# Three lectures on classical and quantum hydrodynamics applied to trapped 1d quantum gases

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My goal is to give an introduction to large-scale approaches to the dynamics and correlations of inhomogeneous quantum systems in 1d. While I focus almost exclusively on the 1d Bose gas with delta repulsion (or Lieb-Liniger gas) in a trapping potential, the basic concepts and methods are more general and can be adapted to other systems, for instance to quantum spin chains.

I am assuming that the reader already possesses a good intuition of such concepts as relaxation or local equilibrium in systems of many particles (these concepts and more will probably be discussed by Bruno Bertini in his Lectures). On the technical side, I am assuming some familiarity with (free) quantum field theory, lagrangian formalism, path integrals, Wick's theorem, etc.; the appendix contains an introduction to those techniques, which are used in Lectures 2 and 3.

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# Lecture 1

# Euler-scale classical hydrodynamics applied to a 1d quantum gas

In this first lecture, I introduce some of the basic concepts that underly "hydrodynamic approaches", namely large-scale descriptions of the collective motion of systems with many particles.

# 1.1 A first look at hydrodynamics

Let me start with a simple model of a classical lattice gas: the ASEP (for "Asymmetric Simple Exclusion Process").

The model is defined as follows: particles live at discrete positions  $x \in a\mathbb{Z}$ where a > 0 is the lattice spacing. They jump stochastically from one site to its nearest neighbors, with a hard core constraint (i.e. the number of particles  $n_x$  at each site is 0 or 1). The rules are as follows. From time t to t + dt, a particle at x jumps to x + a with probability  $\frac{1}{a}dt$  if the site x + a is empty, and to x - a with probability  $\frac{q}{a}dt$  (q > 0) if the site x - a is empty. The jump is forbidden if the target site is already occupied.

The rules are represented in Fig. 1.1. Notice that, crucially, for  $q \neq 1$  the transition rates are *not* symmetric under  $x \to -x$ .



Figure 1.1: Allowed jumps in the ASEP, and their transition rates, here for lattice spacing a = 1. [Picture from A. Lazarescu's "Physicist's companion to current fluctuations".]

#### 1.1.1 The continuity equation

Imagine that we start from a distribution of particles with a density profile  $\rho(x) = \langle n_x \rangle$  at time t = 0. We would like to understand how to describe the evolution of that density. With the above rules, one easily arrives at the equation

$$\frac{d}{dt}\langle n_x\rangle + \frac{j(x) - j(x-a)}{a} = 0, \qquad (1.1)$$

with a *current* given by  $j(x) = \langle n_{x-a}(1-n_x) - q(1-n_{x-a})n_x \rangle$ .

Equation (1.1) is a *continuity equation*: it expresses conservation of the total number of particles  $\sum_{x \in a\mathbb{Z}} n_x$ . Indeed, summing over x in (1.1) and assuming that the current vanishes at infinity, one gets  $\frac{d}{dt} \langle \sum_x n_x \rangle = 0$ .

Notice that to evaluate the current j(x) one needs to know the state of the system  $\langle . \rangle$  at time t. So at this point we are still facing an extremely difficult problem: we would in principle need to solve the entire dynamics of the full microscopic system just to be able to calculate the current j(x). The continuity equation alone is not very useful. To make use of it, we need a simple way of evaluating the current j(x) (if not exactly, at least approximately).

#### 1.1.2 Separation of scales

To simplify the description of the system, one relies on the key assumption of separation of length scales. We want to view the density profile  $\rho(x) = \langle n_x \rangle$  as a smooth function of x, such that there are two different length scales in the problem:

- a microscopic scale, of order of the interparticle distance  $d \sim a/\rho$
- a macroscopic scale, which is the typical length on which the density varies, which is roughly of order  $L \sim \rho/\partial_x \rho$ .



Figure 1.2: Separation of scales. The microscopic length scale d (of the order of the interparticle distance) and the macroscopic length scale  $L \sim \rho/\partial_x \rho$ (the typical length on which the density varies) are such that there exist mesoscopic fluid cells of size  $\ell$ , with  $d \ll \ell \ll L$ . The state inside each mesoscopic fluid cell may then be represented by a very small number of thermodynamic quantities. This is a tremendous simplication, compared to a full solution of the original microscopic model. [Picture taken from this viewpoint on the two papers of Bertini, Collura, De Nardis, Fagotti and Castro-Alvaredo, Doyon, Yoshimura which introduced the hydrodynamic approach to quantum integrable models.]

We say that *separation of scales* holds if the former is much smaller than the latter; when this is the case, there is an intermediate *mesoscopic* scale  $\ell$  such that

$$d \ll \ell_{\rm meso} \ll \rho / \partial_x \rho.$$
 (1.2a)

Then, in a mesoscopic fluid cell  $[x, x + \ell_{\text{meso}}]$ , the system looks both homogeneous (because the density does not vary) and thermodynamically large (because it contains a large number of particles).

The existence of that intermediate mesoscopic scale is extremely useful and we will use it repeatedly throughout these notes. It allows to trade the complicated inhomogeneous microscopic system by an assembly of fluid cells  $[x, x + \ell_{\text{meso}}]$ , each of which viewed as a large translation-invariant system (for instance with periodic boundary conditions).

In addition, we rely on the assumption of separation of time scales or

*local stationarity*: we assume that there is a microscopic relaxation time  $\tau_{\text{relax}}$  on which the mesoscopic fluid cell reaches its stationary state, and that the density varies slowly on this time-scale. Then there is a mesoscopic time scale such that

$$\tau_{\rm relax} \ll t_{\rm meso} \ll \rho/\partial_t \rho.$$
 (1.2b)

Consequently, the expectation values  $\langle O(x) \rangle_t$  in the exact state at time t are identical to the ones in the stationary state of the translation-invariant system at the same density  $\rho = \rho(x, t)$ :  $\langle O \rangle_{\text{stat.},\rho}$ . This is a tremendous simplification of the problem.

Notice that, with the notations above, a simple way of satisfying separation of scales in the ASEP is to work with some fixed smooth function for the initial density profile  $\rho(x)$ , and then send  $a \to 0$ . Then separation of length-scales is obvious, and separation of time-scales (local stationarity) is expected to hold because the jumping rates scale as 1/a so for any fixed  $t_{\text{meso}}$  there will be infinitely many jumps as  $a \to 0$  and the local fluid cell will indeed have plenty of time to go to the stationary state.

#### 1.1.3 The equation of state

Assuming separation of scales, the expectation value that needs to be calculated is now the one in the stationary state at density  $\rho$ . It turns out that this is an elementary exercise for the ASEP.

For a chain of M sites with periodic boundary conditions, the stationary states are easily obtained: they are product states  $[(1 - \rho) |0\rangle + \rho |1\rangle]^{\otimes M}$ parametrized by the density  $\rho$ . Using the formula below (1.1), we then obtain the current as a function of the density

$$j = F(\rho), \tag{1.3}$$

with  $F(\rho) = (1 - q)\rho(1 - \rho)$ . This is the equation of state for the ASEP.

#### 1.1.4 Summary

In that basic example of the ASEP, we have encountered two key ideas. The first one is that, because the total number of particles is conserved, the evolution of the particle density is governed by a *continuity equation*. The second is that, assuming separation of length- and time-scales, one can "chop" the system into *locally stationary mesoscopic fluid cells*, and this allows to derive an *equation of state*, namely a relation between the current j and the density  $\rho$ .

We thus end up with a simple hydrodynamic equation which governs the evolution of the density  $\rho(x)$ , by combining the continuity equation with the equation of state,

$$\partial_t \rho + \partial_x F(\rho) = 0. \tag{1.4}$$

That description is coarse-grained: we have obviously lost a huge amount of information from the microscopic model. Nevertheless it becomes an exact description at large scales (i.e. in the limit  $a \rightarrow 0$  with our notations).

These equations are the hydrodynamic equations at the Euler scale for the ASEP. The name Euler scale refers to the largest length- and time-scale, at which there is no dissipation. It corresponds precisely to the regime of separation of scales that we described in paragraph 1.1.2, with the conclusion that the current j is a function of the density  $\rho$  only (as opposed to a function of  $\rho$  and its derivatives  $\partial_x \rho, \partial_x^2 \rho, \ldots$  which would lead to dissipative terms that would be beyond the Euler scale).

At the Euler scale, two different microscopic models look exactly the same if and only if they have the same equation of state.<sup>1</sup> When they do, they can differ only by microscopic fluctuations.

# 1.2 Hydrodynamics of galilean invariant systems

The ASEP has a single conserved quantity: the number of particles. That is why its hydrodynamic description involves only the particle density  $\rho(x)$ . In these notes I will focus on 1d gases with more conserved quantities such as the momentum or the energy; let me briefly sketch the Euler-scale hydrodynamics of these systems.

# 1.2.1 Systems with conservation of mass, momentum and energy

In a continuous translation-invariant system, the momentum is a conserved quantity with a local density. If the energy is also conserved, then there are three continuity equations expressing conservation of mass, momentum and energy:

$$\partial_t \rho + \partial_x j = 0$$
  

$$\partial_t \rho_P + \partial_x j_P = 0$$
  

$$\partial_t \rho_E + \partial_x j_E = 0.$$
(1.5)

<sup>&</sup>lt;sup>1</sup>This is true as long as no shock appears in the solution of the equations (1.4). If a shock develops (i.e. a singularity of  $\rho(x)$ ), then one needs additional input from the microscopic model in order to properly regularize the shock.

In order to complete this system of equations, one needs to express the three currents  $j, j_P, j_E$  as functions of the three densities  $\rho, \rho_P, \rho_E$ . In other words, one needs to find the equation(s) of state for that system.

The equation of state is of course model-dependent, since it depends on the microscopic details of the model. Finding it is, in general, a difficult task, and in many models this can only be achieved numerically.

However, there are a few general constraints that hold for all systems invariant under galilean boosts and parity symmetry, which are quite useful. I briefly review those now.

#### **1.2.2** Consequence of galilean invariance

In a galilean invariant system, the three densities transform as follows under a galilean boost w of the velocities of all particles,  $v \to v + w$ ,

$$\begin{array}{lll}
\rho(x,t) &\to & \rho(x-wt,t) \\
\rho_P(x,t) &\to & \rho(x-wt,t)w + \rho_P(x-wt,t) \\
\rho_E(x,t) &\to & \rho(x-wt,t)\frac{w^2}{2} + \rho_P(x-wt,t)w + \rho_E(x-wt,t).
\end{array}$$
(1.6)

[This can be viewed as a definition of galilean invariance. To find these equations, take the example of a system with particles with kinetic energy  $\sum_i mv_i^2/2$  and interaction energy  $V(\{x_i - x_j\}_{i < j})$ , then it is clear that only the kinetic term is affected by the boost and that it leads to (1.6).]

In order to be compatible with the continuity equation, the currents must transform as

$$j \rightarrow \rho w + j$$

$$j_P \rightarrow w^2 \rho + w \rho_P + w j + j_P$$

$$j_E \rightarrow \frac{w^3}{2} \rho + w^2 \rho_P + w \rho_E + \frac{w^2}{2} j + w j_P + j_E.$$
(1.7)

If we assume that our system is also parity symmetric, then the densities and currents should transform under parity transformation  $x \to -x$  as

$$\begin{array}{ll}
\rho \rightarrow \rho & j \rightarrow -j \\
\rho_P \rightarrow -\rho_P & j_P \rightarrow j_P \\
\rho_E \rightarrow \rho_E & j_E \rightarrow -j_E.
\end{array}$$
(1.8)

As a result of these symmetries, the expression of the three currents in terms of the three densities simplifies. First, the particle current is simply equal to the momentum density,

$$j = \rho_P. \tag{1.9a}$$

The argument is as follows. The equilibrium state with densities  $(\rho, 0, \rho_E)$  is parity invariant, therefore  $j = 0 = \rho_P$  in that macrostate. But j and  $\rho_P$  transform in the same way under galilean boosts, so  $j = \rho_P$  also in arbitrary macrostates  $(\rho, \rho_P, \rho_E)$ .

Second, let us define the pressure  $\mathcal{P}(\rho, \rho_E)$  as the momentum current  $j_P$ in the equilibrium state with densities  $(\rho, 0, \rho_E)$ . Since other macrostates with  $\rho_P \neq 0$  can always be obtained from that one by a galilean boost with  $w = \rho_P / \rho$ , looking at the transformation law for  $j_P$ , one gets

$$j_P = \frac{\rho_P^2}{\rho} + \mathcal{P}(\rho, \rho_E - \frac{\rho_P^2}{2\rho}).$$
 (1.9b)

Third, by the same kind of argument, the energy current is

$$j_E = \frac{\rho_P}{\rho} \left( \rho_E + \mathcal{P}(\rho, \rho_E - \frac{\rho_P^2}{2\rho}) \right).$$
(1.9c)

In conclusion, as a consequence of galilean invariance, the three currents are expressed in terms of a single unknown function of the densities: the equilibrium pressure  $\mathcal{P}(\rho, \rho_E)$ . It is then that function itself which is referred to as "the equation of state".

## 1.2.3 Standard form of the Euler equations for galilean invariant fluids

Injecting the expression for the three currents in terms of the pressure (1.9) in the continuity equations (1.5), one arrives at the following equations for the mean velocity  $u \equiv \rho_P / \rho$  and the internal energy per unit mass  $e \equiv \left(\rho_E - \frac{\rho_P^2}{2\rho}\right) / \rho$ , which will probably look familiar to most readers:

$$\partial_t \rho + \partial_x (\rho u) = 0$$
  

$$\partial_t u + u \partial_x u + \frac{1}{\rho} \partial_x \mathcal{P} = 0$$
  

$$\partial_t e + u \partial_x e + \frac{\mathcal{P}}{\rho} \partial_x u = 0.$$
  
(1.10)

These are usually named "the Euler equations" and they are the most common hydrodynamic equations in the literature.

## 1.2.4 Fluids with zero entropy: reduction to two equations. Trapping potential.

As we have seen, the third equation in (1.10) comes from conservation of energy. Equivalently, it can also be recast to express the fact that entropy moves with the flow, namely

$$\partial_t s + u \partial_x s = 0, \tag{1.11}$$

where s is the entropy per particle (see below for a derivation). This is important for these lecture notes because we will mostly focus on quantum gases at zero temperature which have zero entropy. In that case, this equation is always trivially satisfied. Consequently, we can drop the third equation in (1.5) or (1.10). It is important to stress that dropping the third equation in (1.5)-(1.10) does not mean that energy is not conserved; on the contrary, it means that it is automatically conserved, as a consequence of the other two conservation laws, for the particular case of zero-entropy fluids.

Finally, we will need one last piece of information about Euler equations for galilean invariant fluids: we want to be able to include a force created by an external potential V(x). The first two equations in (1.10) become

$$\begin{array}{rcl} \partial_t \rho + \partial_x (\rho u) &=& 0\\ \partial_t u + u \partial_x u + \frac{1}{\rho} \partial_x \mathcal{P} &=& -\frac{1}{m} \partial_x V. \end{array}$$
(1.12)

Notice that momentum is no longer conserved in the presence of the force  $-\partial_x V$ . Obviously no one should be surprised by that: this is just Newton's second law.

[To see that (1.11) is equivalent to the third equation in (1.10), one way is to use the thermodynamic relation between the internal energy per unit mass and s,  $de = \frac{T}{m}ds - \mathcal{P}d(1/\rho)$ . Then

$$\frac{T}{m}(\partial_t + u\partial_x)s = (\partial_t + u\partial_x)e - \frac{\mathcal{P}}{\rho^2}(\partial_t + u\partial_x)\rho$$
$$= (\partial_t + u\partial_x)e + \frac{\mathcal{P}}{\rho}\partial_x u,$$

where I used the first equation in (1.10) to go from the first to the second line.]

# 1.3 Classical hydrodynamics of the quantum 1d Bose gas at zero temperature

Now let me turn to the model on which I will focus for the rest of these Lectures: the gas of bosons with delta repulsion, defined by the Hamiltonian

$$H_V = \int dx \left( \Psi^{\dagger} \left( -\frac{\hbar^2 \partial_x^2}{2m} + V(x) \right) \Psi + \frac{g}{2} \Psi^{\dagger 2} \Psi^2 \right).$$
(1.13)

 $\Psi^{\dagger}(x), \Psi(x)$  are the creation/annihilation operators of the bosons, which satisfy the canonical commutation relation  $[\Psi(x), \Psi^{\dagger}(x')] = \delta(x - x')$ . V(x)is an external trapping potential, and g > 0 is the repulsion strength. This model is also known as the *Lieb-Liniger model*. When V(x) = 0, it is solvable by Bethe Ansatz.

#### **1.3.1** Reduction to standard classical hydrodynamics

When V(x) = 0, the Lieb-Liniger hamiltonian  $H_0$  is integrable and has infinitely many conserved charges. We will come back to that in Lecture 3. For now, let us imagine that only the total mass, momentum and energy are conserved. Those correspond to the following charge densities (here I put hats to stress that those densities are now second-quantized operators acting on the Hilbert space)

$$\hat{\rho}(x) = m\hat{\Psi}^{\dagger}(x)\hat{\Psi}(x)$$

$$\hat{\rho}_{P}(x) = -i\hbar\hat{\Psi}^{\dagger}(x)\partial_{x}\hat{\Psi}(x)$$

$$\hat{\rho}_{E}(x) = \frac{\hbar^{2}}{2m}(\partial_{x}\hat{\Psi}^{\dagger}(x))(\partial_{x}\hat{\Psi}(x)) + \frac{g}{2}\hat{\Psi}^{\dagger 2}(x)\hat{\Psi}^{2}(x).$$
(1.14)

Now, even though the system is *quantum*, one can follow the same logic as the one of Sec. 1.1. Namely, since these are conserved quantities, one can in principle construct the corresponding currents, which must be local operators  $\hat{j}(x)$ ,  $\hat{j}_{P}(x)$ ,  $\hat{j}_{E}(x)$  defined such that

$$\hat{j}(x) - \hat{j}(x') = i \left[\hat{H}, \int_{x'}^{x} \hat{\rho}(y) dy\right],$$

and the same relation for  $\hat{j}_P$  and  $\hat{j}_E$ . This ensures that the operators  $\hat{\rho}$  and  $\hat{j}$  satisfy the continuity equation

$$\partial_t \hat{\rho} + \partial_x \hat{j} = 0 \tag{1.15}$$

(same for  $\hat{\rho}_P$ ,  $\hat{j}_P$  and for  $\hat{\rho}_E$ ,  $\hat{j}_E$ ). Clearly, the continuity equations also hold for the expectation values,

$$\partial_t \left< \hat{\rho} \right> + \partial_x \left< \hat{j} \right> = 0 \tag{1.16}$$

(same for  $\langle \hat{\rho}_P \rangle$  and  $\langle \hat{\rho}_E \rangle$ ). Assuming that such current operators  $\hat{j}, \hat{j}_P, \hat{j}_E$  have been constructed, one can again rely on separation of scales, replace the local state of the system inside a mesoscopic fluid cell by a macrostate  $\langle \rangle_{\rho,\rho_P,\rho_E}$  which has the correct expectation values for the densities, i.e.  $\langle \hat{\rho} \rangle_{\rho,\rho_P,\rho_E} = \rho, \langle \hat{\rho}_P \rangle_{\rho,\rho_P,\rho_E} = \rho_P, \langle \hat{\rho}_E \rangle_{\rho,\rho_P,\rho_E} = \rho_E$  (such a macrostate will, in general, be given by a density matrix or by some typical eigenstate of H), and then use the local macrostate to calculate the expectation values of the currents.

The application of that program leads, in principle, to relations of the form

$$\left\langle \hat{j} \right\rangle = F\left(\left\langle \rho \right\rangle, \left\langle \rho_P \right\rangle, \left\langle \rho_E \right\rangle\right)$$
 (1.17)

and similar relation for  $\langle \hat{j}_P \rangle$  and  $\langle \hat{j}_E \rangle$ .

So we're in business, again: we have reached a coarse-grained description, with a huge reduction of the number of degrees of freedom, from a tremendously complicated many-body quantum system to only three continuous real-valued functions: the expectation values  $\rho(x) = \langle \hat{\rho}(x) \rangle$ ,  $\rho_P = \langle \hat{\rho}_P(x) \rangle$ ,  $\rho_E = \langle \hat{\rho}_E(x) \rangle$ . The entire difference between classical models like the ASEP and quantum models like the 1d Bose gas lies in the choice of the local macrostates which one uses to calculate the expectation values of the currents. But, provided this step has been performed, the outcome is exactly the same: a set of Euler-scale hydrodynamic equations which determine the evolution of the charge densities.

Finally, since the Lieb-Liniger model is galilean invariant, and also invariant under parity  $x \to -x$ , the discussion of section 1.2 applies. At zero temperature, the Euler hydrodynamics of the Lieb-Liniger gas in a trap is given by the system of two equations (1.12). The only thing we need is the ground state pressure as a function of mass density,  $\mathcal{P}(\rho)$ , or in other words, the equation of state at zero temperature.

#### 1.3.2 The Equation of Love

The equation of state of the Lieb-Liniger model at zero temperature is obtained from the Bethe Ansatz solution of the model (for V(x) = 0). Here I would like to avoid a discussion of the Bethe Ansatz, so I will just state the result, in order to give a flavour of the kind of equations one is dealing with. To those readers who are eager to learn about the Bethe Ansatz solution of the Lieb-Liniger model, I would recommend Chapter 1 of the book of Korepin, Bogoliubov and Izergin. I will also come back to the Bethe Ansatz in Lecture 3.

The equation of state at zero temperature is given by the following relations. First, the *Equation of Love* (or *Lieb Equation*),

$$\rho_{\rm p}(\theta) = \frac{m}{2\pi\hbar} + \int_{-\theta_{\rm F}}^{\theta_{\rm F}} d\theta' \, \frac{2g/\hbar}{(g/\hbar)^2 + (\theta - \theta')^2} \, \rho_{\rm p}(\theta'), \qquad (1.18a)$$

relates the *density of rapidities*  $\rho_{\rm p}(\theta)$  in the ground state to the "Fermi rapidity"  $\theta_{\rm F}$ . The Fermi rapidity itself is fixed in terms of the mass density  $\rho$  in the ground state, such that

$$\rho = \int_{-\theta_{\rm F}}^{\theta_{\rm F}} d\theta \,\rho_{\rm p}(\theta) \,m. \tag{1.18b}$$

The energy density is then given in terms of  $\rho_{\rm p}(\theta)$  and  $\theta_{\rm F}$  by

$$\rho_E = \int_{-\theta_F}^{\theta_F} d\theta \,\rho_p(\theta) \,\frac{m\theta^2}{2}.$$
(1.18c)

Eqs. (1.18a-b-c) relate the internal energy per unit mass  $e \equiv \rho_E/\rho$  to the mass density  $\rho$  at zero temperature. Then, to get the pressure, one can use the thermodynamic relation  $de = \frac{T}{m}ds - \mathcal{P}d(1/\rho)$  with T = s = 0. This gives

$$\mathcal{P}(\rho) = -\left(\frac{\partial e}{\partial(1/\rho)}\right)_{T=0} = -\rho_E + \rho \left(\frac{\partial \rho_E}{\partial \rho}\right)_{T=0}.$$
(1.19)

This is all we need to do Euler-scale hydrodynamics of the Lieb-Liniger gas at zero temperature.<sup>2</sup>

<sup>&</sup>lt;sup>2</sup>To be slightly more precise, this is all we need before the formation of shocks, i.e. discontinuities of  $\rho(x)$ . When shocks develop, one needs generalized hydrodynamics (see Sec. ?? below) to describe what happens after the shock.

## THE ELECTROSTATIC FIELD OF TWO EQUAL CIRCULAR CO-AXIAL CONDUCTING DISKS

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[Received 31 May 1949]

#### SUMMARY

In the earliest discussion of this problem Nicholson (1) expressed the potential as a series of spheroidal harmonics with coefficients satisfying an infinite system of linear equations, and gave a formula for an explicit solution; but this formula appears to be meaningless and its derivation to contain serious errors.

In the present paper, starting tentatively from Nicholson's infinite system of

THEOREM 1. In the two leading cases described above, the potential at any point  $(\rho, \zeta, \zeta')$ , specified as in Figs. 1 and 2 by its distance  $r = \rho a$  from the axis of the disks and its axial distances  $z = \zeta a$  and  $z' = \zeta' a$  from their planes, is

$$\frac{V_0}{\pi} \int_{-1}^{1} \left\{ \frac{1}{\sqrt{\{\rho^2 + (\zeta + it)^2\}}} \pm \frac{1}{\sqrt{\{\rho^2 + (\zeta' + it)^2\}}} \right\} f(t) \, dt, \tag{1}$$

where each square root has positive real part, and f(t) is the solution of the integral equation

$$f(x) \pm \frac{1}{\pi} \int_{-1}^{1} \frac{\kappa}{\kappa^2 + (x-t)^2} f(t) dt = 1 \quad (-1 \le x \le 1),$$

$$\kappa \text{ is the spacing parameter (Fig. 1).}$$

$$(2)$$

Figure 1.3: The paper of Love, with his Equation. In the Bethe Ansatz literature, this equation is sometimes referred to as the *Lieb equation*. Conveniently enough, "Liebe" means "Love" in German, so the nomenclature is consistent.

#### **1.3.3** An application: 1d expansion from harmonic trap

where

Let me turn to a first application of the Euler-scale hydrodynamic of the Lieb-Liniger gas. Consider an atomic cloud initially prepared at zero temperature in a harmonic potential  $V(x) = \frac{1}{2}m\omega^2 x^2$ . At t = 0, the potential V(x) is suddenly switched off, and the cloud expands. How does it expand? This kind of question is efficiently answered by hydrodynamics.

The initial state. It is described by hydrostatics: we look for a solution to the Euler equations (1.12) with  $\partial_t \rho = 0$  and u = 0:  $\frac{1}{\rho} \partial_x \mathcal{P} = -\frac{1}{m} \partial_x V$ . With the basic thermodynamic relation  $d\mathcal{P} = \rho s dT + \frac{\rho}{m} d\mu$  at T = 0, this implies  $\partial_x (\mu + V) = 0$ . In other words, the density  $\rho(x)$  is obtained by asking that the local chemical potential be  $\mu(x) = \text{const} - V(x)$ . Then, since the density is a function of the chemical potential,  $\rho = \rho(\mu)$ , the hydrostatic density profile is

$$\rho_0(x) = \rho(\text{const.} - V(x)). \tag{1.20}$$

The constant in (1.20) is fixed in order to get the correct total number of particles  $N = \int \rho_0(x) dx$ . This procedure for finding the equilibrium density profile in the trap is widely known as the *Local Density Approximation* (*LDA*). Let me stress that it the same thing as *hydrostatics* at the Euler-scale, and that it relies on the assumption of separation of scales, as all hydrodynamic approaches.

The evolution at t > 0. The expansion of the cloud is then obtained by solving the hydrodynamic equations (1.12). This can be done numerically very efficiently (in a few seconds/minutes on a laptop) for arbitrary values of the parameters  $m, g, \omega$  and for arbitrary large N (the larger the better, because large N gets us closer to the hydrodynamic regime with separation of scales). Notice that this a tremendous simplification of the problem: simulating the trap release directly in the full quantum many-body system (1.13) would simply be impossible for N more than two or three dozens of particles.

There is one point which must be stressed here: for the particular case of an initial *harmonic* potential, no shock appears at finite time. This is what allows to use the Euler equations (1.13) at arbitrary large times. For more general initial potentials, this would not necessarily be correct (I give an example in Sec. ?? below).

In Fig. 1.4, I show some experimental data from M. Schemmer and I. Bouchoule, and a comparison with the numerical solution of the Euler equations.<sup>3</sup>

Density profile at  $t \to \infty$ , and the density of rapidities. There is a very interesting general fact about the mass density profile  $\rho(x,t)$  at large t, which is worth pointing out. Of course, as t increases, the cloud keeps expanding indefinitely, so the density simply goes to zero everywhere. However, when properly rescaled, the density profile shows a remarkable structure. Let us define

$$\rho_{1d,\infty}(\theta) := \lim_{t \to \infty} t \times \frac{\rho(\theta t, t)}{m}.$$
(1.21)

 $<sup>{}^{3}</sup>$ I am cheating a bit, because the temperature is not zero in those data, so the equations that are actually solved to make the plots are Eqs. (1.10) instead of (1.12), but this is just to illustrate the possibilities offered by classical hydrodynamics at the Euler scale.



Figure 1.4: Density profiles of a 1d cloud of <sup>87</sup>Rb atoms released from a harmonic potential ( $N \simeq 4600$  atoms,  $\omega = 2\pi \times 8.8$  Hz,  $g/\hbar = 0.51 \,\mu\text{m/ms}$ ). The experimental data (the noisy lines) are from M. Schemmer and I. Bouchoule, Institut d'Optique Palaiseau. The smooth lines are obtained by integrating the Euler equations (1.12) numerically —more precisely, the Euler equations at finite temperature (1.10), since the temperature is not zero in the experiment—. [Picture from arXiv:1810.07170.]

Then  $\rho_{1d,\infty}(\theta)$  turns out to be related to the density of rapidities in the initial state. Recall that the density of rapidities  $\rho_{\rm p}(\theta)$  for a homogeneous (translation-invariant) gas was defined as the solution of the Equation of Love (Sec. 1.3.2), for some fixed homogeneous mass density  $\rho$ . Now we have instead an inhomogeneous system, with a spatially varying density  $\rho(x)$ . Nevertheless, since we are relying on separation of scales, we can define a distribution of rapidities  $\rho_{\rm p}(x,\theta)$  in each mesoscopic fluid cell [x, x + dx]: it is simply the rapidity distribution which solves the Equation of Love, for the mass density  $\rho(x)$ .

Then the following relation holds:

$$\rho_{1d,\infty}(\theta) = \int dx \,\rho_{\rm p}(x,\theta). \tag{1.22}$$

This result reflects the infinite number of conservation laws in the Lieb-Liniger model. It would be possible to derive it now, in the context of the Euler equations (1.12) for a harmonic trap (and at zero temperature). However this would be tedious. It will become much more natural later; so we will instead derive it in Lecture 3. For now, let me simply emphasize that trap expansions in 1d give access to the distribution of rapidities, a remarkable fact which could lead to direct measurements of those distributions in experiments.

# 1.4 Towards quantum hydrodynamics

So far I talked exclusively about classical hydrodynamics: the continuity equation obeyed by the expectation values of the densities and currents give a classical description of the system. The input from the microscopic quantum model is the equation of state, which relates the expectation values of the currents to the ones of the densities, and that equation might be different for a quantum gas and for a classical gas. For instance, in a classical gas of non-interacting particles, the particles would simply freeze at zero temperature and the pressure would vanish, while this can never be true with quantum particles which always have zero point fluctuations. Nevertheless, apart from the equation of state which may be sensitive to quantum effects, what we have done so far was *classical hydrodynamics*.

In the next two Lectures, I will turn to quantum hydrodynamics in 1d, also known as Tomonaga-Luttinger liquid theory (hence there is obviously some large overlap with the Lectures by Thierry Giamarchi). To conclude this first Lecture, let me briefly explain what is meant by "quantum" hydro-dynamics.

The idea is to take the Euler equations at zero temperature (1.12), and quantize them. Formally, there are various ways of doing that, but in the end they are all more or less equivalent. Since my own goal is to emphasize the relation with the Gaussian Free Field (reviewed in the appendix), the most direct route is through path integrals.

The action. First, one needs to find a variational principle for the Euler equations (1.12), i.e. some action functional  $S[\rho, j]$  whose Euler-Lagrange equation gives back (1.12).

To do that, one has to treat the two equations (1.12) on a different footing. We look at the first one (the continuity equation for the mass density) as a *constraint*, which by definition is always satisfied exactly by

the couple  $(\rho, j)$ . The second one (the Euler equation for u) results from the variational principle, by minimizing  $S[\rho, j]$  under the constraint  $\partial_t \rho + \partial_x j = 0$ . We chose the action as

$$\mathcal{S}[\rho, j] = \int dx dt \left[ \frac{j^2}{2\rho} - \rho_E(\rho) - \rho V \right], \qquad (1.23)$$

where  $\rho_E(\rho)$  is the ground state energy density defined by formulas (1.18.a-b-c).

To vary this action under the constraint  $\partial_t \rho + \partial_x j = 0$ , one can imagine that one has a configuration  $(\rho(x,t), j(x,t))$  which satisfies the continuity equation, and represent fluctuations around that configuration by a "height field" h(x,t),

$$\delta\rho(x,t) = \frac{m}{2\pi}\partial_x h(x,t), \qquad \delta j(x,t) = -\frac{m}{2\pi}\partial_t h(x,t), \qquad (1.24)$$

such that  $(\rho + \delta \rho, j + \delta j)$  automatically satisfies the constraint. Then, varying the action to first order in  $\delta \rho, \delta j$  (substituting (1.24) and integrating by parts), one gets

$$\delta \mathcal{S}^{(1^{\text{st order}})} = \int dx dt \left[ \frac{j \delta j}{\rho} - \frac{j^2}{2\rho^2} \delta \rho - \delta \rho \partial_\rho \rho_E - \delta \rho V \right]$$
$$= \frac{m}{2\pi} \int dx dt \left[ \partial_t \left( \frac{j}{\rho} \right) + \partial_x \left( \frac{j^2}{2\rho^2} \right) + \partial_x (\partial_\rho \rho_E + V) \right] h.$$

The expression inside the bracket must vanish; using  $u = j/\rho$  and  $\partial_x(\partial_\rho\rho_E) = \frac{1}{\rho}\partial_x P$  [this follows from the thermodynamic relations (at T = 0)  $d\rho_E = \mu d\rho$  and  $d\mathcal{P} = \rho d\mu$ ], this gives precisely the second equation in (1.12), as required. So the action (1.23) is indeed a "good" action for the Euler equations (1.12).

Quantum hydrodynamics in 1d. Now that an action has been identified, one can turn classical hydrodynamics into "quantum hydrodynamics" simply by putting the action inside an exponential, and by integrating over all configurations  $(\rho, j)$  that satisfy the constraint. Formally, this would correspond to a path integral of the form

$$\int [d\rho \, dj] \,\delta(\partial_t \rho + \partial_x j) \,\exp\left(\frac{i}{\hbar} \mathcal{S}[\rho, j]\right). \tag{1.25}$$

A more convenient way of writing such a path integral is to use the height field: given some reference configuration  $(\rho, j)$  which satisfies the continuity equation, one can write the path integral as

$$\int [dh] \exp\left(\frac{i}{\hbar} \mathcal{S}\left[\rho + \frac{m}{2\pi} \partial_x h, j - \frac{m}{2\pi} \partial_t h\right]\right).$$
(1.26)

Such path integrals are typically approximated by the saddle-point method, by expanding to second order in h around classical configurations  $(\rho, j)$  — configurations which are solutions to the Euler equations (1.12)—, leading to a gaussian path integral.

Expanding the action to second order gives

$$\delta \mathcal{S}^{(2^{\text{nd order}})} = \frac{1}{2} \int dx dt \left[ \frac{1}{\rho} (\delta j)^2 - 2 \frac{j}{\rho^2} (\delta \rho) (\delta j) + (\frac{j^2}{\rho^3} - \partial_{\rho}^2 \rho_E) (\delta \rho)^2 \right]$$
$$= \frac{\hbar}{8\pi} \int dx dt \left( \begin{array}{c} \partial_t h \\ \partial_x h \end{array} \right) \cdot A \cdot \left( \begin{array}{c} \partial_t h \\ \partial_x h \end{array} \right)$$

for some  $2 \times 2$  symmetric matrix A. This gives a quadratic action for the quantum fluctuations around a classical hydrodynamic solution. Of course, there should also be higher order terms, but those are irrelevant in the renormalization group sense in two spacetime dimensions, so one usually omits them. One concludes that quantum fluctuations of 1d liquids are captured by a quadratic action. This conclusion is known as Luttinger (or Tomonaga-Luttinger) liquid theory.

Luttinger parameters and covariant form of the action. To elaborate, the above symmetric matrix A can be checked to be of the form

$$A = \frac{1}{K} \times \left(\begin{array}{cc} \frac{u^2}{v} - v & -\frac{u}{v} \\ -\frac{u}{v} & \frac{1}{v} \end{array}\right)^{-1}$$

where  $u = j/\rho$  is the local mean velocity of the fluid, and the sound velocity v and the dimensionless Luttinger parameter K are given by<sup>4</sup>

$$v = \sqrt{\rho \,\partial_{\rho}^2 \rho_E}$$
 and  $K = \frac{\pi \hbar \rho}{m^2 v}$ . (1.27)

In terms of these parameters, the above quadratic action for the fluctuations of h becomes

$$\mathcal{S}[h] \equiv \delta \mathcal{S}^{(2^{\text{nd order}})} = \frac{\hbar}{8\pi} \int \frac{\sqrt{-g} \, d^2 \mathbf{x}}{K} g^{ab}(\partial_a h)(\partial_b h)$$
(1.28)

<sup>&</sup>lt;sup>4</sup>The fact that K is related to v here can be traced back to galilean invariance in the action (1.23). This is not a general feature of all Luttinger liquids: in Luttinger liquids that describe the low-energy physics of non-galilean-invariant systems (e.g. critical spin chains), v and K are independent.

where  $\mathbf{x}^0 = t, \mathbf{x}^1 = x$ , and the metric  $g_{ab}$  is

$$ds^{2} = g_{ab}dx^{a}dx^{b} = (dx - (u + v) dt)(dx - (u - v)dt).$$
(1.29)

The action (1.28) is the one of the Gaussian Free Field (GFF), or free massless boson, in the metric g. The physical meaning of g is that it is the natural metric in which sound waves propagate. Indeed, the sound waves propagating on top of the fluid correspond to low-energy excitations of the GFF, which propagate along the null geodesics of g, and those are precisely the right- and left-moving sound waves which propagate at velocity  $u \pm v$ (see exercise B below).

## 1.5 Exercices

#### A. More on Euler-scale hydrodynamics in the ASEP.

1. Consider the ASEP on an infinite line, with the initial density profile  $\rho(x) = \rho_1$  if x < 0 and  $\rho(x) = \rho_2$  if  $x \ge 0$ . Find a function f such that  $\rho(x,t) = f(x/t)$  is a solution to the hydrodynamic equation (1.4) for that initial condition.

[Remark: this is called a "Riemann problem" in standard literature on hydrodynamic equations.]

2. Find a simple change of variable  $(x,t) \mapsto (y(x,t),s(t))$  such that Eq. (1.4) becomes the inviscid Burgers equation  $\partial_s \rho + \rho \partial_y \rho = 0$ . Then show that, for a given initial profile  $\rho_0(y)$  at t = s = 0, the profile at later 'time'  $s, \rho(y,s) = X$ , is a solution of the equation  $X = \rho_0(y - sX)$ , provided such a solution exists.

Use this to construct numerically (for instance with Mathematica) the solution  $\rho(x,t)$  for a gaussian initial profile  $\rho_0(x) = A + Be^{-Cx^2}$ , and observe that typically the gradient  $\partial_x \rho$  increases with time, leading to shock formation.

#### B. Sound waves.

1. Take a constant mass density  $\rho$  and a constant velocity u, and consider an infinitesimal perturbation  $(\rho + \delta\rho(x, t), u + \delta u(x, t))$  which solves the Euler equations (1.12). Express the equation satisfied by  $\delta\rho(x, t)$ and  $\delta u(x, t)$  in the form of a 2 × 2 matrix acting on the vector  $\begin{pmatrix} \delta\rho\\ \delta u \end{pmatrix}$ . 2. Interpret the result as the propagation of two waves with velocities  $u \pm v$  respectively, where v is the sound velocity defined in (1.27).

#### C. Equation of state of the Lieb-Liniger gas at zero temperature.

- 1. By a simple dimensional analysis argument, show that the pressure of the Lieb-Liniger gas at zero temperature is of the form  $\mathcal{P}(\rho) = \frac{\hbar^2 \rho^3}{m^4} f(\gamma)$ , where  $\rho$  is the mass density and  $\gamma = \frac{gm^2}{\hbar^2 \rho}$ , for some unknown function  $f(\gamma)$ .
- 2. The regime  $\gamma \ll 1$  is called the Gross-Pitaevskii regime. The bosons are close to being in a Bose-Einstein condensate, and in the ground state of the gas, their kinetic energy may be neglected. Then, by a simple physical argument, find  $\mathcal{P}$  in that regime.
- 3. The regime  $\gamma \gg 1$  is called the Tonks-Girardeau regime. There, the gas becomes equivalent to a non-interacting Fermi gas (see Lecture 2). Calculate  $\mathcal{P}$  for the one-dimensional Fermi gas at zero temperature and at density  $\rho$ .
- 4. Using the Equation of Love and the other formulas of Sec. 1.3.2, evaluate numerically the pressure at zero temperature. Plot the function  $f(\gamma)$ . Check that it has the expected behavior in the limit  $\gamma \to 0$ , and in the limit  $\gamma \to \infty$ .

# D. Harmonically trapped gas in the Gross-Pitaevskii and Tonks-Girardeau regimes.

Consider an atomic cloud at zero temperature described by the Lieb-Liniger model in a harmonic trap with frequency  $\omega$ , which is either in the Gross-Pitaevskii ( $\gamma \ll 1$ ) or in the Tonks-Girardeau ( $\gamma \gg 1$ ) regime.

1. Using the analytic expression for the pressure  $\mathcal{P}$  found in the previous exercise for the Gross-Pitaevskii and Tonks-Girardeau regimes, show that the Euler equations take the form

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) &= 0\\ \partial_t u + u \partial_x u + A \partial_x (\rho^{\alpha}) &= -\omega^2 x, \end{cases}$$

where  $\alpha = 1$  in the Gross-Pitevskii regime, and  $\alpha = 2$  in the Tonks-Girardeau regime. Indentify the constant A in both regimes.

2. Find the hydrostatic density profile  $\rho_0(x)$  in both regimes.

[Hint: both profiles are very famous in the literature, one is known as "inverted parabola", the other as "Wigner semi-circle"...]

- 3. Now imagine that the frequency of the trap varies with time,  $\omega \rightarrow \omega(t)$ . Consider the "scaling Ansatz"  $\rho(x,t) = \frac{1}{b(t)}\rho_0\left(\frac{x}{b(t)}\right)$ . Find the differential equation satisfied by the scaling factor b(t) such that the scaling Ansatz is a solution of the above Euler equation.
- 4. Consider the case of a small frequency quench:  $\omega(t) = \omega \delta \omega$  for  $t \leq 0$ , and  $\omega(t) = \omega$  for t > 0, with  $|\delta \omega| \ll \omega$ . Such a quench sets the atomic cloud in motion at t > 0: one says that the cloud "breathes". By linearizing the differential equation for the scaling factor (i.e. for  $b(t) = 1 + \epsilon(t)$  with  $|\epsilon(t)| \ll 1$ ), calculate the frequency of the breathing oscillations, in the Gross-Pitaevskii and Tonks-Girardeau regimes.

Can you comment on the result? [Hint: which of the two frequencies is compatible with the motion expected of free particles? Which one is instead compatible with the name "non-linear" (as in "non-linear Schrödinger equation"  $\dots$ )?]

# Lecture 2

# The trapped Tonks-Girardeau gas at zero temperature

In this second Lecture, I focus exclusively on the Tonks-Girardeau gas, i.e. on the hamiltonian (1.13) in the limit of infinite repulsion  $g \to +\infty$ . The main reason for that is a huge technical simplification: in that limit, the model maps to a non-interacting Fermi gas in 1d. One can then get nice, physically meaningful, results with relatively simple calculations. I will come back to the more complicated case of finite repulsion g in Lecture 3.

# 2.1 An experimental motivation: cloud expansions in 3d and momentum distribution

Before we plunge into the formalism, let me briefly sketch one of the motivations for looking at correlation functions of 1d atomic clouds.

**Expansions: 3d vs. 1d.** When they create and manipulate 1d atomic clouds, experimentalists typically like to let their clouds expand before they take a picture of them. They can let the clouds expand in 1d, as I discussed in Sec. 1.3.3, by suddenly switching off the longitudinal confinement (but not the transverse confinement). 1d expansions are typically well described by Euler-scale hydrodynamics in 1d, or more precisely by Generalized HydroDynamics, see Sec. ??.

By instantaneously switching off both the longitudinal and the transverse confinement, experimentalists can also let the clouds expand in 3d. In a 3d expansion, the evolution of the cloud is not described by 1d hydrodynamics. Instead, it is described by a Liouville equation for the 3d distribution of particles,

$$\partial_t W_{3d} + \sum_{a=x,y,z} \frac{p_a}{m} \partial_a W_{3d} = 0.$$
(2.1)

Here  $W_{3d}(x, y, z, p_x, p_y, p_z)$  is the density of atoms at position (x, y, z) and momentum  $(p_x, p_y, p_z)$  at time t. Since initially the cloud is one-dimensional,  $W_{3d}$  at t = 0 is roughly proportional to  $\delta(y)\delta(z)$  (if y and z are the two transverse directions). Because of the uncertainty principle,  $W_{3d}$  is then very widely distributed as a function of  $p_y$  and  $p_z$ . Thus, as the transverse confinement is switched off, the cloud expands very quickly along the transverse directions, and the atoms stop interacting with each other almost immediately after the release. This is the justification for Eq. (2.1), which describes the evolution of a gas of *non-interacting* atoms.

Momentum distribution of the 1d gas. Assuming that one is interested only in the distribution of the cloud along the longitudinal direction x, one needs to focus on the density integrated over the transverse degrees of freedom:

$$W_{3d,\parallel} \equiv \int dy dz dp_y dp_z W_{3d}.$$
 (2.2)

Injecting this into (2.1) gives  $\partial_t W_{3d,\parallel} + \frac{p_x}{m} \partial_x W_{3d,\parallel} = 0$ , with the solution  $W_{3d,\parallel}(x, p_x; t) = W_{3d,\parallel}(x - \frac{p_x}{m}t, p_x; 0)$ . Thus, when the density of atoms along the longitudinal direction x is measured after a very long time of flight, we see that the outcome gives the momentum distribution of the atoms at t = 0,

$$\rho_{3d,\infty}(q) \equiv \lim_{t \to \infty} \left[ \frac{t}{m} \int dp_x W_{3d,\parallel} \left( \frac{qt}{m}, p_x; t \right) \right] \\
= \lim_{t \to \infty} \left[ \frac{t}{m} \int dp_x W_{3d,\parallel} \left( \frac{q-p_x}{m} t, p_x; 0 \right) \right] \\
= \lim_{t \to \infty} \left[ \int dx W_{3d,\parallel} \left( x, q - mx/t; 0 \right) \right] \\
= \int dx W_{3d,\parallel} \left( x, q; 0 \right),$$
(2.3)

where I have used the change of variables  $x = \frac{q-p_x}{m}t$  in the third line. This result is extremely interesting and important because it relates the (longitudinal) density profile observed after a long time of flight in 3d,  $\rho_{3d,\infty}(q)$ , to the momentum distribution of the atoms just before the expansion,  $W_{3d,\parallel}(x,q;0)$ ,

integrated over all positions x in the cloud. The momentum distribution of the atoms is thus an important observable.

**Relation with correlation functions.** On the theory side, to calculate the momentum distribution in a state  $\langle . \rangle$  for 1d bosons (the state might be a pure state or a density matrix), one needs to evaluate the expectation value of the Fourier modes of the bosonic creation/annihilation operators (i.e.  $\Psi^{\dagger}(p) = \int dx \, e^{i \frac{px}{\hbar}} \Psi^{\dagger}(x)$ ),

$$\langle N_q \rangle \equiv \left\langle \Psi^{\dagger}(q)\Psi(q) \right\rangle = \int dx \, \int dy \, e^{i\frac{qy}{\hbar}} \left\langle \Psi^{\dagger}(x+\frac{y}{2})\Psi(x-\frac{y}{2}) \right\rangle. \tag{2.4}$$

One sees that  $W_{3d,\parallel}(x,q)$  can be identified with  $\int dy \, e^{i\frac{qy}{\hbar}} \left\langle \Psi^{\dagger}(x+\frac{y}{2})\Psi(x-\frac{y}{2}) \right\rangle$ . So the basic object that is needed is the *one-particle density matrix* 

$$g_1(x,x') \equiv \left\langle \Psi^{\dagger}(x)\Psi(x') \right\rangle,$$
 (2.5)

which is a two-point correlation function. If one can calculate the oneparticle density matrix, then one can obtain the momentum distribution of the atoms simply by taking its Fourier transform. More generally, one can be interested in higher moments of the momentum distribution,

$$\langle N_{q_1} N_{q_2} \dots N_{q_n} \rangle$$
, (2.6)

which are obtained from the *n*-particle density matrix by Fourier transform,

$$g_n(\{x_i\},\{x'_j\}) \equiv \left\langle \Psi^{\dagger}(x_1)\dots\Psi^{\dagger}(x_n)\Psi(x'_1)\dots\Psi(x'_n) \right\rangle.$$
(2.7)

The higher moments of the momentum distribution are also experimentally accessible, by repeating the experiment a large number of times with the same initial condition, and by analyzing the noise statistics of the resulting set of momentum profiles. See Fig. 2.1 for an experimental measurement of the second-order cumulant  $\langle \delta N_{q_1} \, \delta N_{q_2} \rangle$  in the 1d Bose gas (in the weak repulsion regime).

Small-q vs. large-q behavior of the momentum distribution. In this second Lecture, I will explain how to calculate the one-particle density matrix in the ground state of the Tonks-Girardeau gas. More precisely, I will explain how to obtain the long-distance behavior of g(x, x') from simple calculations in the Gaussian Free Field. Here "long-distance" means  $|x - x'| \gg \rho^{-1}$ . In the language of Sec. 1.1.2, this means distances that are either mesoscopic or macroscopic; the results will be valid for both. But let me



Figure 2.1: Experimental measurement of the momentum correlation  $\langle \delta N_{q_1} \delta N_{q_2} \rangle - \langle \delta N_{q_1} \rangle \langle \delta N_{q_2} \rangle$  in a weakly interacting Bose gas, for three different repulsion strengths g (sets A, B and C). Here the indices  $\alpha$  and  $\beta$  correspond to the momenta  $q_1$  and  $q_2$ . The top row shows the experimental data, the middle row is the theoretical prediction from quantum Monte Carlo, and the bottom row shows the diagonal parts,  $q_1 = q_2$  and  $q_1 = -q_2$ . [Figure from B. Fang et al., PRL 116, 050402 (2016).]

stress that they will not be valid at microscopic distances (i.e. of the order of the interparticle distance).

This does not mean that the microscopic behavior is not interesting; it just means that the approach I will describe, which is a field theory approach, is not designed to capture that behavior.

There is a very interesting fact about the microscopic behavior of the oneparticle density matrix  $g_1(x, x')$ : the third derivative  $\partial_x^3 g_1(x, x')$  is singular at x = x'. Consequently, the Fourier transform decays as  $1/q^4$  at large q,

$$\int dy \, e^{i\frac{qy}{\hbar}} g_1(x+\frac{y}{2},x-\frac{y}{2}) \, \underset{q \to \infty}{\simeq} \, \frac{C(x)}{q^4}.$$

The coefficient C(x) depends on the local stationary state at position x (i.e.

it depends on the local density, the local energy density, etc.), and it is known as *Tan's contact*. One sees from Eq. (2.4) that the large-q behavior of the momentum distribution is then  $\langle N_q \rangle \simeq_{q \to \infty} \frac{1}{q^4} \int dx C(x)$ .

The method I will sketch in this Lecture will teach us nothing about Tan's contact. Instead, it will give us access to the long-range behavior of  $g_1(x, x')$  and therefore to the small-q behavior of the momentum distribution  $\langle N_q \rangle$ .

With that experimental motivation in mind, let me now turn to the gas of hard core bosons (or Tonks-Girardeau gas) and explain why quantum hydrodynamics, as sketched in Sec. 1.4, allows to derive nice exact results about long-range correlations.

# **2.2** Hard core bosons $\Leftrightarrow$ free fermions

#### 2.2.1 Jordan-Wigner

Hard core bosons in 1d are equivalent to free fermions. The mapping from bosons to fermions is done by inserting a Jordan-Wigner string,

$$\Psi_{\rm F}^{\dagger}(x) = e^{i\pi \int_{y < x} \Psi^{\dagger}(y)\Psi(y)dy} \Psi^{\dagger}(x), \qquad (2.8)$$

where  $\Psi^{\dagger}(y)\Psi(y) = \Psi_{\rm F}^{\dagger}(y)\Psi_{\rm F}(y)$  is the density operator, such that the fermion creation/annihilation modes satisfy the canonical anti-commutation relations  $\{\Psi_{\rm F}(x), \Psi_{\rm F}^{\dagger}(x')\} = \delta(x - x')$ . In terms of the fermions, the Hamiltonian (1.13) in the hard core limit is quadratic,

$$(g \to +\infty) \qquad H_V = \int dx \,\Psi_{\rm F}^{\dagger} \left( -\frac{\hbar^2 \partial_x^2}{2m} + V(x) \right) \Psi_{\rm F}.$$
(2.9)

This is a huge simplification: because of that, the many-body problem essentially boils down to a single-particle problem. [This is true of many, but not all, observables. For example, the calculation of the one-particle density matrix does *not* boil down to a trivial single-particle calculation because of the Jordan-Wigner string.]

## 2.2.2 Equation of state of the Tonks-Girardeau gas

As a first application of the mapping to free fermions, let me derive the equation of state of the Tonks-Girardeau gas at zero temperature, i.e. the relation between the energy density  $\rho_E$  and the mass density  $\rho$  in the ground state of the translation-invariant system. The ground state of  $H_0 - \mu \int dx \Psi^{\dagger} \Psi$ (with chemical potential  $\mu$ ) is obtained by filling all the fermion modes with momentum p which have energy  $\frac{p^2}{2m} - \mu < 0$ . Thus, the ground state obeys

$$\Psi_{\rm F}^{\dagger}(p) |\text{g.s}\rangle = 0 \quad \text{if} \quad |p| < p_{\rm F} \equiv \sqrt{2m\mu},$$
  
$$\Psi_{\rm F}(p) |\text{g.s}\rangle = 0 \quad \text{if} \quad |p| > p_{\rm F}.$$
(2.10)

The mass density in that state is

$$\begin{split} \rho &= m \langle \mathrm{g.s.} | \Psi_{\mathrm{F}}^{\dagger}(x) \Psi_{\mathrm{F}}(x) | \mathrm{g.s.} \rangle \\ &= m \int \frac{dp}{2\pi\hbar} \int \frac{dq}{2\pi\hbar} e^{-i\frac{(p-q)x}{\hbar}} \langle \mathrm{g.s.} | \Psi_{\mathrm{F}}^{\dagger}(p) \Psi_{\mathrm{F}}(q) | \mathrm{g.s.} \rangle \\ &= m \int_{-p_{\mathrm{F}}}^{p_{\mathrm{F}}} \frac{dp}{2\pi\hbar} = \frac{mp_{\mathrm{F}}}{\pi\hbar}. \end{split}$$

Similarly, the energy density is  $\rho_E = \int_{-p_F}^{p_F} \frac{dp}{2\pi\hbar} \frac{p^2}{2m} = \frac{p_F^3}{6\pi\hbar m}$ . As a consequence, the equation of state of the Tonks-Girardeau gas at zero temperature reads

$$\rho_E = \frac{\pi^2 \hbar^2 \rho^3}{6m^4}.$$
 (2.11)

Equivalently, the equation of state can be written as  $\mathcal{P} = \frac{\pi^2 \hbar^2 \rho^3}{3m^4}$ , see formula (1.19) in Lecture 1. The sound velocity (see Eq. (1.27)) is equal to the Fermi velocity,

$$v = v_{\rm F},$$
 where  $v_{\rm F} = \frac{\pi \hbar \rho}{m^2} = \frac{p_{\rm F}}{m}.$  (2.12)

This reflects the underlying free fermionic nature of the Tonks-Girardeau gas: it means that low-energy excitations propagate at the group velocity of free particles with Fermi momentum  $p_{\rm F}$ .

## 2.3 Ground state correlations of the trapped gas

Now I consider the ground state of the hamiltonian (2.9) in a trap and with a chemical potential term  $-\mu \int dx \Psi^{\dagger} \Psi$ . I consider that the mass *m*, the potential V(x), and the chemical potential  $\mu$  are *fixed*, and I want to view  $\hbar$  as a *tunable parameter*. Then I am interested in taking the limit  $\hbar \to 0$ .

The reason for that is that taking  $\hbar \to 0$  is a particularly convenient way of taking the thermodynamic limit. Also, it is a way of stressing that the thermodynamic limit is a semiclassical limit, at least at the level of the long-range physics I am addressing in these notes.

#### 2.3.1 Hydrostatic/LDA density profile

To see that the limit  $\hbar \to 0$  is indeed a thermodynamic limit, one can rely on the following self-consistent argument. Assume first that separation of scales holds, so that the density profile in the ground state can be obtained by hydrostatics (or equivalently by the *Local Density Approximation (LDA)*, see Sec. 1.3.3 in Lecture 1). Then the expectation value of the mass density is ('S.o.S' stands for 'separation of scales')

$$\rho_0(x) \equiv \langle g.s. | \Psi^{\dagger}(x)\Psi(x) | g.s. \rangle \underset{\text{S.o.S}}{=} \rho(\mu - V(x)),$$

where  $\rho(\mu - V(x))$  is the density associated with the local value of the chemical potential. Using the formulas of the previous section, one gets  $\rho(\mu) = \frac{mp_{\rm F}(\mu)}{\pi\hbar} = \frac{m^{\frac{3}{2}}\sqrt{2\mu}}{\pi\hbar}$ . Thus, assuming separation of scales, the mass density in the trapped gas is

$$\rho_0(x) \stackrel{=}{\underset{\text{S.o.S}}{=}} \frac{m^{\frac{3}{2}}\sqrt{2(\mu - V(x))}}{\pi\hbar}.$$
(2.13)

But then it is easy to check that, if this is correct, then separation of scales holds everywhere in the region where  $\mu - V(x) > 0$ , as long as  $\hbar$  is sufficiently small. Indeed, the interparticle distance is  $\sim m/\rho_0 \sim O(\hbar)$ , while the typical length of density variations is  $\sim \rho_0/\partial_x \rho_0 \sim O(1)$ . So one can take a mesoscopic scale of order  $\ell_{\text{mesco.}} \sim O(\hbar^{1/2})$ , such that separation of scales (1.2b) holds automatically in the limit  $\hbar \to 0$ .

Since the local mass density diverges in the limit  $\hbar \to 0$ , the total number of particles  $N = \int dx \frac{\rho_0(x)}{m}$  also diverges. One has

$$\boxed{N\hbar = O(1).} \tag{2.14}$$

In that sense, the thermodynamic limit is a semiclassical limit, and taking  $\hbar \to 0$  is exactly like taking  $N \to \infty$ .

In the following, I set

$$m = \mu = 1, \qquad (2.15)$$

and V(x) is viewed as a dimensionless function of position x. I also assume that the region where 1 - V(x) > 0 is a single interval, which I take to be the interval [0, L] without loss of generality.

#### 2.3.2 Ground state quantum fluctuations and the GFF

Now let me come back to the quantization scheme sketched in Sec. (1.4), in the context of *ground state* correlations. For ground state correlations we

need a euclidean action rather than the lorentzian one we have encountered in Sec. (1.4).

The basic idea is that ground state correlations can be obtained as the zero-temperature limit

$$\begin{aligned} \langle O(x_1) \dots O(x_n) \rangle &\equiv \langle \text{g.s.} | O(x_1) \dots O(x_n) | \text{g.s.} \rangle \\ &= \lim_{\beta \to \infty} \frac{\text{tr}[O(x_1) \dots O(x_n) e^{-\beta(H_V - \int dx \Psi^{\dagger} \Psi)}]}{\text{tr}[e^{-\beta(H_V - \int dx \Psi^{\dagger} \Psi)}]}, \end{aligned}$$

and that both the numerator and denominator may be written as path integrals in *imaginary* time. Then, in the thermodynamic limit  $1/N \sim \hbar \rightarrow 0$ , instead of writing these path integrals in terms of the microscopic degrees of freedom, one writes them as integrals over the coarse-grained fields  $\rho(x, \tau)$ and  $j(x, \tau)$ , constrained by the continuity equation. This gives

$$\langle O(x_1)\dots O(x_n)\rangle = \lim_{\hbar \to 0} \frac{\int [d\rho \, dj] \delta(\partial_\tau \rho + \partial_x j) \, O(x_1)\dots O(x_n) \, e^{-\mathcal{S}_E[\rho, j]}}{\int [d\rho \, dj] \delta(\partial_\tau \rho + \partial_x j) \, e^{-\mathcal{S}_E[\rho, j]}}$$

where the euclidean action  $S_E$  is nothing but the energy of configurations  $(\rho(x), j(x))$  at imaginary time  $\tau$ , integrated over  $\tau$  with periodic boundary conditions  $(\rho(x, \tau + \beta\hbar), j(x, \tau + \beta\hbar)) = (\rho(x, \tau), j(x, \tau)),$ 

$$\mathcal{S}_{E}[\rho, j] = \int_{-\beta/2}^{\beta/2} \frac{d\tau}{\hbar} \int_{0}^{L} dx \left[ \frac{j^{2}}{2\rho} + \rho_{E}(\rho) + (V-1)\rho \right].$$
(2.16)

Then, taking the limit  $\beta \to \infty$ , one ends up with an action defined on the domain  $(x, \tau) \in [0, L] \times \mathbb{R}$ , where by definition [0, L] is the interval filled with particles (i.e. the interval where the local chemical potential 1 - V(x) is positive). Notice the signs that have changed compared to the lorentzian action (1.23), and the inclusion of the global chemical potential  $\mu = 1$ .

Expanding that action to second order in  $\delta \rho$ ,  $\delta j$  around the ground state configuration  $(\rho, j) = (\rho_0, 0)$  gives

$$\mathcal{S}_E[\rho_0 + \delta\rho, \delta j] = \mathcal{S}_E[\rho_0, 0] + \int_{-\infty}^{\infty} \frac{d\tau}{\hbar} \int_0^L dx \left[ \left( \partial_\rho \rho_E(\rho_0) + V - 1 \right) \delta\rho + \frac{1}{2} \left( \frac{(\delta j)^2}{\rho_0} + \partial_\rho^2 \rho_E(\rho_0) \left( \delta\rho \right)^2 \right) \right].$$

The first order vanishes because  $\partial_{\rho}\rho_E$  is the local chemical potential  $\mu(x)$  corresponding to the hydrostatic density  $\rho_0$ , equal to  $\mu(x) = 1 - V(x)$ .

Introducing again the height field  $h(x,\tau)$  such that  $\delta\rho = \frac{1}{2\pi}\partial_x h$ ,  $\delta j = -\frac{1}{2\pi}\partial_\tau h$ , the second order gives an action  $\mathcal{S}_E[h] \equiv \mathcal{S}_E[\rho_0 + \delta\rho, \delta j] - \mathcal{S}_E[\rho_0, 0]$  for h,

$$\mathcal{S}_E[h] = \frac{1}{8\pi} \int dx d\tau \frac{1}{K} \left( \frac{1}{v} (\partial_\tau h)^2 + v (\partial_x h)^2 \right),$$

where the sound velocity v and the Luttinger parameter K were both defined in Eq. (1.27) in Lecture 1. By a last manipulation, that action can be put in covariant form,

$$\mathcal{S}_E[h] = \frac{1}{8\pi} \int \frac{\sqrt{g} \, d^2 \mathbf{x}}{K} g^{ab}(\partial_a h)(\partial_b h), \qquad (2.17)$$

in the metric with euclidean signature  $ds^2 = g_{ab}dx^a dx^b = v^2 d\tau^2 + dx^2$ . Here v(x) is the position-dependent sound velocity in the cloud at zero temperature. Having that action in covariant form will turn out to be very useful below: it will allow us to pick a simpler coordinate system where all calculations are elementary (more precisely, the calculations are the standard ones in the GFF that are reviewed in the Appendix).

Now let me emphasize an *extremely important point*. For hard core bosons, we see from the definition (1.27) and from the expression (2.12) for the sound velocity in the Tonks-Girardeau gas that

$$(g \to +\infty) \qquad K = 1. \tag{2.18}$$

This is again an artifact of the free fermion nature of the problem: as explained in the appendix, the GFF is related to the free fermion CFT if and only if K = 1; in that case, the fermionic and bosonic descriptions of the CFT are equivalent. Otherwise, for other values of K, there is no straightforward relation between the compactified boson and free fermions. Here, since we know that the Tonks-Girardeau gas is related to free fermions, one should naturally expect to see the K=1 free boson CFT (or GFF) emerge. This is precisely what is happening.

In summary, ground state correlations in the thermodynamic limit  $1/N \sim \hbar \to 0$  are related to the ones of the GFF on an infinite strip  $(x, \tau) \in [0, L] \times \mathbb{R}$  equipped with a metric g,

$$\langle \mathbf{g.s.} | O(x_1) \dots O(x_n) | \mathbf{g.s.} \rangle \stackrel{=}{=} \frac{\int [dh] O(x_1) \dots O(x_n) e^{-\mathcal{S}_E[h]}}{\int [dh] e^{-\mathcal{S}_E[h]}}. \quad (2.19)$$

The metric g encodes the information about the local speed of sound, which is fixed in terms of the background hydrostatic density  $\rho_0(x)$ :  $v(x) = \pi \hbar \rho_0(x)$ .

## 2.3.3 Identification of $\Psi^{\dagger}$ , $\Psi$ with local operators in the CFT

To calculate the one-particle density matrix, or any other correlation function involving the boson creation/annihilation operators, we need to identify those operators with the ones in the CFT. In general, any local operator O(x)in the microscopic model can be written as a sum of local operators in the CFT,

$$O(x) = \sum_{j} \tilde{A}_{O,\Phi_j} \Phi_j(x).$$
(2.20)

Here the  $\Phi_j$ 's are the CFT operators, and the  $\tilde{A}_{O,\phi}$ 's are coefficients. Such an expansion acquires a meaning when it is inserted in (2.19); then it relates a ground state correlation function in the microscopic model to a linear combination of correlators in the CFT,

$$\langle \mathbf{g.s.} | O(x_1) \dots O(x_n) | \mathbf{g.s.} \rangle \underset{1/N \sim \hbar \to 0}{=} \sum_{j_1, \dots, j_n} \tilde{A}_{O_1, \Phi_{j_1}} \dots \tilde{A}_{O_n, \Phi_{j_n}} \langle \Phi_1(x_1) \dots \Phi_n(x_n) \rangle_{\mathrm{CFT}}.$$

The reason why such a formula is interesting is because the main building blocks in the r.h.s are CFT correlators which are known (or at least, easily calculable). Thus, the calculation of *any* correlation function, no matter how many points it involves, boils down to identifying the coefficients  $\tilde{A}_{O,\Phi_i}$ .

The sum in (2.20) can be organized in increasing order of scaling dimensions  $\Delta_{\Phi_j}$ : the lower the scaling dimension, the more important the contribution to the sum. Indeed, by dimensional analysis,  $\tilde{A}_{O,\Phi_j}$  must be homogeneous to a length to the power  $\Delta_{\Phi_j} - \Delta_O$ . Since the natural microscopic scale in the problem is the inverse density  $\rho_0^{-1}(x)$ , one must have  $\tilde{A}_{O,\Phi_j} = A_{O,\Phi_j}/\rho^{\Delta_{\Phi_j}-\Delta_O}$  where  $A_{O,\Phi_j}$  is now a *dimensionless* coefficient. This leads to

$$\langle \mathbf{g.s.} | O(x_1) \dots O(x_n) | \mathbf{g.s.} \rangle =_{1/N \sim \hbar \to 0}$$

$$\sum_{j_1,\dots,j_n} \frac{A_{O_1,\Phi_{j_1}}}{\rho_0(x_1)^{\Delta_{\Phi_{j_1}} - \Delta_O}} \dots \frac{A_{O_n,\Phi_{j_n}}}{\rho_0(x_n)^{\Delta_{\Phi_{j_n}} - \Delta_O}} \langle \Phi_1(x_1,\tau_1) \dots \Phi_n(x_n,\tau_n) \rangle_{\mathrm{CFT}}$$

Since  $\rho_0(x)$  scales as  $1/\hbar$ , the r.h.s is of order  $O(\hbar^{\Delta_{\Phi_1}+\dots+\Delta_{\Phi_n}-n\Delta_O})$ , so we see that the terms are less and less relevant as the dimension  $\Delta_{\Phi_j}$  increases, as claimed.



Figure 2.2: Vaidya and Tracy obtained the asymptotic expansion of the oneparticle density matrix (their Eq. (1.3)) in the translation-invariant case, including the constant  $|A_{\Psi^{\dagger},\mathcal{V}_{0,1}}|^2$  (our constant  $|A_{\Psi^{\dagger},\mathcal{V}_{0,1}}|^2$  is  $\rho_{\infty}/\sqrt{\pi}$  in their notations). The higher order terms in their Eq. (1.3) correspond to the less relevant operators in our expansion (2.21).

Now let me specialize this discussion to the case of the creation/annihilation operators  $\Psi^{\dagger}$ ,  $\Psi$ . The coefficients  $A_{\Psi^{\dagger},\Phi_{j}}$  are constrained by symmetry: they are non-zero only if  $\Phi_{j}$  has the same U(1) charge as  $\Psi^{\dagger}$ . In other words,  $\Psi^{\dagger}$ creates a particle, so the CFT operators  $\Phi_{j}$  appearing in its expansion must create a charge as well. In the language of the height field, this means they a to create a  $2\pi$ -discontinuity. As reviewed in the Appendix, this is precisely what the vertex operators  $\mathcal{V}_{p,1}$  do. So the CFT operators  $\Phi_j$  appearing in the expansion of  $\Psi^{\dagger}$  must be those vertex operators and their derivatives. In particular, the operator with the lowest scaling dimension is necessarily  $\mathcal{V}_{0,1}$ , which has dimension 1/4. This leads to

$$\Psi^{\dagger}(x) = A_{\Psi^{\dagger}, \mathcal{V}_{0,1}} \rho_0(x)^{1/4} \mathcal{V}_{0,1}(x) + \text{ less relevant operators.}$$
(2.21a)

Similarly,

$$\Psi(x) = A_{\Psi, \mathcal{V}_{0,-1}} \rho_0(x)^{1/4} \mathcal{V}_{0,-1}(x) + \text{ less relevant operators.}$$
(2.21b)

To fix the dimensionless coefficients  $A_{\Psi^{\dagger},\mathcal{V}_{0,1}}$ ,  $A_{\Psi,\mathcal{V}_{0,-1}}$ , one needs some input from the microscopic model. Because  $\Psi^{\dagger}$  is the adjoint of  $\Psi$ , the two coefficients should be complex-conjugated:  $A_{\Psi^{\dagger},\mathcal{V}_{0,1}} = (A_{\Psi,\mathcal{V}_{0,-1}})^*$ . Moreover, in the final result for correlation functions, only the product of the two matters; their phase is unimportant because of global U(1) invariance. The amplitude  $|A_{\Psi^{\dagger},\mathcal{V}_{0,1}}|$  is known from exact results for the homogeneous Tonks-Girardeau gas, see Fig. 2.2:

$$|A_{\Psi^{\dagger},\mathcal{V}_{0,1}}|^2 = \frac{G^4(3/2)}{\sqrt{2\pi}} \simeq 0.521409, \qquad (2.22)$$

where G(.) is Barnes' G-function.

#### **2.3.4** The one-particle density matrix $g_1(x, x')$

The one-particle density matrix in the ground state is then given by

$$g_{1}(x,x') \equiv \langle \mathbf{g.s.} | \Psi^{\dagger}(x,t)\Psi(x',t') | \mathbf{g.s.} \rangle$$
  
= 
$$|A_{\Psi^{\dagger},\mathcal{V}_{0,1}}|^{2}\rho(x)^{1/4}\rho(x')^{1/4} \langle \mathcal{V}_{0,1}(x)\mathcal{V}_{0,-1}(x') \rangle_{\mathrm{CFT}}.$$

What we need to do now is to evaluate the CFT correlator in the r.h.s. Recall that it is a correlator in the infinite strip  $[0, L] \times \mathbb{R}$  equipped with the metric  $ds^2 = dx^2 + v(x)^2 d\tau^2$ .

The easiest way to handle the metric is to change coordinates: using the "stretched position"

$$\tilde{x} = \int_0^x \frac{ds}{v(s)},\tag{2.23}$$

the metric becomes

$$ds^{2} = dx^{2} + v(x)^{2} d\tau^{2} = v(x)^{2} [d\tilde{x}^{2} + d\tau^{2}].$$
(2.24)
Because the action of the GFF is invariant under Weyl transformations of the metric, this brings us back to the case of the euclidean metric. So all the formulas from the Appendix apply, provided we work in the coordinates  $(\tilde{x}, \tau)$  instead of  $(x, \tau)$ . The only thing we need to keep track of is the Jacobian,

$$\left\langle \mathcal{V}_{0,1}(x)\mathcal{V}_{0,-1}(x')\right\rangle_{\mathrm{CFT},g} = \left(\frac{d\tilde{x}}{dx}\right)^{1/4} \left(\frac{d\tilde{x}'}{dx'}\right)^{1/4} \left\langle \mathcal{V}_{0,1}(\tilde{x})\mathcal{V}_{0,-1}(\tilde{x}')\right\rangle_{\mathrm{CFT,flat}}.$$
(2.25)

The correlator of vertex operators on the strip of width  $\tilde{L} = \int_0^L \frac{ds}{v(s)}$  is calculated in the Appendix. The result can be expressed in terms of the Green's function on the strip with von Neumann boundary conditions (as well as its regularized part, see the Appendix):

$$\langle \mathcal{V}_{0,1}(\tilde{x})\mathcal{V}_{0,-1}(\tilde{x}') \rangle_{\text{CFT,flat}} = e^{-\frac{1}{2}G_{1/4K}^N(\tilde{x}) - \frac{1}{2}G_{1/4K}^N(\tilde{x}') + G_{1/4K}^N(\tilde{x},\tilde{x}')} (2.26)$$

Since the Green's function with Neumann boundary conditions is  $G_N(\tilde{x}, \tilde{x}') = -\log\left(\left(\frac{2\tilde{L}}{\pi}\right)^2 \sin^2\left(\frac{\pi(\tilde{x}-\tilde{x}')}{2\tilde{L}}\right) \sin^2\left(\frac{\pi(\tilde{x}+\tilde{x}')}{2\tilde{L}}\right)\right)$ , and the regularized Green's function is  $G_N(x) = -\log\left(\sin^2\left(\frac{\pi\tilde{x}}{\tilde{L}}\right)\right)$ , this gives

$$\left\langle \mathcal{V}_{0,1}(\tilde{x})\mathcal{V}_{0,-1}(\tilde{x}')\right\rangle_{\text{CFT,flat}} = \frac{\left|\sin\left(\frac{\pi \tilde{x}}{\tilde{L}}\right)\sin\left(\frac{\pi \tilde{x}'}{\tilde{L}}\right)\right|^{1/4}}{\left|\frac{2\tilde{L}}{\pi}\sin\left(\frac{\pi(\tilde{x}-\tilde{x}')}{2\tilde{L}}\right)\sin\left(\frac{\pi(\tilde{x}+\tilde{x}')}{2\tilde{L}}\right)\right|^{1/2}}.$$

Putting everything together, one arrives at the general formula for the oneparticle density matrix,

$$g_1(x,x') = \frac{|A_{\Psi^{\dagger},\mathcal{V}_{0,1}}|^2}{\sqrt{\pi\hbar}} \frac{\left|\sin\left(\frac{\pi \tilde{x}}{\tilde{L}}\right)\sin\left(\frac{\pi \tilde{x}'}{\tilde{L}}\right)\right|^{1/4}}{\left|\frac{2\tilde{L}}{\pi}\sin\left(\frac{\pi(\tilde{x}-\tilde{x}')}{2\tilde{L}}\right)\sin\left(\frac{\pi(\tilde{x}+\tilde{x}')}{2\tilde{L}}\right)\right|^{1/2}}, \qquad (2.27)$$

where I have used  $\left(\rho_0(x)\frac{d\tilde{x}}{dx}\right)^{1/4} = (\pi\hbar)^{-1/4}$ , see Eq. (2.12). This is a remarkable exact formula, valid for arbitray potentials V(x). If one specializes to the harmonic trap  $V(x) = \frac{1}{2}(x - \sqrt{2})^2$  (I set  $\omega = 1$  and I chose the position of the center such that the left boundary of the cloud is at x = 0), then the formula simplifies even further. Indeed, in that case  $\rho_0(x) = \frac{1}{\pi\hbar}\sqrt{2 - (x - \sqrt{2})^2}$ ,  $v(x) = \sqrt{2 - (x - \sqrt{2})^2}$ ,  $\tilde{x} = \frac{\pi}{2} + \arcsin(\frac{x}{\sqrt{2}} - 1)$ ,

and  $\tilde{L} = \pi$ ; then (2.27) becomes

(harmonic V) 
$$g_1(x, x') = |A_{\Psi^{\dagger}, \mathcal{V}_{0,1}}|^2 \frac{\rho_0(x)^{1/4} \rho_0(x')^{1/4}}{|x - x'|^{1/2}}.$$
  
(2.28)

#### 2.3.5 The *n*-particle density matrix

The exact same technique can be applied to obtain the n-particle density matrix:

$$g_n(x_1, \dots, x_n, x'_1, \dots, x'_n) \equiv \langle \mathbf{g.s.} | \Psi^{\dagger}(x_1) \dots \Psi^{\dagger}(x_n) \Psi(x'_1) \dots \Psi(x'_n) | \mathbf{g.s.} \rangle$$

$$= |A_{\Psi^{\dagger}, \mathcal{V}_{0,1}}|^{2n} \prod_{j=1}^n \rho(x_j)^{1/4} \rho(x'_j)^{1/4}$$

$$\times \langle \mathcal{V}_{0,1}(x_1) \dots \mathcal{V}_{0,1}(x_n) \mathcal{V}_{0,-1}(x'_1) \dots \mathcal{V}_{0,-1}(x'_n) \rangle_{\mathrm{CFT}}$$

The CFT correlator is again taken in the CFT with metric g, and I relate it to the one in the flat metric by switching to coordinates  $(\tilde{x}, \tau)$  as before. In the flat metric, the correlator of vertex operators is

$$\begin{aligned} &\left\langle \mathcal{V}_{0,1}(x_{1})\dots\mathcal{V}_{0,1}(x_{n})\mathcal{V}_{0,-1}(x_{1}')\dots\mathcal{V}_{0,-1}(x_{n}')\right\rangle_{\mathrm{CFT,flat}} \\ &= \prod_{p=1}^{n} e^{-\frac{1}{8}G_{N}(\tilde{x}_{p})-\frac{1}{8}G_{N}(\tilde{x}_{p}')}\prod_{k$$

Then putting everything together like we did for the one-particle density matrix, one gets

$$g_{n}(\{x_{i}\},\{x_{j}'\}) = \frac{|A_{\Psi^{\dagger},\mathcal{V}_{0,1}}|^{2n}}{(\pi\hbar)^{n/2}} \times \frac{\prod_{p=1}^{n} \left|\sin\frac{\pi\tilde{x}_{p}}{\tilde{L}} \sin\frac{\pi\tilde{x}_{p}'}{\tilde{L}}\right|^{\frac{1}{4}} \prod_{k

$$(2.29)$$$$

which is yet another remarkable formula, valid for an arbitrary trapping potential V(x). When one specializes to a harmonic trap, then the formula simplifies again,

(harm. V) 
$$g_n(\{x_i\}, \{x'_j\}) = \left|A_{\Psi^{\dagger}, \mathcal{V}_{0,1}}\right|^{2n} \prod_i \rho_0(x_i)^{\frac{1}{4}} \prod_j \rho_0(x'_j)^{\frac{1}{4}} \\ \times \frac{\prod_{p_1 < p_2} |x_{p_1} - x_{p_2}|^{\frac{1}{2}} \prod_{q_1 < q_2} |x'_{q_1} - x'_{q_2}|^{\frac{1}{2}}}{\prod_{p,q} |x_p - x'_q|^{\frac{1}{2}}}.$$

$$(2.30)$$

The point I want to emphasize here is that, as far as I know, there exists no other method that allows to derive such results in such generality (i.e. for arbitrary trapping potentials and for arbitrary number of points). Moreover, the calculations leading to these formulas are quite light: it's basically all about using the formulas for correlation functions given in the Appendix. So this method is indeed quite powerful.

# 2.4 Signal propagation and sound waves



Figure 2.3: Density plot of the amplitude bosonic two-point function at different times,  $|\langle \Psi^{\dagger}(x,t)\Psi(x',0)\rangle|$  for  $x' = \ell/2$  where  $2\ell$  is the length of the cloud. Left: exact numerical result for N = 160 bosons. Right: formula (??). [Figure from this paper.]



Figure 2.4: Sound waves propagating on top of a breathing gas. It is always possible to find a change of coordinates  $(x,t) \mapsto (\xi,\tau)$  such that in the new coordinates the propagation is the same as in the homogeneous problem in a box, with waves reflected at the boundary. *[Figure from this paper.]* 

# 2.5 Dynamics: the example of the breathing cloud

#### 2.5.1 The Wigner function

It is enlightening to describe the dynamics of the Tonks-Girardeau gas in terms of the Wigner function of free fermions. The Wigner function is defined as

$$n(x,k,t) = \int dy \, e^{iky} \left\langle \Psi_{\rm F}^{\dagger}(x+y/2,t) \Psi_{\rm F}(x-y/2,t) \right\rangle.$$
(2.31)

It has the following semiclassical interpretation: it may be thought of as the probability to find a fermion at position (x, k) in phase space. The Wigner function satisfies the evolution equation

$$\partial_t n(x,k,t) + \frac{\hbar k}{m} \partial_x n(x,k,t) = \frac{1}{\hbar} (\partial_x V(x,t)) \,\partial_k n(x,k,t). \tag{2.32}$$

This equation is *exact* because V(x) is *harmonic*. For more general potentials, Eq. (2.32) would be the leading order term in an expansion in  $\hbar$  and in higher order derivatives  $\partial_x^p V$ ,  $p \ge 2$ .

What is important is that Eq. (2.32) is a classical evolution equation, so the quantumness of the problem must be entirely hidden in the initial condition n(x, k, t = 0). In fact, the quantumness appears in the form of oscillations of the Wigner function around the positions where the density varies quickly, typically near the boundary of the system (see exercise A below). Those oscillations can be strong enough such that the Wigner function takes values smaller than 0 or larger than 1, preventing one from viewing it semi-classically as a probability distribution.

However, since we are assuming separation of scales, such oscillations are not present because the density always varies very slowly at the interparticle distance.

The problem can be made even more classical by considering a particular class of *zero-entropy* initial states, which then leads to an entirely classical description of the problem. Quantum fluctuations will be reconstructed later, by re-quantizing the resulting classical hydrodynamic description.

- 2.5.2 Reduction to standard Euler hydrodynamics
- 2.5.3 Example: quench of the frequency  $\omega_0 \rightarrow \omega_1$
- 2.5.4 Propagation of sound waves from the Wigner function

# Appendix A

# 2d Gaussian Free Field, bosonization/fermionization, and conformal invariance

The goal of this Appendix is to provide a brief introduction to the technical tools that are necessary for the three Lectures above. This is all standard material, but the presentation and notational conventions may differ from the ones found in classic physics textbooks.

[Remark: I am using the name "Gaussian Free Field" (GFF) instead of "massless free boson" or "massless scalar field" or "free boson CFT" or other names used by physicists. This is because the name "GFF" has become the standard nomenclature in mathematics. In modern probability theory, it is a very basic object which plays a fundamental role analogous to the one of the Brownian motion. So I think we physicists had better get used to the name "GFF".]

# A.1 The Gaussian Free Field (GFF): definition

[Remark: this is a basic introduction for physics students. To see how contemporary mathematicians think about the GFF, see for instance the lecture notes by Zeitouni or the ones by Sheffield.]

#### A.1.1 Gaussian probability distributions

Our starting point is the fact that the gaussian integral is finite,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2\gamma}h^2} dh < +\infty \tag{A.1}$$

for any  $\gamma > 0$ . This allows to define the gaussian probability distribution (centered on 0). For any function f(.) we define the expectation value

$$\langle f(h) \rangle \equiv \frac{\int_{-\infty}^{\infty} f(h) e^{-\frac{1}{2\gamma}h^2} dh}{\int_{-\infty}^{\infty} e^{-\frac{1}{2\gamma}h^2} dh}.$$
 (A.2)

In particular,

$$\left\langle e^{ah} \right\rangle = \exp\left(\frac{\gamma a^2}{2}\right),$$
 (A.3)

or equivalently, for non-negative integer n (by expanding in powers of a),

$$\langle h^n \rangle = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{n!}{2^{n/2}(n/2)!} \gamma^{n/2} & \text{if } n \text{ is even.} \end{cases}$$
(A.4)

The combinatorial factor has a nice interpretation: it counts the number of possible pairings in a set of n elements (n even). This is sometimes represented graphically as

$$\underbrace{\langle hh\dots hh\rangle}_{n \text{ times}} = \langle \overline{hhhh}\dots \overline{hh}\rangle + \langle \overline{hhhh}\dots \overline{hh}\rangle + \langle \overline{hhhhh}\dots \overline{hh}\rangle + \langle \overline{hhhhh}\dots \overline{hh}\rangle + \dots$$
(A.5)

where the right hand side is the sum over all possible pairings, with the rule that whenever two 'h' are paired, they get replaced by the two-point function  $\langle hh \rangle = \gamma$ . This is known as "Wick's theorem"; another way of stating that result is to say that all *n*-point connected correlations vanish for n > 2.

This goes through for higher-dimensional variables  $h = (h_1, h_2, \dots, h_N)^t$ : any positive symmetric  $N \times N$  matrix  $\Gamma$  defines an N-dimensional gaussian distribution

$$\langle f(h) \rangle \equiv \frac{\int_{-\infty}^{\infty} f(h) e^{-\frac{1}{2}h^t \Gamma^{-1}h} d^N h}{\int_{-\infty}^{\infty} e^{-\frac{1}{2}h^t \Gamma^{-1}h} d^N h}.$$
 (A.6)

One way of looking at this is to pick the basis where  $\Gamma$  is diagonal,  $\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_N)$ . That distribution is obviously the one of the product of N independent 1-dimensional gaussian variables.

In particular, for any N-dimensional vector  $a = (a_1, a_2, \ldots, a_N)^t$ ,

$$\left\langle e^{a^t h} \right\rangle = \exp\left(\frac{1}{2}a^t \Gamma a\right).$$
 (A.7)

Equivalently (by differentiating w.r.t the components  $a_{i_1}, a_{i_2}, \ldots$ ), one can write the *n*-point correlator of components  $h_{i_1}, h_{i_2}, \ldots$  as

$$\underbrace{\langle h_{i_1}h_{i_2}h_{i_3}h_{i_4}\dots\rangle}_{n \text{ times}} = \langle h_{i_1}h_{i_2}h_{i_3}h_{i_4}\dots\rangle + \langle h_{i_1}h_{i_2}h_{i_3}h_{i_4}\dots\rangle + \langle h_{i_1}h_{i_2}h_{i_3}h_{i_4}\dots\rangle + \dots$$

where the rule is the same as above: each connected pair gets replaced by the two-point function

$$|h_i h_j\rangle = \Gamma_{ij}.$$
 (A.9)

(A.8)

This is known as "Wick's theorem", or simply as the fact that all connected correlation functions vanish for n > 2.

In summary, what is most important is that the gaussian distribution is entirely defined by the two-point function  $\Gamma_{ij} = \langle h_i h_j \rangle$ , also sometimes called "propagator" or "covariance matrix" or "covariance kernel". All higherpoint correlations are given by Wick's theorem. It is completely equivalent to define the gaussian distribution out of a gaussian integral involving the positive quadratic form  $h \mapsto h^t \Gamma^{-1} h$ , or directly by specifying the two-point function  $\Gamma$ .

#### A.1.2 2d Gaussian Free Field in the plane

Next, we focus on a gaussian distribution on an infinite dimensional space: the random variable h now has infinitely many components labeled by a continuous variable  $\mathbf{x} \in \mathbb{R}^d$ . The two-point function is chosen as the Green's function —recall that "Green's function" means the inverse of a linear differential operator— of minus the Laplacian " $-\nabla^2$ " in d dimensions,

$$\langle h(\mathbf{x})h(\mathbf{x}')\rangle \equiv G(\mathbf{x},\mathbf{x}')$$

with 
$$G(\mathbf{x}, \mathbf{x}') \propto \begin{cases} -|\mathbf{x} - \mathbf{x}'| & \text{if } d = 1 \\ -\log|\mathbf{x} - \mathbf{x}'| & \text{if } d = 2 \\ \frac{1}{|\mathbf{x} - \mathbf{x}'|^{d-2}} & \text{if } d \ge 3. \end{cases}$$
 (A.10)

Notice that, according to the previous paragraph, the two-point function should be the inverse of a *positive* operator; minus the laplacian " $-\nabla^2$ " is indeed positive.

The choice (A.10) defines the *d*-dimensional Gaussian Free Field (GFF). Notice that when d = 1 it is nothing but the usual brownian motion.

We are particularly interested in d = 2. For convenience and consistency with the main parts of these lecture notes, I use the convention

$$G(\mathbf{x}, \mathbf{x}') = -K \log |\mathbf{x} - \mathbf{x}'|^2$$
 (A.11)

with K a positive constant, such that

$$-\frac{1}{4\pi K}\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta^{(2)}(\mathbf{x} - \mathbf{x}').$$
 (A.12)

At this point the normalization K could simply be removed by rescaling h, but it will become important in Sec. A.2.

Formally, the expectation value of any functional f[h] in the 2d GFF corresponds to a path integral

$$\langle f[h] \rangle = \frac{\int [dh] f[h] e^{-\frac{1}{8\pi K} \int d^2 \mathbf{x} (\nabla h)^2}}{\int [dh] e^{-\frac{1}{8\pi K} \int d^2 \mathbf{x} (\nabla h)^2}}.$$
 (A.13)

#### A.1.3 Short-distance regularization: normal ordering

When one tries to evaluate the expectation value of some power  $h^n(\mathbf{x})$  with  $n \geq 2$ , one faces the following problem: the result involves the Green's function  $G(\mathbf{x}, \mathbf{x}')$  evaluated at  $\mathbf{x}' = \mathbf{x}$ , which is divergent. One needs a recipe to remove such divergences. This is the purpose of *normal ordering*, usually written with double-dots ":  $h^n(\mathbf{x})$  :", which can be defined inductively as follows.

First, one defines

$$h^{0}(\mathbf{x}) := 1$$
 and  $h^{1}(\mathbf{x}) := h(\mathbf{x})$ 

and then by induction on  $n \ (n \ge 2)$ 

$$: h^{n}(\mathbf{x}) := \lim_{\mathbf{x}' \to \mathbf{x}} \left[ :h^{n-1}(\mathbf{x}) : h(\mathbf{x}') + (n-1)K \log |\mathbf{x} - \mathbf{x}'|^{2} : h^{n-2}(\mathbf{x}) : \right].$$
(A.14)

The second term in (A.14) cancels the divergence.

With that definition, one can check that one arrives at the **very useful** formula

$$\left\langle :e^{\alpha_1 h(\mathbf{x}_1)} :: e^{\alpha_2 h(\mathbf{x}_2)} : \dots :e^{\alpha_n h(\mathbf{x}_n)} : \right\rangle = \prod_{1 \le i,j \le n} e^{\alpha_i \alpha_j G(\mathbf{x}_i,\mathbf{x}_j)}$$
$$= \prod_{1 \le i,j \le n} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|^{2\alpha_i \alpha_j K}}.$$
 (A.15)

[See exercise A below for a derivation.]

# A.2 Compactification

In many physics problems, the GFF appears as an effective theory describing the long-distance correlations of some 2d classical critical point, or some 1d quantum critical point. However, it often appears in a slightly modified form compared to what we described in the previous section: usually  $h(\mathbf{x})$  takes values in  $\mathbb{R}/(2\pi\mathbb{Z})$  rather than in  $\mathbb{R}$ .

The main consequence of that "compactification of the target space" is on the operator content of the theory —i.e. it affects the set of local operators that are found in the theory—. In this section I review the operator content of the compactified GFF (or compactified boson), and I give the general formula for correlation functions of local observables in the plane. But before I do that, let me briefly highlight the analogy with 2d electrostatics, which is often a useful way of thinking about correlation functions in the GFF.

### A.2.1 Analogy with electrostatics and electric/magnetic duality

Consider a static distribution of pointlike electric charges  $\alpha_j$  at positions  $x_j$  in the plane (j = 1, ..., n). The Coulomb electrostatic potential at point x is

$$V(\mathbf{x}) = -\sum_{j=1}^{n} \alpha_j K \log |\mathbf{x} - \mathbf{x}_j|^2$$
 (A.16)

such that the electric field  $E(\mathbf{x}) = -\nabla V(\mathbf{x})$  satisfies

$$\begin{cases} \nabla \cdot \frac{1}{K} E(\mathbf{x}) &= \sum_{j=1}^{n} 4\pi \, \alpha_j \, \delta^{(2)}(\mathbf{x} - \mathbf{x}_j) \\ \nabla \times E(\mathbf{x}) &= 0. \end{cases}$$
(A.17)

[I am using notations such that the analogy with the GFF is clear.] The first equation is Gauss's law for a 2d homogeneous medium with a dielectric constant 1/K. The second one is Faraday's law in the absence of magnetic fluxes through the plane.

The total electrostatic energy of that distribution of charges is

$$\mathcal{E} \equiv \int_{\text{reg.}} \frac{d^2 \mathbf{x}}{8\pi K} |E(\mathbf{x})|^2$$
  
= 
$$\int_{\text{reg.}} \frac{d^2 \mathbf{x}}{8\pi} V \nabla \cdot \frac{1}{K} E$$
  
= 
$$-\sum_{j < k} \alpha_j \alpha_k K \log |\mathbf{x}_j - \mathbf{x}_k|^2.$$
(A.18)

Here "reg." stands for the short-distance regularization which corresponds to discarding the Coulomb self-energy of each pointlike charge. Looking back at formula (A.15), one sees that the correlation function  $\langle : e^{\alpha_1 h} : \cdots : e^{\alpha_n h} : \rangle$  is nothing but the exponential of the Coulomb energy  $\mathcal{E}$ . This provides an alternative way of thinking about normal ordering: it corresponds to dropping the Coulomb self-energy of pointlike charges, as one always does in electrostatics.

Now imagine that the electric charges are also tied to infinitely thin solenoids piercing the plane, with a constantly increasing magnetic field passing through them. This is like adding a source term to Faraday's law,

$$\begin{cases} \nabla \cdot \frac{1}{K} E(\mathbf{x}) &= \sum_{j=1}^{n} 4\pi \,\alpha_j \,\delta^{(2)}(\mathbf{x} - \mathbf{x}_j) \\ \nabla \times E(\mathbf{x}) &= \sum_{j=1}^{n} 2\pi \,\beta_j \,\delta^{(2)}(\mathbf{x} - \mathbf{x}_j). \end{cases}$$
(A.19)

The  $\beta_j$ 's can then be called *"magnetic charges*". An electric field satisfying these equations is

$$E_{a}(\mathbf{x}) = \sum_{j=1}^{n} \left[ \alpha_{j} \, 2K \, \frac{\delta_{ab}(\mathbf{x}^{b} - \mathbf{x}_{j}^{b})}{|\mathbf{x} - \mathbf{x}_{j}|^{2}} - \beta_{j} \frac{\varepsilon_{ab}(\mathbf{x}^{b} - \mathbf{x}_{j}^{b})}{|\mathbf{x} - \mathbf{x}_{j}|^{2}} \right].$$
(A.20)

Magnetic and electric charges are exchanged under the duality transformation

$$\begin{aligned} (\alpha, \beta) &\mapsto (\tilde{\alpha}, \tilde{\beta}) = (\beta, -\alpha) \\ E_a &\mapsto \tilde{E}_a = \frac{1}{2K} \varepsilon_{ab} E_b \\ K &\mapsto \tilde{K} = \frac{1}{4K}, \end{aligned} \tag{A.21}$$

which is defined such that  $\nabla \cdot \frac{1}{\tilde{K}} \tilde{E} = 2\nabla \times E$  and  $\nabla \times \tilde{E} = -\frac{1}{2} \nabla \cdot \frac{1}{K} E$ . Notice that this duality transformation squares to -1 (i.e.  $\tilde{\tilde{E}} = -E$  and  $(\tilde{\tilde{\alpha}}, \tilde{\tilde{\beta}}) = (-\alpha, -\beta)$ ), and also that the total electrostatic energy  $\mathcal{E} = \int \frac{d^2 \mathbf{x}}{8\pi K} |E(\mathbf{x})|^2$  is invariant under that duality transformation (at least, this is true naively, if one overlooks the short-distance regularization).



Figure A.1: Left: electric field  $E(\mathbf{x})$  created by an electric charge  $(\alpha_1, \beta_1) = (1,0)$  at  $\mathbf{x}_1 = (-5, -2)$  and a magnetic charge  $(\alpha_2, \beta_2) = (0,1)$  at  $\mathbf{x}_2 = (5,2)$ . Right: the dual electric field  $\tilde{E}(\mathbf{x})$ .

The electrostatic potential  $V(\mathbf{x})$  corresponding to the electric field (A.20) has branch-cuts, but it can be written locally as

$$V(\mathbf{x}) = -\sum_{j=1}^{n} [\alpha_j K \log |\mathbf{x} - \mathbf{x}_j|^2 + \beta_j \arg(z - z_j)] + \text{const.}, \qquad (A.22)$$

where the second term is the argument of the complex coordinate  $z = x^1 + ix^2$ , defined only modulo  $2\pi$ .

The total electrostatic energy is not well defined as it also has branch cuts, but formally it reads

$$\mathcal{E} = -\sum_{j < k} \alpha_j \alpha_k K \log |\mathbf{x}_j - \mathbf{x}_k|^2 + \sum_{j < k} \beta_j \beta_k \frac{1}{4K} \log |\mathbf{x}_j - \mathbf{x}_k|^2 - \sum_{j < k} (\alpha_k \beta_j + \alpha_j \beta_k) \arg(z_k - z_j) + \text{const.}$$
(A.23)

This can be obtained as follows. Assuming that we know the energy  $\mathcal{E}_{n-1}$  of the configuration with the n-1 first charges, one first adds the last electric

charge  $\alpha_n$ . This changes the total energy to  $\mathcal{E}_{n-1} \to \mathcal{E}'_n = \mathcal{E}_{n-1} + \alpha_n V(\mathbf{x}_n)$ . Then one adds the last magnetic charge  $\beta_n$ , and exploiting the duality this gives  $\mathcal{E}'_n \to \mathcal{E}'_n - \beta_n \tilde{V}(\mathbf{x}_n)$ . In total, the additional energy due to the electric-magnetic pointlike charge  $(\alpha_n, \beta_n)$  is  $\alpha_n V(\mathbf{x}_n) - \beta_n \tilde{V}(\mathbf{x}_n)$ , and this is what is expressed by Eq. (A.23).

One can then ask: do the magnetic charges have a meaning in the GFF? Is the exponential of (A.23) a correlation function of some observable in the GFF? This is basically what happens with compactification —up to small differences due to the fact that some charges become imaginary—.

#### A.2.2 Local operators in the compactified GFF

Now let me come back to the GFF, and to its compactification: instead of taking values in  $\mathbb{R}$ ,  $h(\mathbf{x})$  now takes values in  $\mathbb{R}/(2\pi\mathbb{Z})$ .

First, the compactification puts a strong constraint on those local operators that are otherwise found in the non-compact GFF. Indeed, observables should now be invariant under  $h \to h + 2\pi$ . A local observable that depends on the value of  $h(\mathbf{x})$  at a given point  $\mathbf{x}$  is not allowed in the compactified theory unless it satisfies  $f(h(\mathbf{x})) = f(h(\mathbf{x}) + 2\pi)$ . All such observables (properly regularized by normal ordering) can be written as a Fourier series involving the exponentials

$$: e^{iph(\mathbf{x})} : \quad \text{for} \quad p \in \mathbb{Z},$$
 (A.24)

which I write as " $\mathcal{V}_{p,0}(\mathbf{x})$ " below. Those normal ordered exponentials and their linear combinations are the only local operators involving the value  $h(\mathbf{x})$  that survive in the compactified theory. The other operators that survive are the derivatives  $\partial_a h(\mathbf{x}), \partial_a \partial_b h(\mathbf{x}), \ldots$ 

Second, in the compactified theory, a new type of local operator appears. This is clear if one imagines that the field  $h(\mathbf{x})$  lives on a plane with a few punctures,  $\mathbb{R}^2 \setminus \{y_1, \ldots, y_m\}$ . Then the field  $h : \mathbb{R}^2 \setminus \{y_1, \ldots, y_m\} \to \mathbb{R}^2/(2\pi\mathbb{Z})$  is a random map, with an integer winding  $q_i$  around each puncture  $y_i$ , namely

$$\oint_{C_{y_i}} d\mathbf{x} \cdot \nabla h(\mathbf{x}) = 2\pi q_i \tag{A.25}$$

for any (oriented) contour  $C_{y_i}$  enclosing the puncture  $y_i$  (and only that one). The new operator allowed in the compactified theory is the one which creates a new puncture at position y and fixes the winding of h around it. This operator will be written " $\mathcal{V}_{0,q}(y)$ ". When q = 0 this is just the identity operator.

Finally, by bringing  $\mathcal{V}_{0,q}(\mathbf{y})$  to the same position as the one of the normal ordered exponential (A.24), one a gets a local operator which is the "fusion" of the two, which I write as

$$\mathcal{V}_{p,q}(\mathbf{x}).$$
 (A.26)

All local operators in the compactified GFF are linear combinations of operators  $\mathcal{V}_{p,q}(\mathbf{x})$  or of their derivatives, or of the derivatives  $\partial_a h, \partial_a \partial_b h, \ldots$ 

#### A.2.3 Chiral factorization

It is convenient to think of the height field h as being the sum of two *chiral* fields

$$h(\mathbf{x}) = h(z, \bar{z}) = \sqrt{K} \left[\varphi(z) + \bar{\varphi}(\bar{z})\right]$$
(A.27)

where  $z = x^1 + ix^2$ ,  $\bar{z} = x^1 - ix^2$ . The propagator of the chiral fields are

$$\langle \varphi(z)\varphi(z')\rangle = -\log(z-z') \qquad \langle \bar{\varphi}(\bar{z})\bar{\varphi}(\bar{z}')\rangle = -\log(\bar{z}-\bar{z}') \qquad (A.28)$$

and  $\langle \varphi(z)\bar{\varphi}(\bar{z}')\rangle = 0$ , such that one recovers  $\langle h(z,\bar{z})h(z',\bar{z}')\rangle = -K \log |z - z'|^2$ . For most purposes, one can manipulate the two chiral fields as if they were independent —although for some purposes one needs to be careful about the fact that the zero mode of  $\varphi(z)$  is not independent from the zero mode of  $\bar{\varphi}(\bar{z})$ —. Correlation functions typically factorize into a correlation function involving  $\varphi$  only, and another involving  $\bar{\varphi}$  only. This fact is known as "chiral factorization"; it is a basic feature of all conformal field theories (not only the GFF).

By definition, the vertex operator  $\mathcal{V}_{p,0}(z,\bar{z}) =: e^{iph(z,\bar{z})}:$  can be written in terms of the chiral fields as  $:e^{ip\sqrt{K}[\varphi(z)+\bar{\varphi}(\bar{z})]}:$  What is interesting is that the vertex operator  $\mathcal{V}_{0,q}(z,\bar{z})$  can also be written in a similar way,

$$\mathcal{V}_{0,q}(z,\bar{z}) = :e^{-i\frac{q}{2\sqrt{K}}[\varphi(z)-\bar{\varphi}(\bar{z})]}:.$$
(A.29)

More generally, the operator  $\mathcal{V}_{p,q}(z,\bar{z})$  can be written as

$$\mathcal{V}_{p,q}(z,\bar{z}) = :e^{i\left(p\sqrt{K}-\frac{q}{2\sqrt{K}}\right)\varphi(z)}e^{i\left(p\sqrt{K}+\frac{q}{2\sqrt{K}}\right)\bar{\varphi}(\bar{z})}:.$$
 (A.30)

In the presence of such operators, the field h(x) acquires the expectation

value

$$= \frac{\langle h(z,\bar{z}) \mathcal{V}_{p_{1},q_{1}}(z_{1},\bar{z}_{1}) \dots \mathcal{V}_{p_{n},q_{n}}(z_{n},\bar{z}_{n}) \rangle}{\langle \mathcal{V}_{p_{1},q_{1}}(z_{1},\bar{z}_{1}) \dots \mathcal{V}_{p_{n},q_{n}}(z_{n},\bar{z}_{n}) \rangle}$$

$$= \frac{\langle \sqrt{K}[\varphi(z) + \bar{\varphi}(\bar{z})] : e^{i(p_{1}\sqrt{K} - \frac{q_{1}}{2\sqrt{K}})\varphi(z_{1}) + i(p_{1}\sqrt{K} - \frac{q_{1}}{2\sqrt{K}})\bar{\varphi}(\bar{z}_{1})} : \dots \rangle}{\langle : e^{i(p_{1}\sqrt{K} - \frac{q_{1}}{2\sqrt{K}})\varphi(z_{1}) + i(p_{1}\sqrt{K} - \frac{q_{1}}{2\sqrt{K}})\bar{\varphi}(\bar{z}_{1})} : \dots \rangle}$$

$$= \sum_{j=1}^{n} \left[ -i \left( p_{j}K - \frac{q_{j}}{2} \right) \log(z - z_{j}) - i \left( p_{j}K + \frac{q_{j}}{2} \right) \log(\bar{z} - \bar{z}_{j}) \right]$$

$$= -\sum_{j=1}^{n} \left[ ip_{j}K \log|z - z_{j}|^{2} + q_{j} \arg(z - z_{j}) \right],$$

which is nothing but the electrostatic potential (A.22) created by charges  $(\alpha_j, \beta_j) = (ip_j, q_j)$ . Notice that the "electric charge" is now complex, as could be seen already from Eq. (A.24). Notice also that the field  $h(z, \bar{z})$  is singular at  $z \to \infty$  unless the total electric charge  $\sum_j p_j$  and the total magnetic charge  $\sum_j q_j$  both vanish. If the electric charge is not zero, then  $h(z, \bar{z})$  has a logarithmic divergence, and if the magnetic charge is non-zero, then it is not single-valued at infinity. It is useful to view these possible singularities as coming from another operator  $\mathcal{V}_{-\sum_i p_i, -\sum_j q_j}(\infty)$  located at infinity, which compensates the superfluous electric/magnetic charge.

Now we arrive at one of the **most useful formulas of this Appendix**. Using Wick's theorem, one can evaluate the expectation value

$$\langle \mathcal{V}_{p_1,q_1}(z_1,\bar{z}_1)\dots\mathcal{V}_{p_n,q_n}(z_n,\bar{z}_n)\rangle = \prod_{j$$

or equivalently

$$\langle \mathcal{V}_{p_1,q_1}(z_1, \bar{z}_1) \dots \mathcal{V}_{p_n,q_n}(z_n, \bar{z}_n) \rangle =$$

$$\prod_{j < k} |z_j - z_k|^{2(p_j p_k K + q_j q_k \frac{1}{4K})} \prod_{j < k} e^{-i(q_j p_k + p_j q_k) \arg(z_j - z_k)}.$$
(A.31)

[Notice that it is analogous to the exponential of (A.23), with some additional factors i or -1 that come from the imaginary electric charge  $\alpha = ip$ .] This formula proves to be extremely useful for a number of practical purposes. Basically, it allows to get any interesting correlation function of local observables in microscopic models whose large-scale fluctuations are captured by the GFF.

The scaling dimension of the operator  $\mathcal{V}_{p,q}$  can be read off from Eq. (A.31). Assume there is no operator at infinity, i.e.  $\sum_i p_i = \sum_j q_j = 0$ . Then under a scaling transformation  $\mathbf{x} \to \lambda \mathbf{x}$ , the correlation function behaves as  $\langle \mathcal{V}_{p_1,q_1} \dots \mathcal{V}_{p_n,q_n} \rangle \to \lambda^{-\sum_{i=1}^n \Delta_{p_i,q_i}} \langle \mathcal{V}_{p_1,q_1} \dots \mathcal{V}_{p_n,q_n} \rangle$ , with the scaling dimension

$$\Delta_{p,q} = p^2 K + \frac{q^2}{4K}.$$
(A.32a)

Similarly, the conformal spin is obtained by applying a rotation  $z \to e^{i\theta} z$ . The correlation function transforms as  $\langle \mathcal{V}_{p_1,q_1} \dots \mathcal{V}_{p_n,q_n} \rangle \to e^{-i\theta \sum_{j=1}^n s_{p_j,q_j}} \times \langle \mathcal{V}_{p_1,q_1} \dots \mathcal{V}_{p_n,q_n} \rangle$ , with

$$s_{p,q} = pq. \tag{A.32b}$$

Finally, notice that all these formulas are invariant under the duality

$$\begin{array}{cccc}
\mathcal{V}_{p,q} & \leftrightarrow & \mathcal{V}_{q,p} \\
K & \leftrightarrow & \frac{1}{4K}
\end{array}$$
(A.33)

which is analogous to the electric/magnetic duality. Notice, however, that the duality transformation squares to 1 now, instead of -1 for the electric/magnetic duality above. This is because of the additional multiplication by *i* due to the complex electric charges, so that applying the transformation twice now gives  $i^2 \times -1 = 1$ .

To summarize: the compactified GFF with coupling constant K and the one with coupling constant 1/(4K) are dual to each other. Most interesting correlation functions are given by formula (A.31).

## A.3 Fermionization/Bosonization

The GFF, or free boson CFT, is not the only useful free CFT. Another very useful one is the free Dirac fermion CFT. Here I briefly introduce that theory and discuss its relation to the GFF through bosonization.

#### A.3.1 The free (Dirac) fermion CFT

The free Dirac fermion CFT is defined by the two-point function

$$\left\langle \psi^{\dagger}(z)\psi(z')\right\rangle = \frac{1}{z-z'}, \qquad \left\langle \bar{\psi}^{\dagger}(\bar{z})\bar{\psi}(\bar{z}')\right\rangle = \frac{1}{\bar{z}-\bar{z}'},$$
(A.34)

and  $\langle \psi^{\dagger}(z)\bar{\psi}(\bar{z}')\rangle = \langle \bar{\psi}^{\dagger}(z)\psi(z')\rangle = \langle \psi(z)\psi(z')\rangle = \langle \psi^{\dagger}(z)\psi^{\dagger}(z')\rangle = \langle \bar{\psi}(\bar{z})\bar{\psi}(\bar{z}')\rangle = \langle \bar{\psi}^{\dagger}(\bar{z})\bar{\psi}^{\dagger}(\bar{z}')\rangle = 0$ , and by the fact that any higher-point correlation is given by Wick's theorem *for fermions*. This is analogous to Wick's theorem for bosons, up to the minus signs that arise from anti-commuting the fermionic fields  $\psi, \psi^{\dagger}, \bar{\psi}, \bar{\psi}^{\dagger}$ . For instance, for four points, Wick's theorem for fermions reads

$$\langle \psi^{\dagger}(z_{1})\psi^{\dagger}(z_{2})\psi(z_{2}')\psi(z_{1}')\rangle = \langle \psi^{\dagger}(z_{1})\psi^{\dagger}(z_{2})\psi(z_{2}')\psi(z_{1}')\rangle + \langle \psi^{\dagger}(z_{1})\psi^{\dagger}(z_{2})\psi(z_{2}')\psi(z_{1}')\rangle = \frac{1}{z_{1}-z_{1}'}\frac{1}{z_{2}-z_{2}'} - \frac{1}{z_{1}-z_{2}'}\frac{1}{z_{2}-z_{1}'}.$$
 (A.35)

The second term picks up a minus sign because one has to anti-commute  $\psi^{\dagger}(z_2)$  and  $\psi(z'_2)$  before one can replace the pairs by the two-point functions (A.34).

More generally, for arbitrary number of points, Wick's theorem for fermions is summarized by the following compact formula,

$$\left\langle \psi^{\dagger}(z_{1})\psi^{\dagger}(z_{2})\dots\psi^{\dagger}(z_{n})\psi(z_{n})\dots\psi(z'_{2})\psi(z'_{1})\right\rangle$$

$$=\sum_{\text{perm.}\sigma}(-1)^{|\sigma|}\frac{1}{z_{1}-z'_{\sigma(1)}}\frac{1}{z_{2}-z'_{\sigma(2)}}\dots\frac{1}{z_{n}-z'_{\sigma(n)}}$$

$$=\det\left(\left[\frac{1}{z_{i}-z'_{j}}\right]_{1\leq i,j\leq n}\right).$$
(A.36)

This fixes all correlation functions of the fermionic fields in the free Dirac fermion theory.

[Equivalently, the free Dirac fermion theory could be defined by the action  $S = \frac{1}{2\pi} \int d^2 z [\psi^{\dagger} \partial_{\bar{z}} \psi + \bar{\psi}^{\dagger} \partial_{z} \bar{\psi}]$  and then by a path integral with Grassmann variables. But we will not need to deal with the Dirac action in these notes, so it is more straightforward to define the theory directly in terms of its correlation functions.]

#### A.3.2 Dirac fermion = compact boson with K = 1

The free Dirac fermion theory is actually not really different from the GFF. In fact, it can be represented as a compactified GFF, provided that we allow operators with half-integer spin (namely, fermions) in the GFF.

Indeed, notice that all local operators in the GFF have *integer* spin. This is obvious for all operators  $\mathcal{V}_{p,q}$  from Eq. A.32b because p and q are integers, but it is also obvious for all their derivatives, or for the derivatives of h. In the Dirac fermion theory, the operators  $\psi^{\dagger}, \psi$  have scaling dimension  $\frac{1}{2}$  and spin  $\frac{1}{2}$ , while the operators  $\bar{\psi}^{\dagger}, \bar{\psi}$  have scaling dimension  $\frac{1}{2}$  and spin  $-\frac{1}{2}$ .

In order to get fermions in the compactified GFF, one therefore needs to slightly relax one of the constraints imposed above when compactifying the GFF, such that operators with half-integer spins are allowed. There are two possibilities.

The first possibility is to allow observables that are functions  $f(h(\mathbf{x}))$ not strictly invariant under  $h \to h + 2\pi$ , but only invariant up to a sign,  $f(h(\mathbf{x}) + 2\pi) = \pm f(h(\mathbf{x}))$ . It is then clear from Eq. (A.24) that operators  $\mathcal{V}_{p,q}$  with half-integer p must also be allowed. Then there are two operators with spin  $+\frac{1}{2}$ , and two operators with spin  $-\frac{1}{2}$  in the theory,

$$\mathcal{V}_{\frac{1}{2},1}, \; \mathcal{V}_{-\frac{1}{2},-1}\,, \qquad \text{and} \qquad \mathcal{V}_{\frac{1}{2},-1}, \; \mathcal{V}_{-\frac{1}{2},1},$$

which have scaling dimension  $\Delta_{\frac{1}{2},1} = \Delta_{-\frac{1}{2},-1} = \Delta_{-\frac{1}{2},-1} = \Delta_{-\frac{1}{2},-1} = \frac{1}{4} + \frac{1}{4K}$ . In order for this scaling dimension to be  $\frac{1}{2}$ , we must have

$$K = 1. \tag{A.37}$$

The other possibility is to keep p integer, but to allow "magnetic" operators  $\mathcal{V}_{0,q}$  around which the field h winds by  $\pm \pi$  instead of  $\pm 2\pi$ . This basically has the same consequence as the first option: some operators in the theory are then defined only up to a sign. Following that other possibility, one sees that the operators which are now allowed are  $\mathcal{V}_{p,q}$  with half-integer q (and integer p). Then, requiring that there are four such operators with spins  $\pm \frac{1}{2}$ and scaling dimension  $\frac{1}{2}$ , one arrives at the conclusion that

$$K = 1/4.$$

Of course, we have just picked the dual theory.

So we see that the "two possibilities" are really just one possibility, since the other is just the dual one. From now on, let me focus on  $K = 1, p \in \frac{1}{2}\mathbb{Z}$  and  $q \in \mathbb{Z}$ . It is tempting to make the following identifications between operators in the Dirac CFT and operators in the compactified GFF,

$$\begin{split} \psi^{\dagger}(z) &= \mathcal{V}_{\frac{1}{2},-1}(z) = :e^{-i\varphi(z)}: & \bar{\psi}^{\dagger}(\bar{z}) &= \mathcal{V}_{\frac{1}{2},1}(z) = :e^{i\bar{\varphi}(\bar{z})}: \\ \psi(z) &= \mathcal{V}_{-\frac{1}{2},1}(z) = :e^{i\varphi(z)}: & \bar{\psi}(\bar{z}) &= \mathcal{V}_{-\frac{1}{2},-1}(z) = :e^{-i\bar{\varphi}(\bar{z})}: \\ & (A.38) \end{split}$$

This identification, however, is not entirely benign. Indeed, if it were correct, then it would imply a non-trivial identity between correlation functions in the Dirac fermion CFT and correlation functions in the GFF with K = 1:

$$\left\langle \psi^{\dagger}(z_{1})\psi^{\dagger}(z_{2})\dots\psi^{\dagger}(z_{n})\psi(z_{n}')\dots\psi(z_{2}')\psi(z_{1}')\right\rangle \stackrel{?}{=} \\ \left\langle :e^{i\varphi(z_{1})}::e^{-i\varphi(z_{2})}:\dots:e^{-i\varphi(z_{n})}::e^{i\varphi(z_{n}')}:\dots:e^{i\varphi(z_{2}')}:e^{i\varphi(z_{2}')}:\right\rangle.$$
(A.39)

The left-hand side is given by Wick's theorem for fermions (A.36) while the right-hand side is given by Eq. (A.31). In order for this equation to be correct, one would need to have

$$\det\left(\left[\frac{1}{z_i - z'_j}\right]_{1 \le i, j \le n}\right) \stackrel{?}{=} \frac{\prod_{i < j} (z_i - z_j)(z'_j - z'_i)}{\prod_{p,q} (z_p - z'_q)}.$$
 (A.40)

But is this formula correct?

The answer is: yes. It is known as the Cauchy determinant formula.

So, one can safely conclude that the free Dirac fermion theory is equivalent to the GFF at K = 1 (again, provided we include operators with half-integer spins in the GFF). The possibility of turning fermionic correlation functions into bosonic ones, and vice-versa, is the essence of bosonization/fermionization.

# A.4 The GFF in the upper half-plane: the method of images

So far I have focused exclusively on correlation functions in the plane. But for many practical purposes, one needs to have access to correlation functions on other surfaces, in particular on surfaces with boundaries.

In this section I explain how to define the GFF and calculate its correlation functions on the simplest such surface: the upper half-plane  $\mathbb{H} \equiv \{z \in \mathbb{C}, \text{ Im } z \geq 0\}$ , whose boundary is the real axis. In section A.1 I emphasized that the GFF is defined by its two-point function  $\langle h(z, \bar{z})h(z', \bar{z}')\rangle$ , which is chosen as the Green's function of minus the laplacian. This definition also works in the upper half-plane  $\mathbb{H}$ , however one needs to pay attention to the boundary condition satisfied by the field  $h(z, \bar{z})$  along the real axis.

#### A.4.1 Dirichlet and Neumann boundary conditions

# Green's function in the upper half-plane with Dirichlet/Neumann boundary conditions

The two most natural boundary conditions that can be imposed on  $h(z, \bar{z})$  are the *Dirichlet* and *Neumann* b.c. along the real axis

(Dirichlet b.c.) 
$$h = 0$$
  
(Neumann b.c.)  $\nabla_{\perp} h = 0,$  (A.41)

where  $\nabla_{\perp}$  is the derivative in the direction perpendicular to the boundary, i.e.  $\nabla_{\perp} = \partial/\partial x^2 = i(\partial_z - \partial_{\bar{z}})$ , with  $z = x^1 + ix^2$ ,  $\bar{z} = x^1 - ix^2$ . There exist other interesting boundary conditions in the GFF, but those two are the easiest and most natural ones. They are the only boundary conditions needed in the context of these lectures notes.

The GFF in the upper half-plane  $\mathbb{H}$  is defined in terms of the Green's function of minus the laplacian with either of these two boundary conditions. To obtain the Green's function in  $\mathbb{H}$ , one can rely on the *method of images*. The idea is the same as in 2d electrostatics (for an introduction to the method of images in that context see for instance Jackson's textbook).

One views  $G((z, \bar{z}), (z', \bar{z}'))$  as the electrostatic potential at  $(z, \bar{z})$  created by a pointlike charge at  $(z', \bar{z}')$ . Then the trick is to replace the boundary by the mirror image of the charge at  $(\bar{z}', z')$ —i.e. the reflection of the point  $(z', \bar{z}')$  across the real axis—. The mirror charge is adjusted in order to get the correct boundary condition. This gives

(Dirichlet) 
$$G^{\mathrm{D}}((z,\bar{z}),(z',\bar{z}')) = -K \log |z-z'|^2 + K \log |z-\bar{z}'|^2$$

(Neumann)

) 
$$G^{N}((z,\bar{z}),(z',\bar{z}')) = -K \log |z-z'|^2 - K \log |z-\bar{z}'|^2,$$
  
(A.42)

such that in both cases  $-\frac{1}{4\pi K}\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta^{(2)}(\mathbf{x} - \mathbf{x}')$ , and

$$G^{\mathcal{D}}((z,\bar{z}),(z',\bar{z}')) = 0 \quad \text{if} \quad z \in \mathbb{R}$$
$$\nabla_{\perp} G^{\mathcal{N}}((z,\bar{z}),(z',\bar{z}')) = 0 \quad \text{if} \quad z \in \mathbb{R}.$$

Notice that both Green's functions are symmetric:  $G^{D}(x, x') = G^{D}(x', x)$ and  $G^{N}(x, x') = G^{N}(x', x)$ .

#### Dirichlet/Neumann b.c. and duality $K \leftrightarrow \frac{1}{4K}$

It is important to observe that Dirichlet and Neumann b.c. are *exchanged* by the duality transformation above. This is most easily seen from the electrostatic analogy, see Fig. A.1. Since the duality acts as a rotation of  $-\frac{\pi}{2}$  on the electric field E, and the latter is analogous to  $-\nabla h$ , we see that the perpendicular and parallel components of  $\nabla h$  are exchanged at the boundary:  $\nabla_{\perp}h \rightarrow \nabla_{\parallel}h$  and  $\nabla_{\perp}h \rightarrow -\nabla_{\parallel}h$ .

Then the Dirichlet b.c. h = 0 implies  $\nabla_{\parallel} h = 0$ , which becomes the Neumann b.c.  $\nabla_{\perp} h = 0$  under the duality transformation.

The Neumann b.c.  $\nabla_{\perp}h$  becomes  $\nabla_{\parallel}h = 0$  under duality. This only implies that h is *constant* along the boundary, not necessarily that it is zero. However, when that constant is non-zero we can just shift h globally,  $h \rightarrow h - \text{const.}$ , to get back to the case h = 0. Such a global shift acts trivially on the correlation functions of local observables. So without loss of generality we can assume that h = 0, and we are back at the Dirichlet b.c. as defined above.

To summarize: in the upper half-plane, the compactified GFF at K with *Dirichlet b.c.* is dual to the compactified GFF at  $\frac{1}{4K}$  with *Neumann b.c.* 

#### A.4.2 Chiral factorization and the method of images

Recall that the chiral and anti-chiral components of h are defined such that  $h(z, \bar{z}) = \sqrt{K}(\varphi(z) + \bar{\varphi}(\bar{z}))$ . In terms of the chiral fields  $\varphi$  and  $\bar{\varphi}$ , the Dirichlet and Neumann b.c. read

(Dirichlet) 
$$\varphi(z) = -\bar{\varphi}(\bar{z}) \text{ if } z \in \mathbb{R}$$
  
(Neumann)  $\varphi(z) = \bar{\varphi}(\bar{z}) \text{ if } z \in \mathbb{R}.$  (A.43)

Since  $\varphi(z)$  and  $\bar{\varphi}(\bar{z})$  are coupled along the boundary, they must no longer be regarded as independent. In particular, the two-point function  $\langle \varphi(z)\bar{\varphi}(\bar{z}')\rangle$ is no longer zero. It is often convenient to regard the field  $\bar{\varphi}$  as the analytic continuation of  $\mp \varphi$  to the lower half-plane (with -/+ for D/N). Then the two-point functions of the chiral fields are

$$\langle \varphi(z)\varphi(z')\rangle = -\log(z-z'), \qquad \langle \bar{\varphi}(\bar{z})\bar{\varphi}(\bar{z}')\rangle = -\log(\bar{z}-\bar{z}'),$$

$$\text{(A.44)}$$

$$\text{and} \quad \langle \varphi(z)\bar{\varphi}(\bar{z}')\rangle = \mp \langle \varphi(z)\varphi(\bar{z}')\rangle = \pm \log(z-\bar{z}').$$

Notice that this is consistent with the Green's functions (A.42).

The trick of regarding the anti-chiral field as the analytic continuation of the chiral one is an alternative way of looking at the method of images.

# General formula for correlation functions in the upper half-plane with Dirichlet b.c.

From now on, I focus exclusively on the GFF with Dirichlet boundary condition.

Using the expression (A.30) for  $\mathcal{V}_{p,q}$  in terms of  $\varphi$  and  $\bar{\varphi}$ , one can revisit the calculation of the correlation function  $\langle \mathcal{V}_{p_1,q_1}(z_1, \bar{z}_1) \dots \mathcal{V}_{p_n,q_n}(z_n, \bar{z}_n) \rangle$  see Eq. (A.31)—, this time in the upper half-plane. One gets [to emphasize that I'm now working in the half-plane, I add a subscript  $\mathbb{H}$  to the correlation functions]

$$\begin{array}{l} \langle \mathcal{V}_{p_{1},q_{1}}(z_{1},\bar{z}_{1})\ldots\mathcal{V}_{p_{n},q_{n}}(z_{n},\bar{z}_{n})\rangle_{\mathbb{H}} \\ = & \left\langle \prod_{j=1}^{n}:e^{i(p\sqrt{K}-\frac{q}{2\sqrt{K}})\varphi(z)}e^{i(p\sqrt{K}+\frac{q}{2\sqrt{K}})\bar{\varphi}(\bar{z})}:\right\rangle_{\mathbb{H}} \\ = & \left\langle \prod_{j=1}^{n}:e^{i(p\sqrt{K}-\frac{q}{2\sqrt{K}})\varphi(z)}e^{-i(p\sqrt{K}+\frac{q}{2\sqrt{K}})\varphi(\bar{z})}:\right\rangle_{\mathbb{C}} \\ = & \prod_{j$$

To go from the second to the third line, I use the fact that  $\bar{\varphi}$  can be viewed as the analytic continuation of  $\varphi$  to the lower half-plane. Thus, a correlation function in  $\mathbb{H}$  involving both  $\varphi$  and  $\bar{\varphi}$  becomes a correlation function in the whole plane  $\mathbb{C}$  involving only the chiral field  $\varphi$  (see Fig. A.2). This trick, which is basically the method of images, is not specific to the GFF. It is the standard way of dealing with *any* CFT in the upper half-plane. [The interested reader can have a look at this classic paper by Cardy.]

The final result for the correlators of  $\mathcal{V}_{p_j,q_j}$  in the GFF can be written directly in terms of the Green's functions in the upper half-plane (up to a



Figure A.2: GFF on the upper half-plane  $\mathbb{H}$  with Dirichlet boundary condition  $h = \sqrt{K}(\varphi + \bar{\varphi})$  along the boundary. The anti-chiral field  $\bar{\varphi}$  may be viewed as the analytic continuation of  $-\varphi$  to the lower half-plane. Consequently, a correlation function in  $\mathbb{H}$  of observables involving  $\varphi$  and  $\bar{\varphi}$  is equivalent to a correlation function in the plane  $\mathbb{C}$  involving only the chiral field  $\varphi$ .

global phase independent of the positions):

$$\langle \mathcal{V}_{p_{1},q_{1}}(z_{1},\bar{z}_{1})\dots\mathcal{V}_{p_{n},q_{n}}(z_{n},\bar{z}_{n})\rangle_{\mathbb{H}} = \exp\left(-\sum_{l} [\frac{p_{l}^{2}}{2}G_{K}^{D}(z_{l},\bar{z}_{l}) + \frac{q_{l}^{2}}{2}G_{1/4K}^{N}(z_{l},\bar{z}_{l})]\right) \\ \times \exp\left(-\sum_{j < k} [p_{j}p_{k}G_{K}^{D}((z_{j},\bar{z}_{j}),(z_{k},\bar{z}_{k})) + q_{j}q_{k}G_{1/4K}^{N}((z_{j},\bar{z}_{j}),(z_{k},\bar{z}_{k}))]\right) \\ \times \exp\left(-i\sum_{l \neq m} p_{l}q_{m}F((z_{l},\bar{z}_{l}),(z_{m}',\bar{z}_{m}'))\right).$$
(A.45)

The different functions in the exponentials are defined as follows. The two Green's functions for Dirichlet and Neumann b.c. are the ones we constructed above —the one with Neumann b.c. now appears with the dual coupling constant 1/4K—,

$$G_K^D((z,\bar{z}),(z',\bar{z}')) = -K \log \left| \frac{z-z'}{z-\bar{z}'} \right|^2$$
  
$$G_{1/4K}^N((z,\bar{z}),(z',\bar{z}')) = -\frac{1}{4K} \log \left| (z-z')(z-\bar{z}') \right|^2.$$

The corresponding *regularized Green's functions*, which depend on a single

position, are defined as

$$G_K^D(z,\bar{z}) \equiv \lim_{z'\to z} [G_K^D((z,\bar{z}),(z',\bar{z}')) + K \log |z-z'|^2] = K \log |z-\bar{z}|^2,$$

$$\begin{aligned} G_{1/4K}^N(z,\bar{z}) &\equiv \lim_{z'\to z} [G_{1/4K}^N((z,\bar{z}),(z',\bar{z}')) + K\log|z-z'|^2] \\ &= -\frac{1}{4K}\log|z-\bar{z}'|^2. \end{aligned}$$

I do not know if there is a standard name for the function  $F((z, \bar{z}), (z', \bar{z}'))$ . It has the following properties: it satisfies Dirichlet b.c. for the first position z and Neumann b.c. for the second position z'. It is harmonic everywhere except at  $z' \to z$ . There it has a winding  $2\pi$  when z rotates around z'. It reads

$$F((z, \bar{z}), (z', \bar{z}')) = \arg[(z - z')(z - \bar{z}')].$$

To summarize, the GFF in the upper half-plane with Dirichlet b.c. is very similar to the GFF in the plane. It is defined in terms of the Green's function of minus the laplacian with Dirichlet b.c.,  $G_K^D(\mathbf{x}, \mathbf{x}')$ , which is easily obtained from the method of images. More generally, all correlation functions of local operators can be obtained from the method of images, using chiral factorization. The results can all be expressed in terms of only three basic objects: the Green's function  $G_K^D(\mathbf{x}, \mathbf{x}')$  and the one of the dual problem  $G_{1/4K}^N(\mathbf{x}, \mathbf{x}')$ , and a "mixed" harmonic function  $F(\mathbf{x}, \mathbf{x}')$  with the above properties.

### A.5 Conformal invariance

Now that I have explained how to deal with the GFF in the plane or in the upper half-plane, it is not a big deal to generalize the theory to other surfaces. This is possible thanks to *conformal invariance*. Correlation functions of the GFF transform covariantly under the very large group of conformal maps. This allows to calculate them on any simply-connected surface, by relating them to the ones in the plane or in the upper half-plane.

#### A.5.1 Conformal maps in 2d

A conformal map (in arbitrary dimensions) is a map from one Riemannian manifold (M, g) to another one (M', g') which preserves the angles (but not necessarily the distances). Locally, a conformal transformation always looks like a rotation times a scaling transformation.



Figure A.3: A conformal transformation f. Locally, it looks like a scaling transformation times a rotation. [Picture from the Wikipedia article on conformal maps.]

In two dimensions, this is more easily expressed using complex coordinates. For local complex coordinates  $(z, \bar{z})$  on M such that the metric g is  $ds^2 = 2g_{z\bar{z}}dzd\bar{z}$ , and local coordinates  $(w,\bar{w})$  on M' such that the metric g'is  $ds^2 = 2g'_{w\bar{w}}dwd\bar{w}$ , the mapping  $(z,\bar{z}) \mapsto (w(z,\bar{z}),\bar{w}(z,\bar{z}))$  is conformal iff it is locally a scaling transformation times a rotation, i.e. iff it is of the form  $w(z,\bar{z}) = \lambda e^{i\theta}z, \ \bar{w}(z,\bar{z}) = \lambda e^{-i\theta}\bar{z}$  for some scaling factor  $\lambda > 0$  and some angle  $\theta$ . This is equivalent to requiring  $\partial_{\bar{z}}w = 0$  and  $\partial_z w \neq 0$ . So, in 2d, and making use of complex coordinates, a map  $(z,\bar{z}) \mapsto (w(z,\bar{z}), \bar{w}(z,\bar{z}))$  is conformal iff  $z \mapsto w$  is holomorphic and its derivative does not vanish.

First example: conformal map from the sphere to the plane. The sphere can be parametrize by the angles  $(\theta, \varphi)$ , with the metric  $ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$ . One can define the complex coordinate z as

$$z = 2e^{i\varphi}\tan(\theta/2),$$

such that the metric is

$$ds^2 = \frac{dz d\bar{z}}{(1+z\bar{z}/4)^2}.$$

. .

Then one has the following map from the sphere —more, precisely, the sphere minus the point at  $\theta = \pi$ — to the plane parametrized by  $(w, \bar{w})$  and equipped with the euclidean metric  $ds^2 = dw d\bar{w}$ :

$$z \mapsto w(z) = z.$$

Of course, in this example the mapping is trivial; the non-trivial step is to find the change of coordinates  $(\theta, \varphi) \mapsto (z, \overline{z})$ .

Second example: from the cylinder to the plane. Consider an infinitely long cylinder of circumference L, parametrized by  $(x, y) \in [0, L] \times \mathbb{R}$ , where the coordinate x is periodic. The metric on that cylinder is the euclidean metric  $ds^2 = dx^2 + dy^2$ . We can use the complex coordinate z with

$$z = x + iy$$

The euclidean metric is simply  $ds^2 = dz d\bar{z}$ . A conformal mapping from the cylinder to the plane —more precisely, the plane minus the point at the origin— is given by

$$z \mapsto w(z) = \exp\left(i\frac{2\pi z}{L}\right).$$

The cylinder is one of the most important surfaces in conformal field theory. It appears both in the basic developments of the theory (in the context of radial quantization), and in many concrete applications. It is the natural geometry that arises in problems with 1d quantum critical systems of finite size, with periodic boundary conditions. It also appears in the context of 1d quantum critical systems of infinite size at finite temperature (in that case L is proportional to the inverse temperature  $\beta$ ).

Third example: special conformal transformations. Instead of trying to conformally map one surface to another, one can ask whether there exist mappings from a surface onto itself. For instance, are there conformal mappings from the plane onto itself (apart from the obvious global scaling transformations and rotations)?

It can be shown that all such conformal mappings are of the form

$$z \mapsto w(z) = \frac{az+b}{cz+d}$$

with  $a, b, c, d \in \mathbb{C}$ . These mappings from the complex plane  $\mathbb{C}$  onto itself are known as the *special conformal transformations*.

An example with a boundary: conformal mapping from the infinite strip to the plane. Another surface which is particularly important in conformal field theory is the infinite strip,  $(x, y) \in [0, L] \times \mathbb{R}$ . It is similar to the infinite cylinder, but the coordinate x is no longer periodic, i.e. the points at x = 0 and x = L are no longer identified.

This is a surface with a boundary. Because it has a boundary, the strip cannot be conformally mapped onto the complex plane (which does not have a boundary), but it can be conformally mapped onto the upper half-plane  $\mathbb{H} = \{w \in \mathbb{C}; \operatorname{Im} w > 0\}.$ 

The mapping reads

$$z \mapsto w(z) = \exp\left(i\frac{\pi z}{L}\right).$$
 (A.46)

The formula looks very similar to the mapping from the cylinder to the plane, however notice the absence of the '2' in the numerator.

The Riemann mapping theorem. So far, I have listed a few particular examples of conformal mappings. It is useful to be aware of general theorems that ensure the existence of such mappings, such as the Riemann mapping theorem or the uniformisation theorem for Riemann surfaces. Let me state the Riemann mapping theorem, which is a fundamental result of complex analysis (for a proof, see for instance S. Lang's "Complex Analysis" or some other textbook):

Theorem. Let U be a non-empty simply connected open subset of the plane  $\mathbb{C}$ , which is not  $\mathbb{C}$  itself. Then there exists a conformal mapping from U onto the upper half-plane  $\mathbb{H}$  which is invertible and whose inverse is also holomorphic.

In practice, this allows one to replace any simply connected surface with a boundary by the upper half-plane, since one is ensured that one can be conformally mapped onto the other. Of course, for many concrete applications in physics, one needs to have a technique to construct such mappings; knowing that they exist is not quite enough. But such techniques exist: for instance, the Schwarz-Christoffel transformation gives a general formula for polygonal domains.

For these Lectures, I will not need complicated mappings; the one from the strip to the upper half-plane will basically be enough.

### A.5.2 Conformal invariance of the GFF

The GFF can be constructed on any surface, as soon as one knows the Green's function of the laplacian on that surface. For instance, if one takes a domain U wich satisfies the hypotheses of the Riemann mapping theorem, then a Green's function on that domain with Dirichlet boundary conditions can be obtained as follows.

# The Green's function with Dirichlet b.c. on arbitrary simply connected domain ${\cal U}$

Let be  $z \mapsto w(z)$  be the conformal bijection from U to  $\mathbb{H}$  provide be the Riemann mapping theorem. Then one can define

$$G^{D}((z,\bar{z}),(z',\bar{z}')) \equiv -K \log \left| \frac{w(z) - w(z')}{w(z) - \bar{w}(\bar{z}')} \right|^{2}.$$
 (A.47)

This is a Green's function for the laplacian with Dirichlet boundary conditions, because  $G^D((z, \bar{z}), (z', \bar{z}'))$  vanishes if z or z' lies on the boundary  $\partial U$ —because then w(z) or w(z') is along the real axis—, and because

$$\begin{aligned} -\frac{1}{4\pi K} \nabla^2 G^D((z,\bar{z}),(z',\bar{z}')) &= -\frac{1}{\pi K} \partial_z \partial_{\bar{z}} G^D((z,\bar{z}),(z',\bar{z}')) \\ &= \frac{1}{\pi} \partial_z \partial_{\bar{z}} \log \left| \frac{w(z) - w(z')}{w(z) - \bar{w}(\bar{z}')} \right|^2 \\ &= \frac{1}{\pi} \left| \frac{\partial w}{\partial z} \right|^2 \partial_w \partial_{\bar{w}} \log \left| \frac{w - w'}{w - \bar{w}'} \right|^2 \\ &= \frac{1}{\pi} \left| \frac{\partial w}{\partial z} \right|^2 \left[ \partial_w \partial_{\bar{w}} \log(w - w') + \partial_w \partial_{\bar{w}} \log(\bar{w} - \bar{w}') \right] \\ &= \frac{1}{\pi} \left| \frac{\partial w}{\partial z} \right|^2 \left[ 0 + \partial_w \frac{1}{\bar{w} - \bar{w}'} \right] \\ &= \left| \frac{\partial w}{\partial z} \right|^2 \delta^{(2)}(w - w', \bar{w} - \bar{w}') \\ &= \delta^{(2)}(z - z', \bar{z} - \bar{z}'). \end{aligned}$$

Then the GFF on U is defined by its two-point function  $\langle h(z, \bar{z})h(z', \bar{z}')\rangle_U \equiv G^D((z, \bar{z}), (z', \bar{z}'))$  and by the fact that all higher-point correlation functions are given by Wick's theorem.

One again, formally this is completely equivalent to defining the GFF on U as a path integral,

$$\langle f[h] \rangle_U = \frac{\int [dh] f[h] e^{-\frac{1}{8\pi K} \int_U d^2 (\nabla h)^2}}{\int [dh] f[h] e^{-\frac{1}{8\pi K} \int_U d^2 (\nabla h)^2}}$$

where the path integral runs over all function h on the domain U which satisfy the Dirichlet b.c. h = 0 along the boundary  $\partial U$ .

#### **Primary operators**

Now that the GFF is defined on arbitrary simply-connected domains U, one can study its correlation functions. In conformal field theory, one defines the so-called *primary operators* as the operators  $\Phi(z, \bar{z})$  which transform as follows under a conformal mapping  $z \mapsto \zeta$  from a domain U to a domain V:

$$\Phi(z,\bar{z}) = \left(\frac{\partial\zeta}{\partial z}\right)^h \left(\frac{\partial\bar{\zeta}}{\partial\bar{z}}\right)^{\bar{h}} \Phi(\zeta,\bar{\zeta}).$$
(A.48)

Here h and  $\bar{h}$  are the conformal dimensions of the operator  $\Phi$ , related to its scaling dimension and spin by  $\Delta = h + \bar{h}$  and  $s = h - \bar{h}$ . As usual, such an identity, valid at the operator level, means that it is true inside any correlation function. In particular, if one has n primary operators, then their correlation functions in U and V are related by

$$\langle \Phi_1(z_1, \bar{z}_1) \dots \Phi_n(z_n, \bar{z}_n) \rangle_U = \prod_j \left( \frac{\partial \zeta_j}{\partial z_j} \right)^{h_j} \left( \frac{\partial \bar{\zeta}_j}{\partial \bar{z}_j} \right)^{\bar{h}_j} \left\langle \Phi_1(\zeta_1, \bar{\zeta}_1) \dots \Phi_n(\zeta_n, \bar{\zeta}_n) \right\rangle_V.$$
 (A.49)

Importantly, if an operator  $\Phi$  is *primary*, then its derivatives  $\partial_z \Phi$ ,  $\partial_{\bar{z}} \Phi$ ,  $\partial_z^2 \Phi$ ,  $\partial_z^2 \Phi$ ,  $\partial_z \partial_{\bar{z}} \Phi$ , etc., cannot be primary, because the derivatives would act on the Jacobian factors in (A.49) and generate additional terms.

The idea is that the operators which transform in that way are *the most local operators* in the theory. All the other local operators are *less local*, and should be obtainable as linear combinations involving derivatives of the primary operators.

In the (compactified) GFF, it turns out that the primary operators are precisely the operators  $\mathcal{V}_{p,q}$ , plus the two components of the gradient  $\nabla_a h$ ,

a = 1, 2—or equivalently  $\partial_z h$  and  $\partial_{\bar{z}} h$ —. All correlation functions involving these primary operators can thus be calculating by relating them to the ones in the upper half-plane, which we have studied above.

Let me briefly explain why  $\mathcal{V}_{p,q}(z,\bar{z})$  satisfies the transformation law (A.48). It is convenient to use chiral factorization and focus only on the chiral part of  $\mathcal{V}_{p,q}(z,\bar{z})$ , see Eq. (A.30). Then one needs to be careful about normal ordering. For the chiral field  $\varphi$ , normal ordering can be defined as

$$: \varphi^0(z) := 1$$
 and  $: \varphi^1(z) := \varphi(z),$ 

and for  $n \geq 2$ ,

$$: \varphi^{n}(z) := \lim_{z' \to z} \left[: \varphi^{n-1}(z) : \varphi(z') + (n-1)\log(z-z') : \varphi^{n-2}(z) :\right].$$

[This definition mimics the one I gave for  $: h^n(z, \bar{z}) :$  above.] The key point is that the conformal map  $z \mapsto \zeta$  does not commute with normal ordering (see also exercise B below). Indeed, the definition leads to

$$\begin{aligned} :\varphi^{0}(\zeta) &:= :\varphi^{0}(z) :\\ :\varphi^{1}(\zeta) &:= :\varphi^{1}(z) :\\ :\varphi^{2}(\zeta) &:= :\varphi^{2}(z) :+ \log\left(\frac{\partial\zeta}{\partial z}\right) \\ :\varphi^{3}(\zeta) &:= :\varphi^{3}(z) :+ 3\log\left(\frac{\partial\zeta}{\partial z}\right)\varphi(z) \\ :\varphi^{4}(\zeta) &:= :\varphi^{4}(z) :+ 6\log\left(\frac{\partial\zeta}{\partial z}\right) :\varphi^{2}(z) :+ 3\left(\log\left(\frac{\partial\zeta}{\partial z}\right)\right)^{2} \end{aligned}$$

and in general

$$:\varphi^n(\zeta):=\sum_{0\leq p\leq \frac{n}{2}}\frac{n!}{2^p\,p!(n-2p)!}(\log{(\partial\zeta/\partial z)})^p:\varphi^{n-2p}(z):.$$

This can be proved by induction on n, using the above definition of normal

ordering. This is exactly what is needed to get

$$: e^{ia\varphi(\zeta)} := \sum_{n\geq 0} \frac{(ia)^n}{n!} : \varphi^n(\zeta) :$$

$$= \sum_{n\geq 0} \frac{(ia)^n}{n!} \sum_{0\leq p\leq \frac{n}{2}} \frac{n!}{2^p p! (n-2p)!} (\log (\partial \zeta/\partial z))^p : \varphi^{n-2p}(z) :$$

$$= \sum_{p\geq 0} \sum_{q\geq 0} \frac{(-a^2/2)^p (\log (\partial \zeta/\partial z))^p}{p!} \frac{(ia)^q}{q!} : \varphi^q(z) :$$

$$= \left(\frac{\partial \zeta}{\partial z}\right)^{-\frac{a^2}{2}} : e^{ia\varphi(z)} :,$$

which is Eq. (A.48) for the chiral part of the operator  $\mathcal{V}_{p,q}(z,\bar{z})$ . [To go from the second to the third line, I introduce the variable q = n - 2p.]

### A.5.3 Application: general correlation function of the compactified GFF on the infinite strip

As an application of formula (A.49), let me derive the formula for the correlation function  $\langle \mathcal{V}_{p_1,q_1}(z_1,\bar{z}_1)\ldots\mathcal{V}_{p_n,q_n}(z_n,\bar{z}_n)\rangle_{\text{strip}}$  in the infinite strip  $(x,y) \in [0,L] \times \mathbb{R}$ , which is needed for these Lectures. Again, in order for this correlation function to be non-zero, the total "magnetic charge"  $\sum_j q_j$  must vanish for Dirichlet b.c. (contrary to the "electric charge" which does not need to vanish).

Recall that the conformal mapping from the strip to the upper-half plane

is given by  $z = x + iy \mapsto w = e^{i\frac{\pi z}{L}}$ . Then

$$\begin{split} &\langle \mathcal{V}_{p_{1},q_{1}}(z_{1},\bar{z}_{1})\dots\mathcal{V}_{p_{n},q_{n}}(z_{n},\bar{z}_{n})\rangle_{\text{strip}} \\ &= \prod_{j=1}^{n} \left(\frac{dw}{dz}\right)^{h_{j}} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{h}_{j}} \langle \mathcal{V}_{p_{1},q_{1}}(z_{1},\bar{z}_{1})\dots\mathcal{V}_{p_{n},q_{n}}(z_{n},\bar{z}_{n})\rangle_{\mathbb{H}} \\ &= \prod_{j=1}^{n} \left(\frac{dw}{dz}\right)^{\frac{1}{2}(p_{j}\sqrt{K}-\frac{q_{j}}{2\sqrt{K}})^{2}} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\frac{1}{2}(p_{j}\sqrt{K}+\frac{q_{j}}{2\sqrt{K}})^{2}} \\ &\times \prod_{j$$

In fact, this result is of the same form as the formula (A.45) that we found above for correlations in the upper half-plane, namely

$$\langle \mathcal{V}_{p_{1},q_{1}}(z_{1},\bar{z}_{1})\dots\mathcal{V}_{p_{n},q_{n}}(z_{n},\bar{z}_{n})\rangle_{\text{strip}} = \exp\left(-\sum_{l} \left[\frac{p_{l}^{2}}{2}G_{K}^{D}(z_{l},\bar{z}_{l}) + \frac{q_{l}^{2}}{2}G_{1/4K}^{N}(z_{l},\bar{z}_{l})\right]\right) \\ \times \exp\left(-\sum_{j(A.50)$$

The Green's functions in the strip are given by

$$\begin{aligned} G_K^D((z,\bar{z}),(z',\bar{z}')) &\equiv -K \log \left| \frac{\sin \frac{\pi(z-z')}{2L}}{\sin \frac{\pi(z+\bar{z}')}{2L}} \right|^2, \\ G_{1/4K}^N((z,\bar{z}),(z',\bar{z}')) &\equiv -\frac{1}{4K} \log \left| \frac{4L^2}{\pi^2} \sin \frac{\pi(z-z')}{2L} \sin \frac{\pi(z+\bar{z}')}{2L} \right|^2, \end{aligned}$$

and their regularized parts are

$$\begin{aligned} G_K^D((z,\bar{z})) &\equiv \lim_{z' \to z} \left[ G_K^D((z,\bar{z}),(z',\bar{z}')) + K \log |z - z'|^2 \right] \\ &= K \log \left| \frac{2L}{\pi} \sin \frac{\pi(z + \bar{z}')}{2L} \right|^2, \\ G_{1/4K}^N((z,\bar{z})) &\equiv \lim_{z' \to z} \left[ G_{1/4K}^N((z,\bar{z}),(z',\bar{z}')) + K \log |z - z'|^2 \right] \\ &= -\frac{1}{4K} \log \left| \frac{2L}{\pi} \sin \frac{\pi(z + \bar{z}')}{2L} \right|^2. \end{aligned}$$

The function  $F((z, \overline{z}), (z', \overline{z'}))$  is harmonic in z and in z' everywhere except at  $z' \to z$ . It satisfies Dirichlet b.c. for the first variable and Neumann b.c. for the second variable:

$$F((z,\bar{z}),(z',\bar{z}')) \equiv \arg\left[\sin\frac{\pi(z-z')}{2L}\sin\frac{\pi(\bar{z}+\bar{z}')}{2L}\right].$$

These formulas in the infinite strip are useful in these lecture notes because they give access to all ground state correlations.

# A.6 Exercices

Warning: the exercises B, C, D are *not* independent, you need to do them *in that order*.

#### A. Normal ordering and correlator of vertex operators.

- 1. Using the definition of normal-ordering and Wick's theorem, derive Eq. (A.15) for n = 2 points.
- Consider the GFF in the plane. Find a simple extension of the definition of normal ordering (A.14) which also applies to expectation values at different points. In other words, find a meaningful definition of : h(x<sub>1</sub>)h(x<sub>2</sub>) :, : h(x<sub>1</sub>)h(x<sub>2</sub>)h(x<sub>3</sub>) :, : h(x<sub>1</sub>)h(x<sub>2</sub>)h(x<sub>3</sub>) :, : h(x<sub>1</sub>)h(x<sub>2</sub>)h(x<sub>4</sub>) :, etc.

3. Take two functions  $f_1(\mathbf{x})$  and  $f_2(\mathbf{x})$  which have disjoint supports, i.e.  $f_1(\mathbf{x})f_2(\mathbf{x}) = 0$  for all  $\mathbf{x}$ . Using your definition of normal ordering at different points, prove the following identity,

$$: e^{\int d^2 \mathbf{x} f_1(\mathbf{x}) h(\mathbf{x})} :: e^{\int d^2 \mathbf{x}' f_2(\mathbf{x}') h(\mathbf{x}')} := \\ \exp\left(\int d^2 \mathbf{x} \, d^2 \mathbf{x}' f_1(\mathbf{x}) G(\mathbf{x}, \mathbf{x}') f_2(\mathbf{x}')\right) : e^{\int d^2 \mathbf{x} (f_1(\mathbf{x}) + f_2(\mathbf{x})) h(\mathbf{x})} :,$$

where  $G(\mathbf{x}, \mathbf{x}')$  is the Green's function (A.11).

4. Use this to prove formula (A.15) for arbitrary n.

#### B. The conformal anomaly.

Short-distance regularization (normal ordering) does not generally commute with conformal transformations. This typically results in anomalous transformation laws for normal-ordered operators. The goal of this exercise is to derive the anomalous transformation law of the stress-tensor.

The chiral component of the stress-tensor is given by  $T(z) = -\frac{1}{2K}$ :  $(\partial_z h)^2$ :. Its anti-chiral component is  $\overline{T}(\overline{z}) = -\frac{1}{2K} : (\partial_{\overline{z}} h)^2$ :.

- 1. Consider a conformal transformation  $z \mapsto w(z)$ . How is  $\partial_w h$  related to  $\left(\frac{dz}{dw}\right) \partial_z h$ ? [This question is very simple, it is just to prepare for the next one...]
- 2. Write carefully the definition of T(w), and the definition of  $\left(\frac{dz}{dw}\right)^2 T(z)$ . Then express the difference  $T(w) - \left(\frac{dz}{dw}\right)^2 T(z)$  as a certain limit.
- 3. Evaluate that limit and show that the difference  $T(w) \left(\frac{dz}{dw}\right)^2 T(z)$  can be expressed in a compact way in terms of the Schwarzian derivative

$$S(z;w) \equiv \left(rac{d^3z}{dw^3}{dz\over dw}
ight) - rac{3}{2} \left(rac{d^2z}{dw^2}{dz\over dw}
ight)^2.$$

[Remark: the result you have arrived at is valid beyond the free boson CFT (or GFF). Multiply your result by a constant "c" called the *central charge*, and this is the general transformation law of the stress-tensor in *all* CFTs. It is a very fundamental law on which the entire formalism of 2d conformal field theory is relying.]

#### C. CFT partition functions on the cylinder.

Consider a very long cylinder  $(x, y) \in [0, L] \times [-\ell/2, \ell/2]$ , where x is the periodic direction (i.e. all points at x = 0 and x = L are identified). In the

limit  $\ell \to \infty$  this is the infinite cylinder considered in the text, which can be conformally mapped to the plane minus the point at the origin. For most calculations, considering the case of the infinite cylinder would be enough. However in this exercise one is interested in the free energy  $F = -\log Z$ (where Z is the partition function on the cylinder), which is expected to be proportional to  $\ell$ , so it is useful to consider  $\ell$  large but finite.

1. It can be shown that the CFT free energy  $F(\ell, L) = -\log Z(\ell, L)$  on the very long cylinder satisfies the following differential equation (for fixed L):

$$\frac{dF}{d\ell} = -\frac{1}{2\pi} \int_0^L dx [\langle T(z) \rangle + \text{c.c.}],$$

where T(z) is the holomorphic component of the stress-tensor from the previous exercise, and 'c.c.' means 'complex conjugate'.

Using the anomalous transformation law of the stress-tensor you found in the previous exercise, the conformal map  $z \mapsto w(z)$  from the infinite cylinder to the plane, and the fact that  $\langle T(w) \rangle_{\mathbb{C}} = 0$  in the plane, show that the partition function goes as

$$Z \sim \exp\left(\frac{\pi c}{6}\frac{\ell}{L}\right)$$

when  $\ell \gg L$ . Here c is the central charge of the CFT (for the free boson or GFF, c = 1).

2. Imagine now a primary operator  $\Phi(x, y)$  is inserted at  $y = +\frac{\ell}{2}$ , and another one at  $y = -\frac{\ell}{2}$  (say at x = 0 for both). This changes the partition function from Z to  $Z_{\Phi}$ , where

$$Z_{\Phi} = \langle \Phi(0, \ell/2) \Phi(0, -\ell/2) \rangle_{\text{cylinder}} \times Z.$$

Using the conformal mapping from the infinite cylinder to the plane, calculate the two-point function of the primary operator  $\Phi$  on the cylinder. From this, deduce the asymptotic behavior of  $Z_{\Phi}$  when  $\ell \gg L$ , in terms of the central charge c and of the scaling dimension  $\Delta_{\Phi}$ .

#### D. Conformal scaling of Renyi entropies.

The Renyi entropy of the reduced density matrix  $\rho_A$  of an interval A = [0, x] in an infinitely long 1d quantum critical system can be expressed as the two-point function of a certain primary operator in a CFT in the plane:

$$S_n(x) \equiv \frac{1}{1-n} \log[\operatorname{tr} \rho_A^n] = \frac{1}{1-n} \log \langle \Phi_n(0,0) \Phi_n(x,0) \rangle$$

(see for instance the classic paper of Calabrese and Cardy for a derivation). Here the CFT in which the two-point function is to be evaluated is a CFT with central charge  $n \times c$ , where c is the central charge of the "basic" CFT that captures the long-range correlations of local observables at that quantum critical point.

- 1. Using scale invariance, express  $\langle \Phi_n(0,0)\Phi_n(x,0)\rangle$  in terms of x and of the scaling dimension  $\Delta_n$  of the operator  $\Phi_n$ .
- 2. It remains to evaluate the scaling dimension  $\Delta_n$ . To do this, one can use the results of the previous exercise, which relate the scaling dimension of a primary operator  $\Phi$  to the asymptotic behavior of a ratio of partition functions  $Z_{\Phi}/Z$  on a long cylinder.

What is the asymptotic behavior of the partition function Z on a long cylinder when  $\ell \gg L$  (without insertions of  $\Phi$ ) now?

3. Assume that the effect of the insertion of the primary operator  $\Phi_n$  at the ends of the long cylinder is to change the geometry of the cylinder, in the following sense. Without the insertions of  $\Phi$  at  $y = \pm \ell/2$ , one has a CFT with central charge nc on a cylinder of circumference L. With those insertions, one has a CFT with central charge c on a cylinder of circumference nL.

What is the asymptotic behavior of the partition function  $Z_{\Phi_n}$  when  $\ell \gg L$ ?

4. From that, deduce the scaling dimension  $\Delta_n$ , and write your final result for the Renyi entropy  $S_n(x)$ .
## Further reading

On general aspects of classical hydrodynamics, I would recommend some of the historical references,

- Landau and Lifschitz, Vol. 6, Fluid Mechanics (1959).
- Kadanoff and Martin, "Hydrodynamic equations and correlation functions", Annals of Physics 24, 419-469 (1963).

On the lagrangian of hydrodynamics, the reader can also have a look at

• Itô, "Variational Principle in Hydrodynamics", Progress of Theoretical Physics, 9, 117–131 (1953).

I also recommend the following lecture notes of A. Abanov, which I personally find very inspiring,

• Abanov, "Hydrodynamics of correlated systems", Les Houches Lecture Notes (2004).

On microscopic fluctuations in classical systems, see for instance

 Lazarescu, "The Physicist's Companion to Current Fluctuations: One-Dimensional Bulk-Driven Lattice Gases", J. Phys. A: Math. Theor. 48, 503001 (2015).

On theoretical aspects of 1d quantum gases, a good review is

• Cazalilla, Citro, Giamarchi, Orignac, Rigol "One dimensional Bosons: From Condensed Matter Systems to Ultracold Gases" (2011).

Classic textbooks with an introduction to Luttinger liquids include

- Giamarchi, Quantum Physics in One Dimension (2003),
- Tsvelik, Quantum Field Theory in Condensed Matter Physics (2003).

For introductions to conformal field theory, see for instance

- Ginsparg, Les Houches Lecture Notes on Applied Conformal Field Theory (1988),
- Polchinski, String Theory, Vol. 1, Chapter 2 (1998),
- Mussardo, Statistical Field Theory (2010).

For introductions to the Bethe Ansatz and to the exact solution of the Lieb-Liniger model, see for instance

- Korepin, Bogoliubov and Izergin, Quantum Inverse Scattering Method and Correlation Functions, Chapter 1 (1993),
- Gaudin, The Bethe Wavefunction (1983) —English translation by J.-S. Caux (2014)—,
- Franchini, "An introduction to integrable techniques for one-dimensional quantum systems" (2016).