $\langle c^\dagger_{i_1} c^\dagger_{j_1} \rangle_{\rho_{\{k\}}} = \langle c_{i_1} c_{j_1} \rangle_{\rho_{\{k\}}} = 0$ and that $|\Psi_{\{k\}}\rangle$ are Gaussian (Wick's theorem holds) we then conclude that: <u>$\rho(k)$ completely characterises all expectation values of local observables</u>.

In analogy with $\rho(k)$, it is useful to introduce the density of "non-occupied momenta" $\rho^h(k)$. Such that, for large finite L, we have

$$2\pi L \, \rho^h(k) \Delta k \approx \# \text{ of non-occupied momenta in } [k, k+\Delta k]$$

Using that the difference of two neighbouring allowed values of the momentum is $\frac{2\pi}{L}$ we have

$$\rho(k)+\rho^h(k)=\frac{1}{2\pi}$$ "total" density

Note that many different eigenstates correspond to the same $\rho(k)$, their number can be expressed in terms of $\rho(k)$

$$\divideontimes = \exp\left[L\, S_{YY}[\rho]\right] + \text{subleading terms}$$

$$S_{YY}[\rho]=\int_{-\pi}^{\pi} dk \left\{ \frac{1}{2\pi}\log\frac{1}{2\pi} - \rho(k)\log\rho(k) - \left(\frac{1}{2\pi}-\rho(k)\right)\log\left(\frac{1}{2\pi}-\rho(k)\right)\right\}$$

Exercise 4 : Find this formula.

In the free fermionic chain under exam

$$\rho(k)=\lim_{th}\frac{1}{2\pi}\langle\Psi_{k_1\cdots k_N}|\,\hat{n}_k\,|\Psi_{k_1\cdots k_N}\rangle$$

Exercise 5 : Prove it.

Finally, we note that also translational invariant and stationary Gaussian mixed states $\hat{\rho}_{mix}$ (e.g. thermal states) are fully characterised by a root density in the thermodynamic limit

$$\rho(k)=\lim_{th}\frac{1}{2\pi}\,\mathrm{tr}\left[\hat{\rho}_{mix}\,\hat{n}_k\right]$$

In other words $\hat{\rho}_{mix}$ is equivalent, in the thermodynamic limit, to all eigenstates

$|\Psi_{k_1\cdots k_N}\rangle$ with $k_{j+1}-k_j=\frac{1}{L\rho(k_j)}$ (Microcanonical representation)

• Expectation values of charges and currents in the thermodynamic limit

Using the expressions at page ⑧ we immediatelly have

$$\lim_{th} \langle q_j^{(n)} \rangle_{\rho_{\{k\}}} = \int_{-\pi}^{\pi} dk \; \rho(k) \, \underset{\uparrow}{q_n(k)} - \delta_{n,0} \, \pi \qquad (C)$$

$\{ 1, \sqrt{2} \sin(nk), \sqrt{2} \cos(nk) \}$ Complete orthogonal set in $L_2 [-\pi,\pi]$

The expectation values of currents in the Hamiltonian's eigenstates have been determined in Fagotti '16. In the thermodynamic limit they read as

$$\lim_{th} \langle J_j^{(n)} \rangle_{\rho_{\{k\}}} = \int_{-\pi}^{\pi} dk \; \rho(k) \, q_n(k) \, V(k) + \underset{\substack{\uparrow \\ \text{does not depend on the state} \Rightarrow \text{not useful for us}}}{const} \qquad (J)$$

Here $V(k)$ is the group velocity of an elementary excitation with quasi momentum k on the state $|\Psi_{k_1 \ldots k_N}\rangle$.

It is found as follows:

Consider a stationary state $|\Psi_{k_1 \ldots k_N}\rangle$ in a finite volume L.

Adding an excitation of (allowed) momentum k we have

$$|\Psi_{k_1 \ldots k_N}\rangle \mapsto |\Psi_{k; k_1 \ldots k_N}\rangle = C_k^{\dagger} |\Psi_{k_1 \ldots k_N}\rangle$$

Energy of the excitation

$$\Delta E_k = E_{k; k_1 \ldots k_N} - E_{k_1 \ldots k_N} = \mathcal{E}(k) \qquad \longmapsto \qquad \text{Group Velocity: } V(k) = \frac{\partial \Delta E_k}{\partial \Delta P_k} = \mathcal{E}'(k)$$

$$\Delta P_k = P_{k; k_1 \ldots k_N} - P_{k_1 \ldots k_N} = k$$

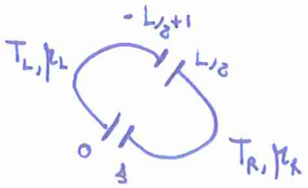
momentum of the excitation

Obs.

Note that k is just a label, ΔP_k is the <u>real</u> momentum change due to the excitations.
We will see that these two quantities do not coincide in the interacting case.

Bipartite Quench

A. Initialise the system in the state $\hat{\rho}_0 = \frac{1}{Z} \exp\left[-\beta_L (H_L - \mu_L N_L) - \beta_R (H_R - \mu_R N_R) \right]$



Where

$$H_L = -\frac{J}{2} \sum_{j=-L/2+1}^{-1} c^\dagger_{j+1} c_j + c^\dagger_j c_{j+1}$$

$$H_R = -\frac{J}{2} \sum_{j=1}^{L/2-1} \quad " \quad\quad "$$

$$N_L = \sum_{j=-L/2+1}^{0} c^\dagger_j c_j$$

$$N_R = \sum_{j=1}^{L/2} \quad " \quad "$$

Obs. $\hat{\rho}_0$ is Gaussian for the fermionic operators $\{c^\dagger_j, c_j\}$. Indeed it can be written as

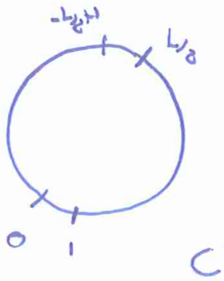
$$\hat{\rho}_0 = \frac{1}{Z} \exp\left[\sum_{i,j=-L/2+1}^{L/2} c^\dagger_i M_{ij} c_j \right]$$

$\uparrow$ Quadratic Form $\Rightarrow$ Wick's theorem applies!

$$M = \left[\begin{array}{c|c} \begin{matrix} \beta_L \mu_L & -\frac{J}{2}\beta_L & 0 \\ -\frac{J}{2}\beta_L & \ddots & \ddots \\ 0 & \ddots & \ddots \end{matrix} & 0 \\ \hline 0 & \begin{matrix} \beta_R \mu_R & -\frac{J}{2}\beta_R & 0 \\ -\frac{J}{2}\beta_R & \ddots & \ddots \\ 0 & \ddots & \ddots \end{matrix} \end{array} \right]$$

M is $L \times L$

B. Evolve with $H \left(= H_R + H_L - \frac{J}{2}(c^\dagger_0 c_1 + c^\dagger_1 c_0) - \frac{J}{2}(c^\dagger_{L/2} c_{-L/2+1} + c^\dagger_{-L/2+1} c_{L/2}) \right)$



Obs 1. Using (*) at page ⑦ (Fourier Transformation) and (*) at page ⑨ (evolution of $c^\dagger(k)$) we have

$$c^\dagger_{j_1}(t) = e^{iHt} c^\dagger_{j_1} e^{-iHt} = \sum_{j_0=-L/2+1}^{L/2} \Big(\underbrace{\sum_{\{k_n\}} \frac{1}{L} e^{i j_1 k_n} e^{i\varepsilon(k_n)t} e^{-i j_0 k_n}}_{\equiv\, G_{j_1 j_0}(t)} \Big) c^\dagger_{j_0}$$

The set $\{c_j, c^\dagger_j\}$ is closed under time evolution!

This is due to the fact that H is quadratic

Since $\hat{\rho}_0$ is quadratic and $\{c^\dagger_j, c_j\}$ is closed under time evolution

$\Rightarrow$ $\hat{\rho}(t) = e^{-iHt} \hat{\rho}_0 e^{iHt}$ is the exponential of a quadratic form $\forall t$

$\Rightarrow$ Wick's theorem applies at all times!

Obs. 2

Since $[H, N] = 0 = [\rho_0, N]$

$$\Rightarrow \quad \langle c_{j_1} c_{j_2} \rangle_{\rho(t)} = \langle c^{\dagger}_{j_1} c^{\dagger}_{j_2} \rangle_{\rho(t)} = 0 \qquad \forall\, t$$

Exercise 6: Prove it!

Hint: Use that

$$c^{\dagger}_{j} \longrightarrow e^{i\phi N} c^{\dagger}_{j} e^{-i\phi N} = e^{i\phi} c^{\dagger}_{j}$$

leaves H and $\hat{\rho}_0$ invariant but changes the above correlators

Obs. 1 and Obs. 2. $\Rightarrow$ to characterise $\underline{\text{all}}$ local observables we only need to compute

$$\langle c^{\dagger}_{j_1} c_{j_2} \rangle_{\hat{\rho}(t)} \underset{\overset{\uparrow}{(*)\text{ page ⑬}}}{=} \sum_{i_1, i_2 = -L/2+1}^{L/2} G_{j_1 i_1}(t)\, \overset{\overset{\text{Complex conjugation}}{\downarrow}}{G^{*}_{j_2 i_2}}(t) \langle c^{\dagger}_{i_1} c_{i_2} \rangle_{\hat{\rho}_0} \qquad (*)$$

Exercise* 7:

Show that

$$\langle c^{\dagger}_{j_1} c_{j_2} \rangle_{\hat{\rho}_0} = \theta(j_1)\theta(j_2) \sum_{\{\tilde{p}_n\}} \frac{2}{L/2+1} \sin(\tilde{p}_n j_1) \sin(\tilde{p}_n j_2)\, f_R(\tilde{p}_n)$$

$$+ \theta(1-j_1)\theta(1-j_2) \sum_{\{\tilde{p}_n\}} \frac{2}{L/2+1} \sin(\tilde{p}_n(1-j_1)) \sin(\tilde{p}_n(1-j_2))\, f_L(\tilde{p}_n) \quad (**)$$

Where

$$\theta(i) = \begin{cases} 1 & i \in \{1, \ldots, L/2\} \\ 0 & \text{otherwise} \end{cases}$$

$$f_{R/L}(k) = \frac{1}{1 + e^{\beta_{R/L}(\varepsilon(k) - \mu_{R/L})}}$$

$$\tilde{p}_n = \frac{2\pi}{L+2} \qquad n = 1, \ldots, L/2$$

$$\sum_{\{\tilde{p}_n\}} f(\tilde{p}_n) \equiv \sum_{n=1}^{L/2} f\left(\frac{2\pi}{L+2} \cdot n\right)$$

Hint.

1. Show that

$$\begin{cases} H_R = \sum_{\{\tilde{p}_n\}} \varepsilon_R(\tilde{p}_n)\, r^\dagger(\tilde{p}_n)\, r(\tilde{p}_n) \\ H_L = \sum_{\{\tilde{p}_n\}} \varepsilon_L(\tilde{p}_n)\, l^\dagger_{\tilde{p}_n} l_{\tilde{p}_n} \end{cases}$$

where

$$\begin{cases} r^\dagger(p) = \sqrt{\frac{2}{L/2+1}} \sum_{j=1}^{L/2} \sin(p_n j)\, c^\dagger_j \\ l^\dagger(p) = \sqrt{\frac{2}{L/2+1}} \sum_{j=1}^{L/2} \sin(p_n j)\, c^\dagger_{1-j} \\ \varepsilon_{R/L}(p) = -J\beta_{R/L}\cos(p) - \beta_{R/L}\mu_{R/L} \\ \{r(\tilde{p}_n), r(\tilde{p}_m)\} = 0 = \{r^\dagger(\tilde{p}_n), r^\dagger(\tilde{p}_m)\} \\ \{l(\tilde{p}_n), l(\tilde{p}_m)\} = 0 = \{l^\dagger(\tilde{p}_n), l^\dagger(\tilde{p}_m)\} \\ \{l^\dagger(\tilde{p}_n), l(\tilde{p}_m)\} = \delta_{n,m} = \{r^\dagger(\tilde{p}_n), r(\tilde{p}_m)\} \end{cases}$$

2. Show that

$$c^\dagger_j = \theta(j)\sqrt{\frac{2}{L/2+1}} \sum_{\{\tilde{p}_n\}} \sin(\tilde{p}_n j)\, r^\dagger(\tilde{p}_n)$$
$$+ \theta(1-j)\sqrt{\frac{2}{L/2+1}} \sum_{\{\tilde{p}_n\}} \sin(\tilde{p}_n(1-j))\, l^\dagger(\tilde{p}_n)$$

3. Show that

$$\langle r^\dagger(\tilde{p}_n)\, r(\tilde{p}_m)\rangle_{\rho_0} = \delta_{n,m}\, f_R(\tilde{p}_m)$$
$$\langle l^\dagger(\tilde{p}_n)\, l(\tilde{p}_m)\rangle_{\rho_0} = \delta_{n,m}\, f_L(\tilde{p}_m)$$

Substituting (**) at page ⑭ in (*) at page ⑭ we have

$$\langle c^\dagger_i c_j\rangle_{\rho(t)} = \sum_{\{\tilde{p}_m\}} \frac{2}{L/2+1} \cdot f_R(\tilde{p}_m) \cdot \left(\sum_{j_0=1}^{L/2} G^*_{jj_0}(t)\sin(\tilde{p}_m j_0)\right)\left(\sum_{i_0=1}^{L/2} G_{ii_0}(t)\sin(\tilde{p}_m i_0)\right)$$
$$+ \sum_{\{\tilde{p}_m\}} \frac{2}{L/2+1} f_L(\tilde{p}_m) \left(\sum_{j_0=-L/2+1}^{0} G^*_{jj_0}(t)\sin(\tilde{p}_m(1-j_0))\right)\left(\sum_{i_0=-L/2+1}^{0} G_{ii_0}(t)\sin(\tilde{p}_m(1-i_0))\right)$$

Performing explicitly the sums over j_0 and i_0 (they are just sums of geometric series) we have

$$\sum_{j_0=1}^{L/2} G_{jj_0}(t)\ \sin(\tilde{p}_m j_0) = F_m(t,j) = \frac{1}{L}\sum_{n=-L/2+1}^{L/2} e^{ik_n j}\, e^{i\varepsilon(k_n)t}\, O_{n,m}$$

$$\sum_{j_0=}^{0} G_{jj_0}(t)\ \sin(\tilde{p}_m i_0) = \tilde{F}_m(t,j) = \frac{1}{L}\sum_{n=-L/2+1}^{L/2} e^{ik_n j}\, e^{i\varepsilon(k_n)t}\, \tilde{O}_{n,m}$$

Where

$$O_{n,m} = \sum_{j_0=1}^{L/2} e^{-ij_0 k_n} \sin(\tilde{p}_m j_0) = e^{-i\frac{(k_n+\tilde{p}_m)}{2}} \frac{\left(e^{i\tilde{p}_m} + (-)^{m+n+1}\right)}{4\sin\left(\frac{\tilde{p}_m - k_n}{2}\right)} + e^{i\frac{(\tilde{p}_m-k_n)}{2}} \frac{\left(e^{-i\tilde{p}_m} + (-)^{n+m+1}\right)}{4\sin\left(\frac{\tilde{p}_m+k_n}{2}\right)}$$

$$\tilde{O}_{n,m} = \sum_{j_0=1}^{L/2} e^{-ik_n} e^{ij_0 k_n} \sin(\tilde{p}_m j_0) = e^{-ik_n} O^*_{n,m}$$

Comments

1. These are nothing but the ovelaps of the single-particle wavefunctions of a periodic chain and an open half chain. We see that there is a dominant contribution when the periodic wavefunction is almost proportional to one of the two counterpropagating components of the wavefunction of the open chain. Namely for $k_n \sim \pm \tilde{p}_m$.

   Note, however, that there can be no divergence since

   $k_n \pm \tilde{p}_m \neq 0 \quad \forall \; n = -L/2+1, \dots, L/2, \quad m = 1, \dots, L/2$

2. $O_{n,m}$ is not a smooth function of k_n and $\tilde{p}_m$, it explicitly depends on the integers n and m. However

   $$O_{2n-s,\,2m-r} = S_{s,r}(k_{2n-s}, \tilde{p}_{2m-r})$$

   with

   $$S_{s,r}(k,p) = e^{i\frac{(k+p)}{2}} \frac{e^{ip} + (-)^{s+r+1}}{-4\sin\left(\frac{k-p}{2}\right)} + e^{i\frac{(p-k)}{2}} \frac{\left(e^{-ip} + (-)^{s+r+1}\right)}{4\sin\left(\frac{p+k}{2}\right)}$$

   So we can write the correlation function in terms of smooth functions of the momenta by separating the sums

   $$\sum_{\{k_n\}} = \sum_{s=0,1} \sum_{\{k_{2n-s}\}}$$

   Explicitly we have

$$\langle c^\dagger_{i_1} c_{j_1} \rangle_{\rho(t)} = \sum_{r,s_1,s_2=0,1} \frac{2}{L/2+1} \sum_{\{\tilde{p}_{2m-r}\}} \frac{1}{L} \sum_{\{k_{2n_1-s_1}\}} \frac{1}{L} \sum_{\{k_{2n_2-s_2}\}} \Big\{ e^{ik_{2n_1-s_1} i_1} e^{i\varepsilon(k_{2n_1-s_1})t} e^{-ik_{2n_2-s_2} j_1} e^{-i\varepsilon(k_{2n_2-s_2})t}$$

$$S_{s_1,r}(k_{2n_1-s_1}, \tilde{p}_{2m-r}) S^*_{s_2,r}(k_{2n_2-s_2}, \tilde{p}_{2m-r})$$

$$f_R(\tilde{p}_{2m-r}) \Big\}$$

$$+ \sum_{r,s_1,s_2=0,1} \frac{2}{L/2+1} \sum_{\{\tilde{p}_{2m-r}\}} \frac{1}{L} \sum_{\{k_{2n_1-s_1}\}} \frac{1}{L} \sum_{\{k_{2n_2-s_2}\}} \Big\{ e^{ik_{2n_1-s_1}(i_1-1)} e^{i\varepsilon(k_{2n_1-s_1})t} e^{-ik_{2n_2-s_2}(j_1-1)} e^{-i\varepsilon(k_{2n_2-s_2})t}$$

$$S^*_{s_1,r}(k_{2n_1-s_1}, \tilde{p}_{2m-r}) S_{s_2,r}(k_{2n_2-s_2}, \tilde{p}_{2m-r}) f_L(\tilde{p}_{2m-r}) \Big\}$$

(*)

1. Integral Representation

Our goal is to take the thermodynamic limit of the above formula. In this limit sums become Integrals

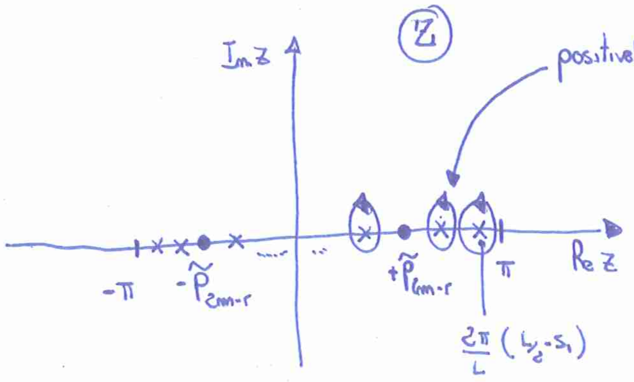
$$\frac{1}{L}\sum'_{p_n} \sim \int \frac{dp}{2\pi}$$

but the functions $S_{B,r}(\kappa,p)$ develop poles. The clean way to proceed is to write the sums in terms of contour integrals. This can be done as follows.

Consider one of the two multiple sums in (*) and call $F_{r,s_1,s_2}(\tilde{p}_{2m-r},\kappa_{2n_1-s_1},\kappa_{2n_2-s_2})$ the summand. The sum over $\{\kappa_{2n_1-s_1}\}$ can be written as a sum of contour integrals

$$\frac{1}{L}\sum_{s_1}^{1}\sum'_{\{\kappa_{2n_1-s_1}\}} F_{r,s_1,s_2}(\tilde{p}_{2m-r},\kappa_{2n_1-s_1},\kappa_{2n_2-s_2}) = \sum_{s_1}^{1}\sum'_{\{\kappa_{2n_1-s_1}\}} \oint_{\mathcal{C}_{\kappa_{2n_1-s_1}}} dz\, f_{s_1,L}(z)\, F_{r,s_1,s_2}(\tilde{p}_{2m-r},z,\kappa_{2n_2-s_2})$$

$\mathcal{C}_{\kappa_{2n_1-s_1}}$ positively oriented circle around $\kappa_{2n_1-s_1}$



$$f_{s_1,L}(z) = \frac{(-)^{s_1}}{4\pi\left(e^{iL_2 z}-(-)^{s_1}\right)}$$

has poles in $\{\kappa_{2n_1-s_1}\}$ with residues $\frac{1}{2i\pi L}$

The last step is to join all these circles to get a contour encircling $[-\pi,\pi]$. To do that we should subtract the contribution of the poles of $F_{r,s_1,s_2}(\tilde{p}_{2m-r},z,\kappa_{2n_2-s_2})$ in $z=\pm\tilde{p}_{2m-r}$, this gives

$$\frac{1}{L}\sum_{s_1}^{1}\sum'_{\{\kappa_{2n_1-s_1}\}} F_{r,s_1,s_2}(\tilde{p}_{2m-r},\kappa_{2n_1-s_1},\kappa_{2n_2-s_2}) = \sum_{s_1}^{1}\oint_{P_+} dz\, f_{s_1,L}(-z)\, F_{r,s_1,s_2}(\tilde{p}_{2m-r},z,\kappa_{2n_2-s_2})$$

P_+: Positively oriented contour encircling $[-\pi,\pi]$

$f_{s_1,L}(-z)$: effect of subtracting the poles in $\pm\tilde{p}$ (after summing over s_1)

Repeating the same treatment for the sum over $\{\kappa_{2n_2-s_2}\}$

$$\Rightarrow \frac{1}{L}\sum_{s_2}^{1}\sum'_{\{\kappa_{2n_2-s_2}\}}\frac{1}{L}\sum_{s_1}^{1}\sum'_{\{\kappa_{2n_1-s_1}\}} F_{r,s_1,s_2}(\tilde{p}_{2m-r},\kappa_{2n_1-s_1},\kappa_{2n_2-s_2}) = \sum_{s_1,s_2=0,1}\oint_{P_+} dz_1 \oint_{P_-} dz_2\, f_{s_1,L}(-z_1)\, f_{s_2,L}(+z_2)\, F_{r,s_1,s_2}(\tilde{p}_{2m-r},z_1,z_2)$$

P_-: Same as P_+ but negatively oriented

Finally, considering the sum over $\{p_m\}$ we have

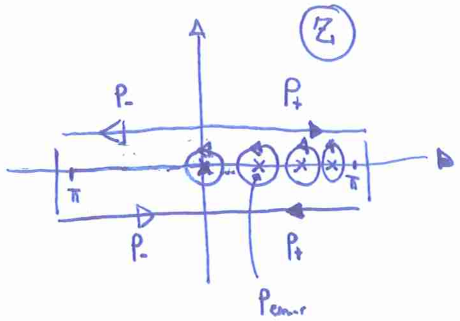
$$\frac{2}{L/2+1}\sum_{r=0,1}{}'\sum_{\{p_{2m-r}\}}{}'\frac{1}{L}\sum_{s_1=0,1}{}'\sum_{\{k_{2n_1-s_1}\}}{}'\frac{1}{L}\sum_{s_2=0,1}{}'\sum_{\{k_{2n_2-s_2}\}}{}' F_{r,s_1,s_2}(\tilde{p}_{2m-r},k_{2n_1-s_1},k_{2n_2-s_2})$$

$$=\sum_{s_1s_2\,r=0,1}{}'\sum_{\{p_{2m-r}\}}{}'\oint_{\mathcal{C}_{p_{2m-r}}}dz\;\varphi_{r,L}(z)\oint_{P_+}dz_1\oint_{P_-}dz_2\;f_{s_1,L}(-z_1)f_{s_2,L}(z_2)\;F_{r,s_1,s_2}(z,z_1,z_2)$$

$$\varphi_{r,L}(z)=\frac{(-)^r}{\pi\left(e^{i\left(\frac{L+2}{2}\right)z}-(-)^r\right)}$$

$\uparrow$ has poles in $\{p_{2m-r}\}$ with residues $\dfrac{1}{2\pi i\left(\frac{L/2+1}{2}\right)}$

If we choose $\mathcal{C}$ inside P_+ and P_- there are no tional poles in $F_{r,s_1,s_2}(z,z_1,z_2)$ as a function of z_1



$\Rightarrow$ we can join the contours

$$\frac{2}{L/2+1}\sum_{r}{}'\sum_{\{p_{2m-r}\}}{}'\frac{1}{L}\sum_{s_1}{}'\sum_{\{k_{2n_1-s_1}\}}{}'\frac{1}{L}\sum_{s_2}{}'\sum_{\{k_{2n_2-s_2}\}}{}' F_{r,s_1,s_2}(\tilde{p}_{2m-r},k_{2n_1-s_1},k_{2n_2-s_2})$$

$$=\sum_{s_1s_2\,r=0,1}{}'\oint_{Q}dz\oint_{P_+}dz_1\oint_{P_-}dz_2\;\varphi_{r,L}(z)\,f_{s_1,L}(-z_1)\,f_{s_2,L}(z_2)\;F_{r,s_1,s_2}(z,z_1,z_2)$$

Q $\uparrow$ Positively oriented contour encircling $[0,\pi]$ <u>inside</u> P_+ and P_-

Repeating the same treatment for the other multiple sum in $(*)$ we find the following <u>exact integral representation</u>

$$\langle c^\dagger_{i_1}c_{j_1}\rangle_{\rho(t)}=\sum_{s_1s_2r=0,1}{}'\oint_{Q}dz\oint_{P_+}dz_1\oint_{P_-}dz_2\Big\{\varphi_{r,L}(z)\underset{\uparrow}{f_{s_1,L}(-z_1)}\underset{\uparrow}{f_{s_2,L}(z_2)}\,e^{iz_1i_1}e^{i\varepsilon(z_1)t}e^{-iz_2j_1}e^{-i\varepsilon(z_2)t}(-)\,S_{s_1,r}(z_1,z)\,S_{s_2,r}(-z_2,-z)\,f_R(z)\Big\}$$

Note the different signs!

$$+\sum_{s_1,s_2,r=0,1}{}'\oint_{Q}dz\oint_{P_+}dz_1\oint_{P_-}dz_2\Big\{\varphi_{r,L}(z)\overset{\downarrow}{f_{s_1,L}(z_1)}\overset{\downarrow}{f_{s_2,L}(-z_2)}\,e^{iz_1(i_1-1)}e^{i\varepsilon(z_1)t}e^{-iz_2(j_1-1)}e^{-i\varepsilon(z_2)t}(-)\,S_{s_1,r}(-z_1,-z)\,S_{s_2,r}(z_2,z)\,f_L(z)\Big\}$$

Now we can safely take the thermodynamic limit!

2. Thermodynamic Limit

To take the thermodynamic limit of the above integral representation we only need to evaluate the infinite volume limit of $\varrho_{r,L}(z)$ and $f_{s,L}(z)$.

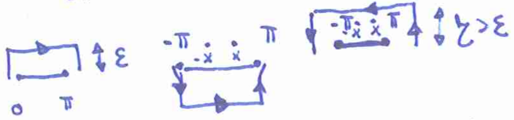
It is immediate to see

$$\lim_{L\to+\infty} f_{s,L}(z) = \begin{cases} 0 & z \in \text{lower half plane} \\ \frac{-1}{4\pi} & z \in \text{upper half plane} \end{cases}$$

$$\lim_{L\to+\infty} \varrho_{r,L}(z) = \begin{cases} 0 & z \in \text{lower half plane} \\ -\frac{1}{\pi} & z \in \text{upper half plane} \end{cases}$$

Carrying out the summations over r, s_1, s_2

$$\Rightarrow \lim_{th} \langle c^{+}_{i_1} c_{j_1} \rangle_{\rho(t)} = -\int \frac{dx}{2\pi} \int \frac{dz_1}{2\pi} \int \frac{dz_2}{2\pi} \Big\{ e^{i(z_1 i_1 + \varepsilon(z_1)t - z_2 j_1 - \varepsilon(z_2)t)} f_R(x)$$



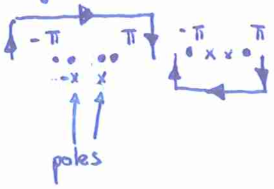
this can be deformed on the real axis!

$$\int_0^{\pi} \frac{dx}{2\pi}$$

$$\times \left[\frac{e^{i\left(\frac{x-z_1}{2}\right)}}{2\sin\left(\frac{x-z_1}{2}\right)} + \frac{e^{-i\left(\frac{x+z_1}{2}\right)}}{2\sin\left(\frac{x+z_1}{2}\right)} \right]$$

$$\times \left[\frac{e^{-i\left(\frac{x-z_2}{2}\right)}}{2\sin\left(\frac{x-z_2}{2}\right)} + \frac{e^{i\left(\frac{x+z_2}{2}\right)}}{2\sin\left(\frac{x+z_2}{2}\right)} \right] \Big\}$$

$$-\int_0^{\pi} \frac{dx}{2\pi} \int \frac{dz_1}{2\pi} \int \frac{dz_2}{2\pi} \Big\{ e^{i(z_1 i_1 + \varepsilon(z_1)t - z_2 j_1 - \varepsilon(z_2)t)} f_L(x)$$



$$\times \left[\frac{e^{-i\left(\frac{x+z_1}{2}\right)}}{2\sin\left(\frac{x-z_1}{2}\right)} + \frac{e^{-i\left(\frac{x-z_1}{2}\right)}}{2\sin\left(\frac{x+z_1}{2}\right)} \right]$$

$$\times \left[\frac{e^{i\left(\frac{x+z_2}{2}\right)}}{2\sin\left(\frac{x-z_2}{2}\right)} + \frac{e^{-i\left(\frac{x-z_2}{2}\right)}}{2\sin\left(\frac{x+z_2}{2}\right)} \right] \Big\}$$

3. Scaling Limit

The final step is to take the scaling limit

$$\lim_{sc,\xi} \equiv \lim_{\substack{t, i_1, j_1 \to +\infty \\ i_1/t = j_1/t = \xi}}$$

To compute the scaling limit we deform the contours of integration in z_1 and z_2 such that the exponents go to $-\infty$ in the limit

$\Rightarrow$ The only surviving contributions are given by the residues of the poles encountered when we cross the real axis.

To carry out this analysis is sufficient to consider a single integral in $(*)$, say the z_1 integration in the first term. We have

$$F(x,t,j) = \int \frac{dz_1}{2\pi} \left\{ \frac{e^{i\left(\frac{x-z_1}{2}\right)}}{2\sin\left(\frac{x-z_1}{2}\right)} + \frac{e^{-i\left(\frac{x+z_1}{2}\right)}}{2\sin\left(\frac{x+z_1}{2}\right)} \right\} e^{i(z_1 j + \varepsilon(z_1)t)}$$

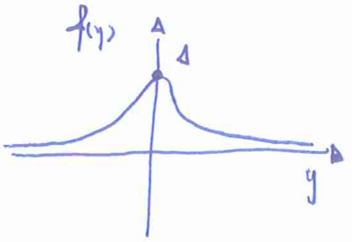
we want this

$$\mathrm{Re}\left[i(z_1 j + \varepsilon(z_1)t)\right] = -y_1 j + \varepsilon'(x_1)\sinh y_1 \cdot t < 0$$

Here $x_1 = \mathrm{Re}\, z_1$
$y_1 = \mathrm{Im}\, z_1$

$$\Rightarrow \begin{cases} \left[\begin{array}{l} \varepsilon'(x_1) < \xi f(y_1) \\ y > 0 \end{array}\right. & (a) \\ \left[\begin{array}{l} \varepsilon'(x_1) > \xi f(y_1) \\ y < 0 \end{array}\right. & (b) \end{cases}$$

where $f(y) \equiv \dfrac{y}{\sinh y}$



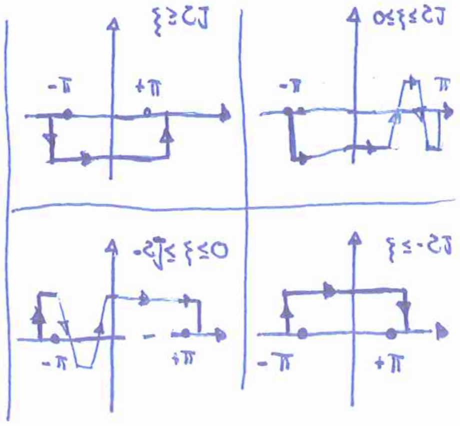
We then see that if $\varepsilon'(x_1) < \xi$ $\exists$ $y_1 > 0$ s.t. (a) is fulfilled $\Rightarrow$ need to cross the real axis!

if $\varepsilon'(x_1) > \xi$ $\exists$ $y_1 < 0$ s.t. (b) is fulfilled $\Rightarrow$ don't need to cross the real axis!

$\boxed{\text{always } t\cdot\alpha + i\beta \text{ on } D}$

$$F(x,t,j) = \int_D \frac{dz_1}{2\pi} \left\{ \frac{e^{-i\left(\frac{z_1-x}{2}\right)}}{2\sin\left(\frac{x-z_1}{2}\right)} + \frac{e^{-i\left(\frac{x+z_1}{2}\right)}}{2\sin\left(\frac{x+z_1}{2}\right)} \right\} e^{i(z_1 j + \varepsilon(z_1)t)}$$

$D =$



$\varepsilon(-x) = \varepsilon(x)$

$$-i\,\theta(-\varepsilon'(x)+\xi)\, e^{i(xj + \varepsilon(x)t)} + i\,\theta(\varepsilon'(x)+\xi)\, e^{i(\varepsilon(x)t - jx)}$$

Treating analogously the other 3 contours we find

$$\lim_{sc,\xi}\lim_{th}\langle c^{\dagger}_{i_1}c_{j_1}\rangle=\lim_{sc,\xi}\int_0^{\pi}\frac{dx}{2\pi}\left\{\vartheta(\xi-\varepsilon'(x))\,e^{ixi_1}-\vartheta(\xi+\varepsilon'(x))\,e^{-ixi_1}\right\}$$

$$\left\{\vartheta(\xi-\varepsilon'(x))\,e^{-ixj_1}-\vartheta(\xi+\varepsilon'(x))\,e^{ixj_1}\right\}f_R(x)$$

$$+\int_0^{\pi}\frac{dx}{2\pi}\left\{\vartheta(\varepsilon'(x)-\xi)\,e^{ix(i_1-1)}-\vartheta(-\varepsilon'(x)-\xi)\,e^{-ix(i_1-1)}\right\}$$

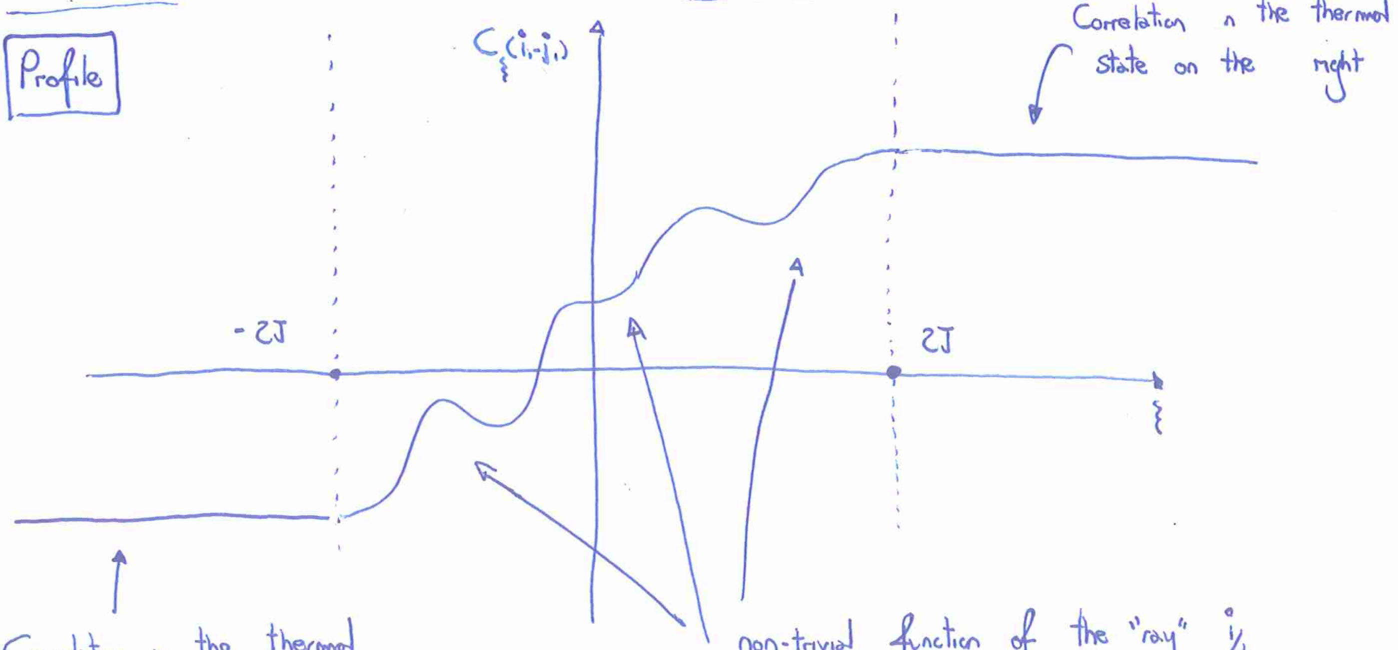
$$\left\{\vartheta(\varepsilon'(x)-\xi)\,e^{-ix(j_1-1)}-\vartheta(-\varepsilon'(x)-\xi)\,e^{ix(j_1-1)}\right\}$$

the limit kills the crossed terms $\sim e^{\pm ix(i_1+j_1)}$

$$=\int_{-\pi}^{\pi}\frac{dx}{2\pi}\left\{\vartheta(\xi-\varepsilon'(x))f_R(x)+\vartheta(\varepsilon'(x)-\xi)f_L(x)\right\}e^{ix(i_1-j_1)}\equiv C_{\xi}(i_1-j_1) \quad Ⓐ$$

i_1-j_1 fixed

Profile



Correlation in the thermal state on the right

Correlation in the thermal state on the left

non-trivial function of the "ray" i/t

Exercise 8:

Show that

$$J_{e_j}=\frac{J^2}{2\sqrt{2}}q^{(5)}_j+\frac{Jh}{\sqrt{2}}q^{(3)}_j$$

energy current

and plot its profile.

- We call this region "light-cone region" as it is contained in a light cone spreading out from the junction at the maximal speed $2J$.

- In real space this region grows with t, while the thermal regions on the left and on the right are of infinite size ⇒ effective baths

Comparing $C_\xi(i_1-j_1)$ with stationary correlators in the thermodynamic limit

Comparing Ⓐ with (✱) at p. ⑩ we see that, at fixed ξ, look the same if we choose

$\rho(u) = \rho_\xi(u)$ with

$$\rho_\xi(u) = \frac{1}{2\pi}\left\{ f_R(u)\,\theta(\xi-\varepsilon'(u)) + f_L(u)\,\theta(\varepsilon'(u)-\xi)\right\} \quad (✱)$$

$\Rightarrow$ In the scaling limit (after the thermodynamic limit is taken) we can represent $\hat{\rho}(t)$ by a translational invariant stationary state $\hat{\rho}^s_\xi$ in the expectation values of all local observables.

"Locally Quasi-Stationary State" LQSS

Crucial question:

Can we find $\hat{\rho}^s_\xi$ without solving the full problem?

Idea: (borrowed from the literature on homogeneous quantum quenches
(see Essler and Fagotti '16))

We can fix the stationary state using the expectation values of local conserved charges!

We have

① $[Q^{(n)}, H] = 0 \quad \forall n \quad \Leftrightarrow \quad \partial_t q^{(n)}_j(t) = J^{(n)}_{j-1}(t) - J^{(n)}_j(t) \quad \forall n$ (OCE)

\+ We assume the existence of a stationary state on each ray

② $\lim\limits_{sc,\xi}\lim\limits_{th}\langle O_j\rangle_{\hat{\rho}(t)} = \langle O_j\rangle_{\hat{\rho}^s_\xi}$ [In the thermodynamic limit is completely characterised by its root density $\rho_\xi(u)$

Use ① and ② to find $\hat{\rho}^s_\xi$ (or equivalently its associated root density $\rho_s(u)$)

Warming up example

Assume $\hat{\rho}(t)$ translationally invariant $\forall t$ ($\underline{\text{not}}$ our case)

Then $\lim\limits_{sc,\xi} = \lim\limits_{t\to+\infty}$ $\forall \xi$

So ② becomes

$$\lim_{t\to+\infty}\lim_{th}\langle O_j\rangle_{\hat{\rho}(t)} = \lim_{th}\langle O_j\rangle_{\hat{\rho}^s} \qquad ②^*$$

While ① gives

$$\partial_t\langle q_j^{(n)}\rangle_{\hat{\rho}(t)} = 0 \Rightarrow \langle q_j^{(n)}\rangle_{\hat{\rho}(t)} = \langle q_j^{(n)}\rangle_{\hat{\rho}(0)} \quad \forall n \qquad ①^*$$

then

$$\Rightarrow \quad \lim_{th}\langle q_j^{(n)}\rangle_{\hat{\rho}^s} = \lim_{th}\langle q^{(n)}\rangle_{\hat{\rho}(0)}$$

$$\Rightarrow \quad \int_{-\pi}^{\pi} dk\, q_n(k)\left(\rho_s(k) - \rho_0(k)\right) = 0$$

where

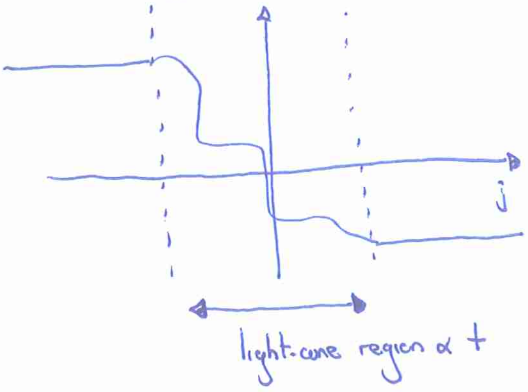
$$\begin{cases} \rho_0(k) = \lim\limits_{th}\frac{1}{2\pi}\,\mathrm{tr}\left[\hat{\rho}(0)\,\hat{n}_k\right] \\ \rho_s(k) = \lim\limits_{th}\frac{1}{2\pi}\,\mathrm{tr}\left[\hat{\rho}^s\,\hat{n}_k\right] \end{cases}$$

Complete set $\Rightarrow$ $\rho_s(k) = \rho_0(k)$

What about the non-translational invariant case? (Our case)

$① \not\Rightarrow ①^*$, only the integrated density $\lim\limits_{th}\frac{1}{L}\langle Q^{(n)}\rangle_{\hat{\rho}(t)}$ is conserved.

The integrated density, however, gives no information on the light-cone region because $t/L \to 0$ in the thermodynamic limit.



$\Rightarrow$ We need some local information!

look directly at

$$\langle q_j^{(n)}\rangle_{\hat{\rho}(t)}$$

Taking the expectation value of (OCE) we find

$$\partial_t \langle q_j^{(n)} \rangle_{\hat\rho(t)} + \langle J_j^{(n)} \rangle_{\hat\rho(t)} - \langle J_{j-1}^{(n)} \rangle_{\hat\rho(t)} = 0 \qquad \forall n$$

Taking $\lim_{th}$ and $\lim_{sc,\xi}$ we have

$$\lim_{sc,\xi} \left(t \left[\partial_t \langle q_j^{(n)} \rangle_{\hat\rho(t)} + \langle J_j^{(n)} \rangle_{\hat\rho(t)} - \langle J_{j-1}^{(n)} \rangle_{\hat\rho(t)} \right] \right) = -\xi \partial_\xi \langle q_j^{(n)} \rangle_{\hat\rho^{ns}_\xi} + \partial_\xi \langle J_j^{(n)} \rangle_{\hat\rho^{ns}_\xi} = 0 \quad \forall n \quad (CE)$$

Here all e.v. are in the thermodynamic limit.

Assuming a <u>finite speed v</u> ~~o~~for the propagation of information (for spin-chains with finite local Hilbert space this is guaranteed by Lieb-Robinson theorem '72)

we have

$$\lim_{\xi \to \pm\infty} \lim_{sc,\xi} \langle q_j^{(n)} \rangle_{\hat\rho(t)} = \langle q_j^{(n)} \rangle_{\hat\rho_{\beta_{\substack{R\\L}},\mu_{\substack{R\\L}}}} \quad \forall n \quad (B)$$

$\xleftarrow{}$ Thermal State

These equations give boundary conditions for (CE)

Plugging (C) and (J) (Page 12) into (CE) we have

$$\int dk \; q_n(k) \left\{ -\xi \partial_\xi \rho_\xi(k) + \partial_\xi (v(k) \rho_\xi(k)) \right\} = 0 \quad \underset{\substack{\{q_n(k)\} \\ \text{complete set}}}{\Rightarrow} \quad \boxed{-\xi \partial_\xi \rho_\xi(k) + \partial_\xi (v(k) \rho_\xi(k)) = 0} \quad (CR)$$

Plugging (C) into (B)

$$\int dk \; q_n(k) \left\{ \rho_{\pm\infty}(k) - \rho_{R/L}(k) \right\} = 0 \quad \Rightarrow \quad \boxed{\rho_{\pm\infty}(k) = \rho_{R/L}(k)} \quad (BR)$$

where

$$\rho_{R/L}(k) = \lim_{th} \frac{1}{2\pi} \mathrm{tr}\left[\hat n_k \hat\rho_{\beta_{R/L},\mu_{R/L}} \right] = \frac{1}{2\pi} f_{R/L}(k)$$

(CR) with the boudary conditions (BR) gives

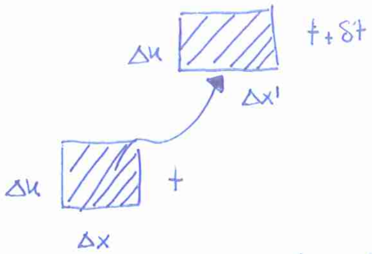
$$\rho_\xi(k) = \rho_L(k) \Theta(v(k) - \xi) + \rho_R(k) \Theta(\xi - v(k))$$

We found the previous result!

- Physical Interpretation

Consider the kinetic theory of free classical particles moving with velocity $v(k)$ and let $\rho^{cl}_{x,t}(\underset{\substack{\uparrow \\ \text{momentum}}}{k})$ be their distribution function.

$\Rightarrow \quad \rho^{cl}_{x,t}(k) \Delta x \Delta k =$ # of particles with momentum in $[k, k+\Delta k]$ and position in $[x, x+\Delta x]$ at time t.



$$\rho^{cl}_{x,t}(u)\,\Delta u\,\Delta x = \rho^{cl}_{x+W(u)\delta t,\,t+\delta t}(u)\,\Delta u'\Delta x'$$

$\Delta u = \Delta u'$ the momentum does not change (no external force)

$$\Delta x = x_1 - x_0$$

$$\Delta x' = x_1 + \cancel{W(u)}\delta t - x_0 - \cancel{W(u)}\delta t = \Delta x$$

$$\Rightarrow \quad \rho^{cl}_{x+V(u)\delta t,\,t+\delta t}(u) - \rho^{ch}_{x,t}(u) = 0 \quad \Rightarrow \quad \left(W(u)\,\partial_x \rho^{cl}_{x,t}(u) + \partial_t \rho^{cl}_{x,t}(u)\right) = 0$$

Same as (E) if we set:

$$\rho^{cl}_{x,t}(u) \longrightarrow \rho_{x,t}(u) \quad \text{root density}$$

$$W(u) \longrightarrow V(u) \quad \text{velocity of excitations}$$

$$(\;)_{cl} \longrightarrow \langle \dots \rangle_{\rho_\xi}$$

Change of variables to $\xi = x/t$

$$\Rrightarrow \quad -\xi\,\partial_\xi \rho_\xi(u) + V(u)\,\partial_\xi \rho_\xi(u) = 0$$

Note that if we imagine each particle to carry an amount $q_n(x)$ of charge $Q^{(n)}$

$$(q^{(n)})_{x,t} = \int_{-\pi}^{\pi} du\; q_n(u)\,\rho_{x,t}(u)$$

Classical average

$(j^{(n)})_{x,t} =$ net flux of $Q^{(n)}$ through the point x at time t

$=$ Charge passing from left to right in the time interval $[t, t+\Delta t]/\Delta t$

$-$ Charge passing from right to left in the time interval $[t, t+\Delta t]/\Delta t$

$$= \int_{-\pi}^{\pi} du\; q_n(u)\,V(u)\,\rho_{x,t}(u)$$

Our quantum problem in the scaling limit becomes equivalent to that of a gas of non-interacting classical "excitations" with distribution function $\rho_{x,t}(u)$

This treatment has been named Generalised Hydrodynamics because it deals with the infinite number of continuity equations of the conserved charges. What we did above can be immediatelly generalised to all non-interacting quantum many-body systems. Namely systems where the Hamiltonian can be brought to the following form

$$H = \sum_p \varepsilon_p \hat{n}_p \qquad [\hat{n}_p, \hat{n}_k] = 0$$

- Fermionic $\hat{n}_p^2 = \hat{n}_p$ (In this case the current formula (J) has been rigorously proven (Fagotti, '16))
- Bosonic $\hat{n}_p = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \infty \end{pmatrix}$ In this case it works but to the best of my knowledge there is no proof of (J).

What about interacting integrable models?

Consider Bethe-Ansatz-integrable models

One dimensional systems with nontrivial interactions, but the interactions can be factorised into a sequence of two-body processes.

In these models one can

1- Construct all eigenstates of the Hamiltonian
2- Define root densities in analogy with the free case
3- Characterise exactly the excitations over a generic macrostate

$\Rightarrow$ We can generalise the above treatment

I will now briefly explain how to perform 1.-3. focussing on the example of the XXZ spin-$\frac{1}{2}$ chain

$$H = J \sum_{j=-L/2+1}^{L/2} S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \qquad [S_i^\alpha, S_j^\beta] = \delta_{ij} \varepsilon^{\alpha\beta\gamma} S_j^\gamma$$

$$(S_j^\alpha)^2 = 1/4$$

1. Eigenstates

Algebraic Bethe Ansatz (ABA) or co-ordinane Bethe Ansatz (BA)

XXZ:

$$|\Psi_{k_1\dots k_N}\rangle = \sum_{1\le i_1<\dots<i_N\le L} \phi_{k_1\dots k_N}(i_1,\dots,i_N)\, S^-_{i_1}\cdots S^-_{i_N}\,|\uparrow\dots\uparrow\rangle \qquad S^\pm_j = \frac{S^x_j \pm i S^y_j}{2}$$

[Since the number of particles is consered we diagonalise H in sectos of fixed N.]

$\uparrow$ N parameters ~ moments ~ free systems

(Careful though: they can be complex)

$\uparrow$ Particles = Spin flips

Schrödinger's Equation

$$H|\Psi_{k_1\dots k_N}\rangle = E_{k_1\dots k_N}|\Psi_{k_1\dots k_N}\rangle \xRightarrow{\text{project on } \langle\uparrow\dots\uparrow| S^+_{i_1}\cdots S^+_{i_N}} 0 = \frac{J}{2}\sum_{j=1}^{N}\left(1-\delta_{i_j+1,\,i_{j+1}}\right)\Big\{\phi_{k_1\dots k_N}(i_1,\dots,i_j,i_{j+1}-1,i_{j+2},\dots,i_N)$$

$$+\,\phi_{k_1\dots k_N}(i_1,\dots,i_{j-1},i_j+1,i_{j+1},\dots,i_N)\Big\}$$

$$+\left\{\frac{J\Delta L}{4} - \Delta J N - E_{k_1\dots k_N} + J\Delta\sum_{j=1}^{N}\delta_{i_j+1,\,i_{j+1}}\right\}\phi_{k_1\dots k_N}(i_1\dots i_N) \qquad (*)$$

Ansatz (Bethe '31)

$$\phi_{k_1\dots k_N}(i_1\dots i_N) = \sum_{P\in S_N} A(P)\, e^{i\sum_{j=1}^{N} k_{P_j} i_j}$$

(Far from each other the particles are plane waves independent from each other)

$\uparrow$ set of all permutations of N elements

This form solves (*) for

$$A(P) = \prod_{\substack{j<\lambda\\ P_j>P_\lambda}}\left[-\frac{\left(e^{i(k_{P_j}+k_{P_\lambda})}+1-2\Delta e^{ik_{P_j}}\right)}{\left(e^{i(k_{P_j}+k_{P_\lambda})}+1-2\Delta e^{ik_{P_\lambda}}\right)}\right]$$

$$\equiv e^{-i\tilde{\theta}(k_{P_j},k_{P_\lambda})} \quad \text{"scattering phase"}$$

[Exercise 5

Check it for N=2,3]

Simple Interpretation:

Every time two moments are swap we get a scattering phase (in the free fermionic case the phase was -1)

$$\phi_{k_1\dots k_N}(i_1\dots i_N) = \sum_{P\in S_N}\prod_{\substack{j<\lambda\\ P_j>P_\lambda}} e^{-i\tilde{\theta}(k_{P_j},k_{P_\lambda})}\prod_{j=1}^{N} e^{ik_{P_j}i_j} \quad \text{eigenstate wave fuction}$$

$$E_{k_1\dots k_N} = \sum_{j=1}^{N} J\cos(k_j) - J\Delta N + \frac{J\Delta L}{4}$$

In general $\Rightarrow$

$$Q^{(n)}_{k_1\dots k_m} = \sum_{j=1}^{N} q_n(k_j)$$

q_n: Function characterising the charge. Analogous to what we saw for the XX model $\mathcal{Q}^{(n)}$

$Q^{(n)}$: eigenvalue of a generic higher conservation law

$$P_{k_1\dots k_N} = \Big(\sum_{j=1}^{N} k_j\Big) \bmod 2\pi$$

$\uparrow$ momentum eigenvalue

Imposing periodic boundary conditions

$$\phi_{k_1 \dots k_N}(x_1 \dots x_{N-1}, L+1) = \phi_{k_1 \dots k_N}(1, x_1, \dots, x_{N-1})$$

gives

$$e^{i k_j L} = \prod_{l \neq j} e^{-i \tilde{\theta}(k_j, k_l)}$$

"Bethe Equations" :

Quantization conditions generalization of $e^{i k_j L} = 1$ which we had for free fermions

The solutions to these equatios are generally complex numbers !!

difference with the free case!

Complex solutions are interpreted as bound states (see the following)

How do these solutions look like?

To study the Bethe equations it is useful to reparametrise the momenta in terms of new variables called "rapidities" $\{\lambda_j\}_{j=1 \dots N}$

In XXZ the reparametrisation depends on Δ, more precisely on whether $|\Delta| > 1$ or $|\Delta| < 1$.

We consider the 2nd case as an example: $\Delta = \cos\gamma$

$$k_j = f(\lambda_j) \quad \text{where} \quad f(\lambda) = -i \log\left[\frac{\sinh(\lambda + i\gamma/2)}{\sinh(\lambda - i\gamma/2)}\right] = 2 \arctan\left(\frac{\tanh(\lambda)}{\tan(\gamma/2)}\right)$$

The parametrization is convenient because

$$e^{-i \tilde{\theta}(f(\lambda_i), f(\lambda_j))} = \underset{\substack{\uparrow \\ \text{depends only on the difference!}}}{e^{-i \theta(\lambda_i - \lambda_j)}} \quad \text{with} \quad \theta(x) = -2 \arctan\left(\frac{\tanh(x)}{\tan(\gamma)}\right)$$

$\Rightarrow$ The Bethe equations become

$$\left(\frac{\sinh(\lambda_j + i\gamma/2)}{\sinh(\lambda_j - i\gamma/2)}\right)^L = -\prod_{k=1}^{N} \left[\frac{\sinh[\lambda_j - \lambda_k + i\gamma]}{\sinh[\lambda_j - \lambda_k - i\gamma]}\right] \qquad (B)$$

To understand the structure of the solutions is useful to take fixed N and $L \to +\infty$

In this case the l.h.s. of (B) either explodes or goes to 0 for $\mathrm{Im}(\lambda_j) \neq 0$

$\Rightarrow$ in the r.h.s we must have a 0 or a pole

Exercise 9

Verify that for $N=2$ and $L\gg 1$, assuming $\mathrm{Im}(\lambda_1)>0$ we have

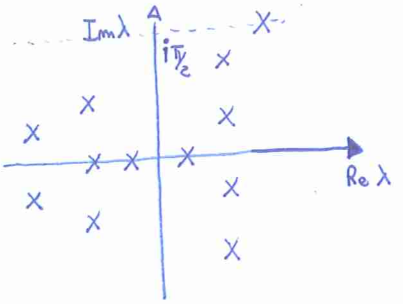
$\lambda_1 - \lambda_2 = i\gamma$

$\mathrm{Im}(\lambda_1 + i\gamma/_2) = \mathrm{Im}(-\lambda_2 + i\gamma/_2)$

$$\Rightarrow \begin{cases} \lambda_1 = x + i\gamma/_2 + \delta^2_{x,1} \\ \lambda_2 = x - i\gamma/_2 + \delta^2_{x,2} \end{cases} \qquad x\in\mathbb{R}, \qquad \delta^2_{x,1,2} \sim e^{-\alpha_{1,2}L}$$ deviations from the perfect string

"2-string": bound state of two spin flips

This generalises for larger N: in general the solutions to (B) can be thought of as the composition of strings of different length (~ bound states of many different spin flips)



This is a generic feature of Bethe-Ansatz integrable models. In general the structure of allowed strings depen on the model and on the values of the parameters.

Let us consider a simple case: XXZ model for $\Delta = \cos(\pi/_\ell)$

In this case we have

$$\lambda_j \longrightarrow \lambda^k_{\alpha,a} = \underset{\substack{\uparrow \\ \text{string centre}}}{\lambda^k_\alpha} + i\gamma/_2(n_k + 1 - 2a) + i\frac{\pi}{4}(1-u_k) + \delta^k_{\alpha,a} \qquad \begin{cases} k = 1,\dots,\ell \\ n_k = \begin{cases} k & k=1,\dots,\ell-1 \\ 1 & k=\ell \end{cases} \\ u_k = \begin{cases} 1 & k=1,\dots,\ell-1 \\ -1 & k=\ell \end{cases} \end{cases} \qquad (S)$$

$\alpha = 1,\dots,M_k$ ← [number of string centres at fixed string k, $N=\sum_k n_k M_k$]

Obs.

The hypothesis according to which all solutions of (B) are compositions of strings is known as "string hypothesis" [Bethe '31 Takahashi '71]. For some states there are non negligible corrections but it is assumed to hold at finite energy density.

Substituting (S) into (B), taking the product of all equations involving the same string centres, and taking $-i\log[\dots]$ we find

$$L\,\theta_j(\lambda^j_\alpha) = 2\pi I^{(j)}_\alpha + \sum_{k=1}^{\ell}\sum_{\substack{\beta=1 \\ (\beta,k)\neq(\alpha,j)}}^{M_k} \Theta_{jk}(\lambda^j_\alpha - \lambda^k_\beta)$$ "Bethe-Takahashi" equations (BT)

$I^{(j)}_\alpha$ ↑ Integers or half-odd integers (depending on M_j) $\qquad |I^j_\alpha| < \frac{1}{2\pi}\left|L\,\theta_j(\infty) - \sum_k M_k \Theta_{jk}(\infty)\right|$

Here we defined

$$\theta_j(x) = \begin{cases} f_j(x) & j = 1, \dots, \nu-1 \\ \tilde{f}_1(x) & j = \nu \end{cases}$$

$$f_j(x) \equiv 2\arctan\left[\frac{\tan[x]}{\tan[\gamma/2 \cdot j]}\right]$$

$$\tilde{f}_j(x) \equiv -2\arctan\left[\tan[x]\tanh[\gamma/2 \cdot j]\right]$$

$$\Theta_{ij}(x) = \begin{cases} \tilde{f}_{j-1}(x) + \tilde{f}_{j+1}(x) & i = \nu \wedge j \neq \nu \\ \tilde{f}_{i-1}(x) + \tilde{f}_{i+1}(x) & i \neq \nu \wedge j = \nu \\ (1-\delta_{ij}) f_{|i-j|}(x) + 2 f_{|i-j|+2}(x) + \dots + 2 f_{i+j-2}(x) + f_{i+j}(x) & \text{otherwise} \end{cases}$$

One defines a state by giving a set of $\{I_\alpha^{(j)}\} \overset{(BT)}{\Rightarrow} \{\lambda_\alpha^{(j)}\}$

1-1 correspondence between the two sets

(BT) give the quantisation conditions for the string-centre rapidities of different strings $\Rightarrow$ analogous to $k_\alpha = \frac{2\pi}{L} I_\alpha$ in the free fermionic chain

these are really analogous to the moments of free particles (they are real)

2. Root Densities

Consider a solution $\{\lambda_\beta^k\}$ of (BT) corresponding to the set $\{I_\beta^k\}$. We can establish a correspondence between the elements of the two sets by introducing

$$Z_j(\lambda | \{\lambda_\beta^k\}) = \theta_j(\lambda) - \frac{1}{L} \sum_{(\beta,k)}' \Theta_{jk}(\lambda_\alpha^j - \lambda_\beta^k)$$

"Counting Functions"

$\uparrow$ solution of BT

$Z_j(\lambda | \{\lambda_\beta^k\})$ monotonic in λ

$$\begin{cases} Z_j'(\lambda | \{\lambda_\alpha^k\}) > 0 & j = 1, \dots, \nu-1 \\ Z_\nu'(\lambda | \{\lambda_\alpha^k\}) < 0 \end{cases}$$

for XXZ at $\Delta = \cos(\pi/\nu)$

By definition we have

$$Z_j(\lambda^j_\alpha | \{\lambda^k_\beta\}) = \frac{2\pi}{L} I^j_\alpha \qquad \alpha = 1, \dots, M_j$$

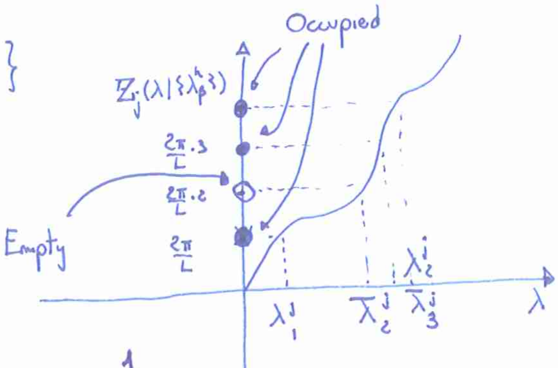
(This is just a rewriting of the BT equations)

but there are also $\bar{\lambda}^j_\alpha$ such that

$$Z_j(\bar{\lambda}^j_\alpha | \{\lambda^k_\beta\}) = \frac{2\pi}{L} \bar{I}^j_\alpha \qquad \bar{I}^j_\alpha \notin \{ I^j_1, \dots, I^j_{M_j} \}$$

$\Rightarrow$ $\bar{\lambda}^j_\alpha$ hole

λ^j_α particle



In the T.L rapidities of particles and holes become dense

$\lambda^j_{\alpha+1} - \lambda_\alpha \sim O(L^{-1})$

$\bar{\lambda}^j_{\alpha+1} - \bar{\lambda}^j_\alpha \sim O(L^{-1})$

define densities $\Rightarrow$

$$\rho_j(\lambda^j_\alpha) = \lim_{L\to+\infty} \frac{1}{L|\lambda^j_{\alpha+1} - \lambda^j_\alpha|}$$

$$\rho^h_j(\lambda^j_\alpha) = \lim_{L\to+\infty} \frac{1}{L|\bar{\lambda}^j_{\alpha+1} - \bar{\lambda}^j_\alpha|}$$

like those of free systems!

These densities characterise the thermodynamic limit of all expectation values of local operators as in free systems

$\Rightarrow$ characterise a macrostate

For large finite L

$L\,\rho_j(\lambda)\,\Delta\lambda \cong$ # particles in $[\lambda, \lambda+\Delta\lambda]$

$L\,\rho^h_j(\lambda)\,\Delta\lambda \cong$ # holes in $[\lambda, \lambda+\Delta\lambda]$

$\downarrow$

$$\rho_j(\lambda) + \rho^h_j(\lambda) = \lim_{\Delta\lambda\to0} \left(\frac{Z_j(\lambda+\Delta\lambda | \{\lambda^k_\beta\}) - Z_j(\lambda|\{\lambda^k_\beta\})}{\Delta\lambda\, 2\pi} \right) \sigma_j$$

$$= \frac{\sigma_j}{2\pi} Z'_j(\lambda|\{\lambda^k_\beta\})$$

for XXZ $\Delta = \cos(\pi/\nu)$

$$\sigma_j = \begin{cases} 1 & j = 1, \dots, \nu-1 \\ -1 & j = \nu \end{cases}$$

Noting that

$$\lim_{th} Z_j(\lambda|\{\lambda^k_\beta\}) = \theta_j(\lambda) - \sum_{k=1}^{\nu} \int d\mu\; \Theta_{jk}(\lambda-\mu)\,\rho_k(\mu)$$

$$\parallel$$

$$Z_j(\lambda|\{\rho_k\})$$

$$\Rightarrow \quad \rho_j(\lambda) + \rho^h_j(\lambda) = \underbrace{\frac{\sigma_j}{2\pi}\theta'_j(\lambda)}_{a_j(\lambda)} - \sum_{k=1}^{\nu} \int d\mu\; \underbrace{\frac{1}{2\pi}\Theta'_{jk}(\lambda-\mu)}_{T_{jk}(\lambda-\mu)}\,\rho_k(\mu)$$

Thermodynamic "Bethe-Takahashi" equations

- Fix $\rho^h_j(\lambda)$ in terms of $\rho_j(\lambda)$
- Analogous to $\rho(k) + \rho^h(k) = \frac{1}{2\pi}$ found for free fermions

3. Conserved - Charge Densities

As for all local operators, the expectation value of conserved-charge densities become functionals of $\{\rho_k\}$ in the thermodynamic limit

To find the explicit expression we recall (*) of page

$$\Rightarrow \frac{1}{L}\langle\{\lambda_j\}|\,Q^{(n)}\,|\{\lambda_j\}\rangle = \langle\{\lambda_j\}|\,q_j^{(n)}\,|\{\lambda_j\}\rangle = \frac{1}{L}\sum_{j=1}^{N} q_n(\lambda_j) = \frac{1}{L}\sum_k\sum_{\alpha=1}^{M_k}\overbrace{\sum_{a\in\text{String}(k,\alpha)} q_n(\lambda_\alpha^k + i\lambda_a)}^{q_{n,k}(\lambda_\alpha)}$$

$$\doteq \frac{1}{L}\sum_k\sum_{\alpha=1}^{M_k} q_{n,k}(\lambda_\alpha^k)$$

For XXZ $\Delta = \cos(\pi/\nu)$ we have

$$q_{n,k}(x) = \sum_{a=1}^{n_k} q_n\left(x + i\tfrac{\gamma}{2}(n_k+1-2a) + i\tfrac{\pi}{4}(1-v_k)\right)$$

"single particle eigenvalue" for the k-th string

In particular

$$\varepsilon_k(x) = -\pi\sin\gamma\; a_k(x)$$

$$p_k(x) = \theta_k(x)$$

single-particle (quasi) momentum

single-particle energy

$$\langle\{\rho_k\}|\,q_j^{(n)}\,|\{\rho_k\}\rangle$$

$$\lim_{\text{th}} \langle\{\lambda_j^\alpha\}|\,q_j^{(n)}\,|\{\lambda_j^\alpha\}\rangle = \sum_k \int d\mu\; q_{n,k}(\mu)\,\rho_k(\mu)$$

Like in the free cases, but with many species of particles!

4. Currents

Since the expectation values of charge densities keep the same form that they have in the free case we make the following ansatz for the currents

$$\langle\{\rho_k\}|\,J_j^{(n)}\,|\{\rho_k\}\rangle = \sum_k \int d\mu\; q_{n,k}(\mu)\,\rho_k(\mu)\,v_k(\mu) \qquad (J)$$

Velocity of the excitation of species k and rapidity λ on the state $|\{\rho_k\}\rangle$

Questions

1. Can we calculate $v_k(\mu)$ in this case?
2. Does (J) hold?

1. Yes, we can!

Change in energy when making the excitation

By definition $$v_k(\lambda) = \frac{\partial\,\Delta E_k(\lambda)}{\partial\,\Delta p_k(\lambda)}$$

Change in momentum when making the excitation

Consider the integers $\{I^j_\alpha\}$ specifying the state

In order not to change M_j we make a particle-hole excitations : replace the integer I^j_a with I^j_b

what happens to $\{\lambda^j_\alpha\}$?

$\lambda^j_\alpha \longmapsto \tilde{\lambda}^j_\alpha \quad \alpha \neq a \qquad \tilde{\lambda}^j_\alpha - \lambda^j_\alpha \sim O(L^{-1})$

λ^j_a is replaced by λ^j_b

So

$$E_{\{\lambda\}} \longrightarrow E_{\{\lambda\},ex} = \sum_{k=1} \sum_{\beta} \varepsilon_k(\tilde{\lambda}^k_\beta) + \varepsilon_j(\lambda^j_b) - \varepsilon_j(\tilde{\lambda}^j_a)$$

$$= \sum_{k} \sum_{\beta} \varepsilon_k(\lambda^k_\beta) + \sum_{k,\beta} \varepsilon'_k(\lambda^k_\beta)\,(\lambda^k_{\beta+1} - \lambda^k_\beta)\, \frac{(\tilde{\lambda}^k_\beta - \lambda^k_\beta)}{(\lambda^k_{\beta+1} - \lambda^k_\beta)} + \varepsilon_j(\lambda^j_b) - \varepsilon_j(\tilde{\lambda}^j_a)$$

$\Rightarrow$ in the thermodynamic limit we have

$$\underbrace{\lim_{th} E_{\{\lambda\},ex} - E_{\{\lambda\}}}_{\Delta E_j(\lambda^j_b;\lambda^j_a)} = \sum_{k} \int d\mu \; \varepsilon'_k(\mu)\, \tilde{F}_{kj}(\mu|\lambda^j_b;\lambda^j_a) + \varepsilon_j(\lambda^j_b) - \varepsilon_j(\lambda^j_a) \qquad (*)$$

where we introduced

$$\tilde{F}_{kj}(\lambda^k_\beta|\lambda^j_b;\lambda^j_a) = \lim_{L\to+\infty} \frac{\tilde{\lambda}^k_\beta - \lambda^k_\beta}{\lambda^k_{\beta+1} - \lambda^k_\beta} \qquad \text{"Shift Function"}$$

Analogously

$$\underbrace{\lim_{L\to+\infty} P_{\{\lambda\},ex} - P_{\{\lambda\}}}_{\Delta P_j(\lambda^j_b;\lambda^j_a)} = \sum_{k} \int d\mu \; p'_k(\mu)\, \tilde{F}_{kj}(\mu|\lambda^j_b;\lambda^j_a) + p_j(\lambda^j_b) - p_j(\lambda^j_a) \qquad (**)$$

An equation for the shift function is found by considering

$$0 = Z_i(\tilde{\lambda}^i_\alpha|\{\lambda^k_\alpha\},ex) - Z_i(\lambda^i_\alpha|\{\lambda^k_\alpha\}) \qquad \begin{matrix} i \neq j \\ \alpha \neq a,b \end{matrix}$$

Writing the difference explictly we have

$$0 = \theta_i(\tilde{\lambda}^i_\alpha) - \theta_i(\lambda^i_\alpha) - \frac{1}{L}\sum_{(k,\beta)}' \left\{ \Theta_{ik}(\tilde{\lambda}^i_\alpha - \tilde{\lambda}^k_\beta) - \Theta_{ik}(\lambda^i_\alpha - \lambda^k_\beta) \right\}$$

$$- \frac{1}{L}\left\{ \Theta_{ij}(\tilde{\lambda}^i_\alpha - \lambda^j_b) - \Theta_{ij}(\tilde{\lambda}^i_\alpha - \lambda^j_a) \right\}$$

$$= \left[\theta'_i(\lambda^i_\alpha) - \frac{1}{L}\sum_{(k,\beta)}' \Theta'_{ik}(\lambda^i_\alpha - \lambda^k_\beta) \right](\tilde{\lambda}^i_\alpha - \lambda^i_\alpha) + \frac{1}{L}\sum_{(k,\beta)}' \Theta'_{ik}(\lambda^i_\alpha - \lambda^k_\beta) \frac{(\tilde{\lambda}^k_\beta - \lambda^k_\beta)(\lambda^k_{\beta+1} - \lambda^k_\beta)}{(\lambda^k_{\beta+1} - \lambda^k_\beta)}$$

$$- \frac{1}{L}\left\{ \Theta_{ij}(\lambda^i_\alpha - \lambda^j_b) - \Theta_{ij}(\lambda^i_\alpha - \lambda^j_a) \right\} + O(\tfrac{1}{L^2})$$

divide by $(\lambda^i_{\alpha+1} - \lambda^i_\alpha)$ and take the limit

$$0 = \tilde{F}_{ij}(\lambda^i_\alpha | \lambda^j_b ; \lambda^j)\, 2\pi \sigma_i \left(\rho_i(\lambda^i_\alpha) + \rho^h_i(\lambda^i_\alpha) \right) + \rho_i(\lambda^i_\alpha) \sum_k' \int d\mu \; 2\pi\, T_{ik}(\lambda^i_\alpha - \mu)\, \tilde{F}_{kj}(\mu | \lambda^j_b ; \lambda^j_a)$$

$$- \rho_i(\lambda^i_\alpha)\left\{ \Theta_{ij}(\lambda^i_\alpha - \lambda^j_b) - \Theta_{ij}(\lambda^i_\alpha - \lambda^j_a) \right\}$$

Since this integral equation is linear for $\tilde{F}_{ij}(\lambda^i ; \lambda^j_b, \lambda^j_a)$

$$\tilde{F}_{ij}(\lambda | \mu_1 ; \mu_2) = \frac{\sigma_i \rho_i(\lambda)}{\rho_i(\lambda) + \rho^h_i(\lambda)} \Big(F_{ij}(\lambda|\mu_1) - F_{ij}(\lambda|\mu_2) \Big)$$

where

$$F_{ij}(\lambda|\mu) = \frac{1}{2\pi}\Theta_{ij}(\lambda - \mu) - \sum_k' \int d\mu' \; T_{ik}(\lambda - \mu')\, F_{kj}(\mu'|\mu) \qquad (***)$$

Going back to (*) and (**) we find

$$\Delta E_j(\lambda) = \varepsilon^d_j(\lambda^j_b) - \varepsilon^d_j(\lambda^j_a) \qquad \text{where} \qquad f^d_j(\lambda) = f_j(\lambda) - \sum_k' \int d\mu \; f'_k(\mu)\, \sigma_k\, \theta_k(\mu)\, F_{kj}(\mu|\lambda)$$

$$\Delta P_j(\lambda) = p^d_j(\lambda^j_b) - p^d_j(\lambda^j_a) \qquad \text{and} \qquad \theta_k(\mu) \equiv \frac{\rho_k(\mu)}{\rho_k(\mu) + \rho^h_k(\mu)}$$

This is as if elementary excitations (j,λ) would carry energy $\varepsilon^d_j(\lambda)$ and momentum $p^d_j(\lambda)$ if particles and $-\varepsilon^d_j(\lambda)$ and $-p^d_j(\lambda)$ if holes. This is exactly what happens in free systems but these functions non-trivially depend on the state $\Rightarrow$ they are "dressed" by interactions with the other particles.

$\Rightarrow$ we then have $V_k(\lambda) = \dfrac{\varepsilon_j^{d'}(\lambda)}{p_j^{d'}(\lambda)}$

Exercise 10

By inverting (***) we can write a closed integral equation for $\varepsilon_j^{d'}(\lambda)$ and $p_j^{d'}(\lambda)$.

Hint

1. Use Compact notation

$[\vec{f}]_{j\lambda} = f_j(\lambda)$ $\quad [\hat{A}\vec{f}]_{j\lambda} = \sum_k' \int d\mu \, A_{jk}(\lambda;\mu) f_k(\mu)$ $\quad \vec{f}\cdot\vec{g} = \sum_k' \int d\lambda \, f_k(\lambda) g_k(\lambda)$

$$\begin{cases} [\hat{1}\vec{f}]_{j\lambda} = f_j(\lambda) \\ [\hat{\sigma}\vec{f}]_{j\lambda} = \sigma_j f_j(\lambda) \\ [\hat{\vartheta}\vec{f}]_{j\lambda} = \vartheta_j(\lambda) f_j(\lambda) \\ [\hat{T}\vec{f}]_{j\lambda} = \sum_k' \int d\mu \, T_{jk}(\lambda-\mu) f_k(\mu) \end{cases}$$

2. Assume $(\hat{1} + \hat{T}\hat{\vartheta}\hat{\sigma})$, $\hat{\vartheta}$, $\hat{\sigma}$ invertible

3. Note that defining the transpose as $\hat{A}^t$

$[\hat{A}^t \vec{f}]_{j\lambda} = \sum_k' \int d\mu \, A_{kj}(\mu;\lambda) f_k(\mu)$

we have

$\hat{\textcircled{H}}^t = -\hat{\textcircled{H}}$

$\hat{T}^t = \hat{T}$

$$\Rightarrow \begin{cases} \varepsilon_j^{d'}(\lambda) = \varepsilon_j(\lambda) - \sum_k' \int d\mu \, T_{jk}(\lambda-\mu)\, \sigma_k \vartheta_k(\mu)\, \varepsilon_k^{d'}(\mu) \\ p_j^{d'}(\lambda) = 2\pi\sigma_j \left(\rho_j(\lambda) + \rho_j^h(\lambda)\right) \end{cases}$$

So we have

$$(\rho_j(\lambda) + \rho_j^h(\lambda))\, V_j(\lambda) = \frac{1}{2\pi}\sigma_j \varepsilon_j'(\lambda) - \sigma_j \sum_k' \int d\mu \, T_{jk}(\lambda-\mu)\, \rho_k(\mu) V_k(\mu) \qquad (V)$$

$\Delta \to 0 \quad V_j(\lambda) \to \begin{cases} \varepsilon_1'(\lambda) \\ -\varepsilon_2'(-\lambda) \end{cases} \quad p_j(\lambda) \to \begin{cases} 2\arctan(\tanh(\lambda)) \\ -2\arctan(\tanh(\lambda)) \end{cases}$ different parametrisation w.r.t. that used for free systems.

Main message of (V): The excitations modify their velocity by interacting among each other

We answered 1. so we can move to 2. ㊱

- (J) has been proven for integrable relativistic quantum field theories with diagonal scattering
  [Castro-Alvaredo, Doyon, Yoshimura '16
  Vu, Yoshimura '18
- Proven for classical systems of hard rods
  → Boldrighini, Dobrushin, Sukhov '82
- Tested numerically with DMRG in XXZ chain
- Passed Consistency checks in XXZ
  ] → B., Collura, De Nardis, Fagotti '16

$$\sum_j J_j^{(2)} \sim Q^{(3)} \quad \Rightarrow \quad \langle \{\rho_k\} | J_j^{(2)} | \{\rho_k\} \rangle = \langle \{\rho_k\} | q_j^{(3)} | \{\rho_k\} \rangle$$

⇒ Assume (J) and use it to study transport!

Remember the logic of GHD:

1. Assume $\quad \langle O_j \rangle_{\hat\rho(t)} \xrightarrow[j/t = \xi]{j,t \to +\infty} \langle O_j \rangle_{\hat\rho^s(\xi)}$

2. Consider the continuity equation for <u>all</u> local (and quasi local) conserved-charge densities

$$0 = +\partial_t \langle q_j^{(n)} \rangle_{\hat\rho(t)} + \partial_x \langle J_j^{(n)} \rangle_{\hat\rho(t)} \xrightarrow[j/t = \xi]{t,j \to +\infty} -\xi \partial_\xi \langle q_i^{(n)} \rangle_{\hat\rho^s_\xi} + \partial_\xi \langle J_j^{(n)} \rangle_{\hat\rho^s_\xi} = 0 \quad \forall n$$

3. Represent $\hat\rho^s(\xi)$ using a set of root densities $\{ \rho_{k,\xi}(\lambda) \}$ and use (J) and (C)

rapidity

species ray

$$\Rightarrow \sum_k \int d\lambda \; q_{n,k}(\lambda) \left\{ -\xi \partial_\xi \rho_{k,\xi}(\lambda) + \partial_\xi (V_{k,\xi}(\lambda) \rho_{k,\xi}(\lambda)) \right\} = 0 \quad \forall n$$

depends on ξ through the state: main difference w.r.t. the free case

This is assumed to be a complete set

(For XXZ this is <u>proven</u>: Ilievski, Quinn, De Nardis, Brockmann '15)

$$\Rightarrow \boxed{ -\xi \partial_\xi \rho_{k,\xi}(\lambda) + \partial_\xi (V_{k,\xi}(\lambda) \rho_{k,\xi}(\lambda)) = 0 }$$

Castro-Alvaredo, Doyon, Yoshimura '16
B., Collura, De Nardis, Fagotti '16 (CE)

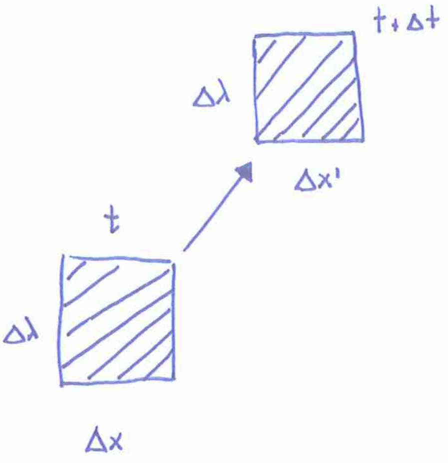
Boundary Conditions (Same as for the free case)

$$\lim_{\xi \to \pm\infty} \rho_{k,\xi}(\lambda) = \rho_{k, R/L}(\lambda)$$

The analogy with a kinetic theory of free classical particles stands!

Difference w.r.t. the free case

- Multiple species of particles → n
- Position dependent velocity → $W_{n,x,t}(\lambda)$



The particles are free: (n,λ) conserved

$$\rho^{cl}_{n,x,t}(\lambda)\,\Delta\lambda\Delta x = \rho^{cl}_{n,x+W_{n,x,t}(\lambda)\Delta t,\,t+\Delta t}(\lambda)\,\Delta\lambda\Delta x'$$

$$\Delta x = x_1 - x_0$$

$$\Delta x' = x_1 + W_{n,x_1,t}(\lambda)\Delta t - x_0 - W_{n,x_0,t}(\lambda)\Delta t$$

$$\simeq \Delta x\left(1 + \partial_x W_{n,x_1,t}(\lambda)\,\Delta t\right)$$

$$\Rightarrow \quad \rho^{cl}_{n,x,t}(\lambda) = \left\{1 + \partial_x W_{n,x,t}(\lambda)\cdot\Delta t\right\}\left\{\rho^{cl}_{n,x,t}(\lambda) + \partial_t \rho^{cl}_{n,x,t}(\lambda)\cdot\Delta t + \partial_x \rho^{cl}_{n,x,t}(\lambda)\cdot W_{n,x,t}(\lambda)\,\Delta t\right\}$$

$$\Rightarrow \quad \partial_t \rho^{cl}_{n,x,t}(\lambda) + \partial_x\left(W_{n,x,t}(\lambda)\,\rho_{n,x,t}(\lambda)\right) = 0$$

Equivalent to (CE) if we set $\rho^{cl} \to \rho$

$w \to V$

(CE) is not easy to solve anymore, but it can be solved implicitly by writing it in terms of

$$\vartheta_{n,\xi}(\lambda) = \frac{\rho_{n,\xi}(\lambda)}{\rho_{n,\xi}(\lambda) + \rho^h_{n,\xi}(\lambda)}\,.$$

Exercise 11

Prove that (CE) $\Leftrightarrow$ $-\xi\partial_t \vartheta_{n,\xi}(\lambda) + V_{n,\xi}(\lambda)\,\partial_\xi \vartheta_{n,\xi}(\lambda) = 0$ (TCE)

Solving (TCE) we have

$$\vartheta_{n,\xi}(\lambda) = \vartheta_{n,L}(\lambda)\,\theta\left(V_{n,\xi}(\lambda) - \xi\right) + \vartheta_{n,R}(\lambda)\,\theta\left(\xi - V_{n,\xi}(\lambda)\right) \qquad (a)$$

but $V_{n,\xi}(\lambda)$ depends on $\vartheta_{n,\xi}(\lambda)$. Must proceed by iteration

1. Guess some $V_{n,\xi}(\lambda)$ (e.g. $\varepsilon'_n(\lambda)$)

2. Find $\vartheta_{n,\xi}(\lambda)$ using (a)

3. Plug it in

$$(\rho_{n,\xi}(\lambda)+\rho^h_{n,\xi}(\lambda))\sigma_n = a_n(\lambda) - \sum_n \int d\mu \; T_{nn}(\lambda-\mu)\,(\rho_{n,\xi}(\mu)+\rho^h_{n,\xi}(\mu))\,\vartheta_{n,\xi}(\mu)$$

$$(\rho_{n,\xi}(\lambda)+\rho^h_{n,\xi}(\lambda))\,V_{n,\xi}(\lambda) = a'_n(\lambda) - \sum_n \int d\mu \; T_{nn}(\lambda-\mu)\,(\rho_{n,\xi}(\mu)+\rho^h_{n,\xi}(\mu))\,\vartheta_{n,\xi}(\mu)\,V_{n,\xi}(\mu)$$

to find $V_{n,\xi}(\lambda)$

The convergence of this algorithm is typically fast.

Remarks

- This method allows one to find the scaling limit values of all local observables of which the explicit expression in terms of $\{\rho_n\}$ is known.
- It can also be applied when the two subchains suddenly joined are not in stationary states

  e.g. $\hat{\rho}_0 = |\Psi_0\rangle\langle\Psi_0|$ with $|\Psi_0\rangle = |\uparrow\downarrow\uparrow\cdots\uparrow\rangle \otimes \left(\frac{|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle}{\sqrt{2}}\right)\cdots\left(\frac{|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle}{\sqrt{2}}\right)$

Outlook

- Prove the assumption 1. at page ㊱ (i.e. that in the scaling limit all local observables are described by $\hat{\rho}^s_\xi$)

  ⇒ Very hard!
- Prove (J) at page ㉜ for a generic TBA-solvable model
- (CE) has been conjectured to apply more broadly than just in the scaling limit

  [See Bulchandani, Vasseur, Karrasch, Moore '17
  Doyon, Dubail, Konik, Yoshimura '17
  Lectures by Jérôme]

One imagines that $\mathrm{tr}[\hat{\rho}(t)\,O_j] \approx \mathrm{tr}[\hat{\rho}^s(j,t)\,O_j]$ for large j and t

$\hat{\rho}^s(j,t) \Leftrightarrow \{\rho_{n,j,t}(\lambda)\}$ ← root densities

$$\Rightarrow \quad \partial_t\,\rho_{n,x,t}(\lambda) + \partial_x\left(V_{n,x,t}(\lambda)\,\rho_{n,x,t}(\lambda)\right) = O\left(\frac{1}{t^\alpha}\cdot\frac{1}{j^\beta}\right)$$

→ How to find these?

De Nardis, Doyon, Bernard '18 have an interesting proposal.... Still in progress!