

Chapter 3 of the book :
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LUTTINGER LIQUIDS

You can know the name of a bird in all the languages of the world, but when you're finished, you'll know absolutely nothing whatever about the bird. You'll only know about humans in different places and what they call the bird. So let's look at the bird and see what it's doing — that's what counts.

Richard Feynman, quoting his father

In the previous chapter, we have seen, for fermions, a careful derivation of the bosonization formulas. The bosonization technique allows to solve the interacting problem and to extract a certain number of properties such as power law behavior of the correlation functions. However, this derivation is, strictly speaking, based on a linearization of the spectrum close to the Fermi energy. One can thus wonder what will remain of the results of the previous chapter when it is impossible to stay confined close to the Fermi level, for example, when the interactions become strong. In order to answer this question let us rederive the bosonization formulas, in a more phenomenological but more general way (Haldane, 1981*b*). I also give in this chapter a more physical interpretation for most of the results that were obtained in a rigorous way in the previous chapter. This allows to show that a similar concept as Fermi liquids exists in one dimension. This concept called Luttinger liquid encompasses most of the physics of simple one-dimensional systems and will be the starting point to study more complex situations.

This chapter is very simple technically and emphasizes the physical interpretations of the various formulas. Mathematical rigor has been set aside. If there is only one chapter you should read this is the one. You can then go back to the previous chapters to have the dots on the i and the crosses on the t of the bosonization.

3.1 Phenomenological bosonization

Let us start with any one-dimensional system (fermions or bosons). The density operator of such a system is

$$\rho(x) = \sum_i \delta(x - x_i) \quad (3.1)$$

where x_i is the position operator of the i th particle. Let us label the position of the i th particle by an 'equilibrium' position R_i^0 that the particle would occupy

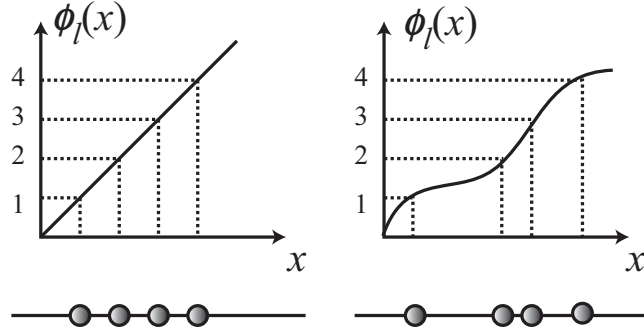


FIG. 3.1. Some examples of the labelling field $\phi_l(x)$. If the particles form a perfect lattice of lattice spacing d , then $\phi_l^0(x) = 2\pi x/d$, and is just a straight line. Different functions $\phi_l(x)$ allow to put the particles at any position in space.

if the particles were forming a perfect crystalline lattice, and the displacement u_i relative to this equilibrium position. Thus,

$$x_i = R_i^0 + u_i \quad (3.2)$$

If ρ_0 is the average density of particles, $d = \rho_0^{-1}$ is the distance between the particles. Then, the equilibrium position of the i th particle is

$$R_i^0 = di \quad (3.3)$$

The density operator written as (3.1) is not very convenient. To rewrite it in a more pleasant form we introduce a labelling field $\phi_l(x)$ (Haldane, 1981b). This field, which is a continuous function of the position, takes the value $\phi_l(x_i) = 2\pi i$ at the position of the i th particle. It can thus be viewed as a way to number the particles. Since in one dimension, contrary to higher dimensions, one can always number the particles in a unique way (e.g. starting at $x = -\infty$ and processing from left to right), this field is always well-defined. Some examples are shown in Fig. 3.1. Using this labelling field and the rules for transforming δ functions

$$\delta(f(x)) = \sum_{\text{zeros of } f} \frac{1}{|f'(x_i)|} \delta(x - x_i) \quad (3.4)$$

one can rewrite the density as

$$\begin{aligned} \rho(x) &= \sum_i \delta(x - x_i) \\ &= \sum_n |\nabla \phi_l(x)| \delta(\phi_l(x) - 2\pi n) \end{aligned} \quad (3.5)$$

It is easy to see from Fig. 3.1 that $\phi_l(x)$ can always be taken as an increasing function of x , which allows to drop the absolute value in (3.5). Using the Poisson summation formula this can be rewritten

$$\rho(x) = \frac{\nabla\phi_l(x)}{2\pi} \sum_p e^{ip\phi_l(x)} \quad (3.6)$$

where p is an integer. It is convenient to define a field ϕ relative to the perfect crystalline solution and to introduce

$$\phi_l(x) = 2\pi\rho_0x - 2\phi(x) \quad (3.7)$$

The density becomes

$$\rho(x) = \left[\rho_0 - \frac{1}{\pi} \nabla\phi(x) \right] \sum_p e^{i2p(\pi\rho_0x - \phi(x))} \quad (3.8)$$

Since the density operators at two different sites commute it is normal to expect that the field $\phi(x)$ commutes with itself. Note that if one averages the density over distances large compared to the interparticle distance d all oscillating terms in (3.8) vanish. Thus, only $p = 0$ remains and the smeared density is

$$\rho_{q \sim 0}(x) \simeq \rho_0 - \frac{1}{\pi} \nabla\phi(x) \quad (3.9)$$

We can go further by looking at the single-particle creation operator $\psi^\dagger(x)$. Such an operator can always be written as

$$\psi^\dagger(x) = [\rho(x)]^{1/2} e^{-i\theta(x)} \quad (3.10)$$

where $\theta(x)$ is some operator. The (anti-) commutation relations between the ψ impose some commutation relations between the density operators and the $\theta(x)$. For bosons, the condition is

$$[\psi_B(x), \psi_B^\dagger(x')] = \delta(x - x') \quad (3.11)$$

Using (3.10) the commutator gives

$$e^{+i\theta(x)} [\rho(x)]^{1/2} [\rho(x')]^{1/2} e^{-i\theta(x')} - [\rho(x')]^{1/2} e^{-i\theta(x')} e^{+i\theta(x)} [\rho(x)]^{1/2} \quad (3.12)$$

If we assume quite reasonably that the field θ commutes with itself ($[\theta(x), \theta(x')] = 0$), the commutator (3.12) is obviously zero for $x \neq x'$ if (for $x \neq x'$)

$$[[\rho(x)]^{1/2}, e^{-i\theta(x')}] = 0 \quad (3.13)$$

A sufficient condition to satisfy (3.11) would thus be

$$[\rho(x), e^{-i\theta(x')}] = \delta(x - x') e^{-i\theta(x')} \quad (3.14)$$

How to satisfy the above relation? One can guess the answer by replacing the exact density in (3.14) by the smeared density (3.9). Physically, one expects that

most of the commutator should come from this term since the fields $\phi(x)$ and $\theta(x)$ are expected to vary slowly at the scale of the interparticle distance ρ_0^{-1} . If the density is only the smeared density then (3.14) is obviously satisfied if

$$\left[\frac{1}{\pi}\nabla\phi(x), \theta(x')\right] = -i\delta(x-x') \quad (3.15)$$

as can be directly checked by using (A.9). The commutation relation (3.15) implies directly that the commutator between ϕ and θ is of the form (2.24) (see also the discussion in Section 3.2). The higher harmonics in the density give commutators of the form

$$\begin{aligned} [e^{-i2p\phi(x)}, e^{-i\theta(x')}] &= e^{-i2p\phi(x)} e^{-i\theta(x')} (1 - e^{2p[\phi(x), \theta(x')]}]) \\ &= e^{-i2p\phi(x)} e^{-i\theta(x')} (1 - e^{ip\pi\text{Sign}(x'-x)}) \end{aligned} \quad (3.16)$$

Thus, for $x = x'$ all higher harmonics vanish (remember that the function $\text{Sign}(x)$ in (2.24) is regularized to give zero at $x = 0$). Equation (3.14) works perfectly. This is not the case when $x \neq x'$ where odd harmonics remain. However, since these terms are multiplied by the oscillating factor $e^{i2p\pi\rho_0 x}$ we can expect them to play no role in the continuum limit. Note in particular that

$$[\rho(x), e^{-i2\theta(x')}] = 2\delta(x-x')e^{-i2\theta(x')} \quad (3.17)$$

is *perfectly* satisfied. So let us proceed for the moment by assuming that (3.15) is indeed the condition that ϕ and θ need to satisfy. Equation (3.15) proves that θ and $\frac{1}{\pi}\nabla\phi$ are canonically conjugate. Note that for the moment this results from totally general considerations and does not rest on a given microscopic model. Integrating by part (3.15) shows that

$$\pi\Pi(x) = \nabla\theta(x) \quad (3.18)$$

where $\Pi(x)$ is the canonically conjugate momentum to $\phi(x)$.

To obtain the single-particle operator one can substitute (3.8) into (3.10). Since the square root of a delta function is also a delta function up to a normalization factor the square root of ρ is identical to ρ up to a normalization factor that depends on the ultraviolet structure of the theory. Thus,

$$\psi_B^\dagger(x) = [\rho_0 - \frac{1}{\pi}\nabla\phi(x)]^{1/2} \sum_p e^{i2p(\pi\rho_0 x - \phi(x))} e^{-i\theta(x)} \quad (3.19)$$

where the index B emphasizes that this is the representation of a *bosonic* creation operator. How to modify the above formulas if we have fermions instead of bosons? The density can obviously be expressed in the same way in terms of the field ϕ . For the single-particle operator one has to satisfy an anticommutation relation instead of (3.11). We thus have to introduce in representation (3.10) something that introduces the proper minus sign when the two fermions operators are commuted. This is known as a Jordan–Wigner transformation and I will

come back to it when looking at spin chains in Chapter 6. Here, the operator to add is easy to guess. Since the field ϕ_l has been constructed to be a multiple of 2π at each particle, $e^{i\frac{1}{2}\phi_l(x)}$ oscillates between ± 1 at the location of consecutive particles. The Fermi field can thus be easily constructed from the boson field (3.10) by

$$\psi_F^\dagger(x) = \psi_B^\dagger(x) e^{i\frac{1}{2}\phi_l(x)} \quad (3.20)$$

This can be rewritten in a form similar to (3.10) as

$$\psi_F^\dagger(x) = [\rho_0 - \frac{1}{\pi} \nabla \phi(x)]^{1/2} \sum_p e^{i(2p+1)(\pi\rho_0 x - \phi(x))} e^{-i\theta(x)} \quad (3.21)$$

The above formulas are a way to represent the excitations of the system directly in terms of variables defined in the continuum limit. It is thus well suited to describe the asymptotic properties of the system.

In addition to the small oscillations, the fields ϕ and θ can have topological excitations. For periodic boundary conditions $\psi(x+L) = \psi(x)$, this implies that, for bosons, $\theta(x+L) = \theta(x) + \pi J$ where J is an even integer. In a similar way the periodicity condition on the density operator imposes $\phi(x+L) = \phi(x) + \pi N$ where N is an integer (Haldane, 1981*b*; Mironov and Zabrodin, 1991). Because of the extra operator for fermions, the boundary condition is now $N+J$ even. Compare (3.21) with the bosonization formula (2.30) of the previous chapter, derived for the Luttinger model. For spinless fermions $k_F/\pi = \rho_0$, thus (3.21) and (3.8) are a generalization of (2.30) and (2.62). They contain the $q \sim 0$ and $q \sim 2k_F$ components (for the density) and $q = \pm k_F$ (for the single-particle) components that are present in the Luttinger model. But in addition they contain all the higher harmonics. The absence of higher harmonics in the Luttinger model is an artefact of the strictly linear dispersion relation. I will come back to this point later.

The field ϕ and θ that we have derived in this chapter have a simple canonical commutation relation (ϕ and $\Pi = \nabla\theta/\pi$ are canonically conjugate). Irrespective of any Hamiltonian ϕ and Π thus have a simple expression in terms of bosonic operators b_p and b_p^\dagger (one boson per momentum mode p since $\phi(x)$ depends on space). Since ϕ is real it should contain both b and b^\dagger . It is exactly equivalent to a standard harmonic oscillator representation (Mahan, 1981). Thus, the fields can be expressed in terms of the bosons b_p and b_p^\dagger as shown in (2.23) in the previous chapter. As before the bosons represent the small oscillations of the density. The topological numbers N and J of this chapter are analogous to $N_+ + N_-$ (total number of particles) and $N_+ - N_-$ (total current) of the previous chapter. Here, the parameter α of (2.23) is a cutoff to regularize the theory at large momentum. It is of the order of the distance above which the continuum description applies, namely here the interparticle distance $d = \rho_0^{-1}$.

What is the Hamiltonian of the system? It should be written only in terms of the variables $\phi(x)$ and $\theta(x)$. It is easy to see that $(\nabla\phi(x))^2$ terms should be present. These terms would come, for example, from the interaction $\int dx \rho(x)^2$.

Similarly $(\nabla\theta(x))^2$ terms should also be there. For example, for bosons, they directly come from the kinetic energy

$$H_K = \int dx \frac{1}{2m} (\nabla\psi^\dagger(x))(\nabla\psi(x)) \quad (3.22)$$

Using

$$\psi^\dagger(x) = \rho_0^{1/2} e^{-i\theta(x)} \quad (3.23)$$

which is the part of the single-particle operator (3.10) containing less powers of $\nabla\phi$ and thus the most relevant, one gets

$$H_K = \int dx \frac{\rho_0}{2m} (\nabla\theta(x))^2 \quad (3.24)$$

The coefficient cannot be obtained reliably since both terms coming from the interactions and renormalization from irrelevant operators can change it. But the important result is that the most relevant terms in the Hamiltonian, that is, the ones that will control the low-energy properties are $(\nabla\theta(x))^2$ and $(\nabla\phi(x))^2$. It is easy to check that cross terms cannot appear. Indeed, if the system has an inversion symmetry, the energy is invariant by $x \rightarrow -x$. Changing $\rho(x) \rightarrow \tilde{\rho}(-x)$ and $\psi(x) \rightarrow \tilde{\psi}(-x)$ imposes that $\phi(x) = -\tilde{\phi}(-x)$ and $\theta(x) = \tilde{\theta}(-x)$. And thus one has $\nabla\phi(x) = \nabla\tilde{\phi}(-x)$ and $\nabla\theta(x) = -\nabla\tilde{\theta}(-x)$. This is nothing but the conservation of density and as we will soon see the inversion of the current. In such an inversion a cross term $\nabla\phi\nabla\theta$ would change sign and thus cannot appear in the energy. Thus, the most general Hamiltonian describing the low-energy properties of a *massless* one-dimensional system is

$$H = \frac{\hbar}{2\pi} \int dx \left[\frac{uK}{\hbar^2} (\pi\Pi(x))^2 + \frac{u}{K} (\nabla\phi(x))^2 \right] \quad (3.25)$$

where I have put back the \hbar for completeness. This leads to the action (see Appendix C)

$$S/\hbar = \frac{1}{2\pi K} \int dx d\tau \left[\frac{1}{u} (\partial_\tau\phi)^2 + u (\partial_x\phi(x))^2 \right] \quad (3.26)$$

Parameters u and K are used to parameterize the two unknown coefficients in front of the two operators. These two coefficients *totally* characterize the low-energy properties of *any* massless one-dimensional system. In the absence of a good perturbation theory (e.g. in the interaction), it is difficult to compute these coefficients. Nevertheless, once they are fixed, all properties of the system are determined.

This is a very important step. It shows that *all* the bosonization formulas that we have established in the previous chapter are in fact non-perturbative. They are totally generic (provided all the harmonics are included). That was the only artefact of the Tomonaga–Luttinger model. Otherwise, all the properties, representation in terms of bosons of the various operators, calculation of the

correlation functions, phase diagram, etc., are generic and apply to *any* one-dimensional model. The bosonic representation and Hamiltonian (3.25) play in fact the same role for one-dimensional systems than the Fermi liquid theory plays for higher-dimensional systems. It is an effective low-energy theory that is the fixed point of all massless theories.¹⁶ This theory, which is known as Luttinger liquid theory, depends only on the two parameters u and K . Provided that the correct value of these parameters are used, *all* asymptotic properties of the correlation functions of the system can be obtained *exactly* using (3.8) and (3.10) for bosons (or (3.21) for fermions).

Let me be more specific and show explicitly how to compute the correlation functions, using the standard operator technique. A calculation using the functional integral is given in Appendix C. Let us compute

$$G_{\phi\phi}(x, \tau) = \langle [\phi(x, \tau) - \phi(0, 0)]^2 \rangle \quad (3.27)$$

Let us absorb the factor K in the Hamiltonian by rescaling the fields (this preserves the commutation relation)

$$\begin{aligned} \phi &= \sqrt{K} \tilde{\phi} \\ \theta &= \frac{1}{\sqrt{K}} \tilde{\theta} \end{aligned} \quad (3.28)$$

The fields $\tilde{\phi}$ and $\tilde{\theta}$ are expressed in terms of bosons operator as (2.23). It is easy to check (see the previous chapter) that Hamiltonian (3.25) with $K = 1$ is simply

$$\tilde{H} = \sum_{p \neq 0} u|p| b_p^\dagger b_p \quad (3.29)$$

This can be checked by a direct substitution of (2.23) in (3.25). In the limit $L \rightarrow \infty$ topological excitations are not important and can be ignored, so only the bosonic part remains. If $K \neq 1$ had still been there in the Hamiltonian the boson expression would have contained the unpleasant terms bb and $b^\dagger b^\dagger$. These terms should then have been diagonalized by a Bogoliubov transformation (Mahan, 1981) on the bosons. The rescaling of the fields is thus the equivalent of this Bogoliubov transformation and much simpler. The time dependence of the field can now be easily computed from (3.29), (2.23), and (A.25). This gives

$$\phi(x, \tau) = -\frac{i\pi}{L} \sum_{p \neq 0} \left(\frac{L|p|}{2\pi} \right)^{1/2} \frac{1}{p} e^{-\alpha|p|/2 - ipx} (b_p^\dagger e^{u|p|\tau} + b_{-p} e^{-u|p|\tau}) \quad (3.30)$$

Beware that for operators the average $\langle \rangle$ means time-ordered product (see Appendix A) so just for the next few lines I put back explicitly the time-ordered

¹⁶There are of course some exceptions, otherwise life would be too simple.

product and denote $\langle \rangle_0$ the averages without the time-ordered product, that is, simply $\text{Tr}[e^{-\beta(H-\mu N)} \dots]/Z$

$$\begin{aligned} G_{\phi\phi}(x, \tau) &= K \langle T_\tau [\tilde{\phi}(x, \tau) - \tilde{\phi}(0, 0)]^2 \rangle_0 \\ &= 2K [\langle \tilde{\phi}(0, 0) \tilde{\phi}(0, 0) \rangle_0 - Y(\tau) \langle \tilde{\phi}(x, \tau) \tilde{\phi}(0, 0) \rangle_0 \\ &\quad - Y(-\tau) \langle \tilde{\phi}(0, 0) \tilde{\phi}(x, \tau) \rangle_0] \end{aligned} \quad (3.31)$$

where Y is the step function (see Appendix A). One then plugs (3.30) in (3.31). The calculation is thus reduced to the averages of factors such as

$$\langle b_p^\dagger b_{p'} \rangle_0 = \delta_{p,p'} f_B(\epsilon_p = u|p|) \quad (3.32)$$

and factors such as $bb^\dagger = 1 - b^\dagger b$ that can be easily reduced to the above form. f_B is the standard Bose factor. At $T = 0$ since $\epsilon_p > 0$ (remember that $p \neq 0$ for the bosons modes) $f_B(\epsilon_q) = 0$. Thus, (3.31) becomes (taking the standard limit $L \rightarrow \infty$)

$$\begin{aligned} G_{\phi\phi}(x, \tau) &= K \int_0^\infty \frac{dp}{p} e^{-\alpha p} [1 - e^{-u|\tau|p} \cos(px)] \\ &= \frac{K}{2} \log \left[\frac{x^2 + (u|\tau| + \alpha)^2}{\alpha^2} \right] \end{aligned} \quad (3.33)$$

Thus, up to the small cutoff α , this is essentially $\log(r)$ where r is the distance in space-time. This invariance by rotation in space-time reflects the Lorentz invariance of the action. One can introduce

$$\begin{aligned} r &= \sqrt{x^2 + y_\alpha^2} \\ y_\alpha &= u\tau + \alpha \text{Sign}(\tau) \end{aligned} \quad (3.34)$$

The same calculation with θ instead of ϕ gives exactly the same result with $1/K$ instead of K . One can either do it directly or notice that the Hamiltonian is invariant by $\phi \rightarrow \theta$ and $K \rightarrow 1/K$. Calculation of the $\phi\theta$ correlation function can be done exactly in the same way to give

$$\begin{aligned} G_{\phi\theta}(x, \tau) &= \langle T_\tau \phi(x, \tau) \theta(0, 0) \rangle_0 \\ &= -\frac{i}{2} \text{Sign}(\tau) \int_0^\infty \frac{dp}{p} e^{-\alpha p} e^{-u|\tau|p} \sin(px) \\ &= -\frac{i}{2} \text{Sign}(\tau) \arctan \left[\frac{x}{u|\tau| + \alpha} \right] \end{aligned} \quad (3.35)$$

Since the time ordering product has a minus sign for fermions and a plus sign for bosons when permuting two operators one should in fact add an extra phase (see Appendix C) when computing correlation functions of objects such as (3.21). As explained in Appendix C this changes (3.35) into

$$G_{\phi\theta}(x, \tau) = -i \text{Arg}(y_\alpha - ix) \quad (3.36)$$

where Arg is the argument of the complex number with a cut on the negative axis (i.e. for $a < 0$ $\text{Arg}(a \pm i\epsilon) = \pm\pi$). This coincides with (3.35) for $x > 0$

and $\tau > 0$ but is different for $x < 0$ and $\tau < 0$. Since this extra phase is $\pm\pi$ it disappears in all other correlation functions of a *physical* quantity since the ϕ and θ fields appear with integer coefficients. One can thus always take (i.e. for $a < 0$ $\text{Arg}(a \pm i\epsilon) = \pm\pi$) (3.36) for the $\phi - \theta$ correlations. More details and the expressions at finite temperature are given in Appendix C.

To compute exponentials of the fields one simply uses that for an operator A that is *linear* in terms of boson fields and a quadratic Hamiltonian one has (see Appendix C)

$$\langle e^A \rangle = e^{\frac{1}{2}\langle A^2 \rangle} \quad (3.37)$$

Thus, for example

$$\begin{aligned} \langle e^{i2\phi(x,\tau)} e^{-i2\phi(0,0)} \rangle &= e^{-2\langle [\phi(x,\tau) - \phi(0,0)]^2 \rangle} \\ &= e^{-2G_{\phi\phi}(x,\tau)} \end{aligned} \quad (3.38)$$

If from (3.8) we compute

$$\langle \rho(x,\tau)\rho(0) \rangle \quad (3.39)$$

we obtain, using (3.8) and the formulas of Appendix C (compare with (2.68))

$$\begin{aligned} \langle \rho(x,\tau)\rho(0) \rangle &= \rho_0^2 + \frac{K}{2\pi^2} \frac{y_\alpha^2 - x^2}{(x^2 + y_\alpha^2)^2} + \rho_0^2 A_2 \cos(2\pi\rho_0 x) \left(\frac{\alpha}{r}\right)^{2K} \\ &\quad + \rho_0^2 A_4 \cos(4\pi\rho_0 x) \left(\frac{\alpha}{r}\right)^{8K} + \dots \end{aligned} \quad (3.40)$$

Here, the lowest distance in the theory is $\alpha \sim \rho_0^{-1}$. The amplitudes A_i are non-universal objects. They depend on the precise microscopic model, and even on the parameters of the model. For example, for the Luttinger model introduced in Section 2.1, all amplitudes except A_2 are strictly zero, since only $q = 0$ and $q = 2k_F$ terms exist (keep in mind that for spinless fermions $\rho_0 = k_F/\pi$). Other examples will be examined in Chapter 7. Contrary to the amplitudes A_n , which depend on the precise microscopic model, the power-law decay of the various terms are *universal*. They *all* depend on the unique Luttinger coefficient K . Similarly, one can compute the single-particle Green's function. For the Luttinger model it had only a power law decay for Fourier components around $\pm k_F$ (see (2.83)). In fact, (3.21) shows that power law decay also occurs at $\pm 3k_F, 5k_F$, etc., albeit with a higher exponent also controlled by K (Ogata and Shiba, 1990; Penc and Sólyom, 1991; Penc *et al.*, 1996).

The fact that *all* asymptotic properties are controlled by the *unique* parameter K is a remarkable result, as important as the theory of Fermi liquids for higher-dimensional systems. This result can be used in two ways that we will illustrate in the remaining of this book. Either one likes to work with a specific microscopic model. In that case, to get the low-energy properties of this model, it is *enough* to compute by one way or the other the two coefficients u and K . Since they control all correlation functions, and in particular the thermodynamics (see (2.46)), it is relatively easy and accurate to do this. This can

be done either analytically, using perturbation if the interactions are small, or by exact solutions as we will see in Chapter 5. Alternatively, if no exact solution is available these coefficients can be determined numerically as we will discuss in more details in Section 5.2. Since these coefficients can be extracted from thermodynamic quantities they are much less sensitive to finite size effects than the correlation functions themselves. The Luttinger liquid theory thus provides, coupled with the numerics, an incredibly accurate way to compute correlations and physical properties of a system.

But, of course, a much more important use of Luttinger liquid theory is to justify the use of the boson Hamiltonian and fermion–boson relations as starting points for any microscopic model. The Luttinger parameters then become some effective parameters. They can be taken as input, based on general rules (e.g. for fermions $K > 1$ means attractive interactions and $K < 1$ repulsive interactions), without any reference to a particular microscopic model. This removes part of the caricatural aspects of any modelization of a true experimental system. This use of the Luttinger liquid is reminiscent of the one made of Fermi liquid theory. Very often calculations are performed in solids starting from ‘free’ electrons and adding important perturbations (such as the BCS attractive interaction to obtain superconductivity). The justification of such a procedure is rooted in the Fermi liquid theory, where one does not deal with ‘real’ electrons but with the quasiparticles, which are intrinsically fermionic in nature. The mass m and the Fermi velocity v_F are then some parameters. The calculations in $d = 1$ proceed in the same spirit with the Luttinger liquid replacing the Fermi liquid. The Luttinger liquid theory is thus an invaluable tool to tackle the effect of perturbations on an interacting one-dimensional electron gas (such as the effect of lattice, impurities, coupling between chains, etc.). I will illustrate such use in the following chapters.

Now might be a good point to read the previous chapter if you skipped it, to get all the details on the bosonization procedure. A summary of the dictionary between bosons and fermions and the Luttinger liquid relations is given in Appendix D.

For systems with spins (or any flavor index) one can of course use the same procedure. Each species can be bosonized separately as we saw in Chapter 2.3 and I refer the reader to this section for details. For two species it is usually useful to introduce the charge and spin combinations $\phi_{\rho,\sigma} = (\phi_{\uparrow} \pm \phi_{\downarrow})/\sqrt{2}$. There is however an important point to note when one computes the correlation functions. Let us look, for example, at the density–density correlation. Using (3.8) one would write for the total density $\rho_0 = \rho_{0,\uparrow} + \rho_{0,\downarrow}$

$$\begin{aligned} \rho(x) = \rho_0 - \frac{\sqrt{2}}{\pi} \nabla \phi_{\rho}(x) + 2\rho_0 [e^{i(2k_F x - \sqrt{2}\phi_{\rho}(x))} \cos(\sqrt{2}\phi_{\sigma}(x)) + \text{h.c.}] \\ + 2\rho_0 [e^{i(4k_F x - 2\sqrt{2}\phi_{\rho}(x))} \cos(2\sqrt{2}\phi_{\sigma}(x)) + \text{h.c.}] + \dots \end{aligned} \quad (3.41)$$

So in the higher harmonics both ϕ_{ρ} and ϕ_{σ} appear with the same coefficient. One

would thus naively think that in the correlation functions the $4k_F$ component would decay with a power law

$$\left(\frac{\alpha}{r}\right)^{4K_\rho+4K_\sigma} \quad (3.42)$$

in the same way that the $2k_F$ component was decaying with the exponent $K_\rho + K_\sigma$. This would be true if the spin part of the Hamiltonian was strictly quadratic, that is, if $g_{1\perp}$ was zero. However, for a generic interaction, due to the presence of $g_{1\perp}$ there is a $\cos(\sqrt{8}\phi_\sigma)$ term in the Hamiltonian (see Chapter 2). If we compute the density–density correlation function this term generates in perturbation in power of $g_{1\perp}$ terms of the form

$$\rho(x, \tau) \int dx_1 d\tau_1 \cos(\sqrt{8}\phi_\sigma(x_1, \tau_1)) \quad (3.43)$$

which contains, from the $4k_F$ components of the density, terms of the form

$$\cos(\sqrt{8}\phi_\sigma(x, \tau)) \int dx_1 d\tau_1 \cos(\sqrt{8}\phi_\sigma(x_1, \tau_1)) \quad (3.44)$$

If the points (x, τ) and (x_1, τ_1) are separated this term leads after averaging to an additional power law decay and is thus in general less relevant than the terms containing only the $\cos(\sqrt{8}\phi_\sigma)$ term coming from the density. However, if (x_1, τ_1) is very close (within a cutoff distance α) of (x, τ) then one can approximate (3.44) as

$$\frac{\alpha^2}{u} \cos(\sqrt{8}\phi_\sigma(x, \tau))^2 \simeq \frac{\alpha^2}{2u} [1 + \cos(2\sqrt{8}\phi_\sigma(x, \tau))] \quad (3.45)$$

$\cos(2\sqrt{8}\phi_\sigma)$ is an operator that leads to correlations decaying rapidly (with an exponent $16K_\sigma$) and is thus quite irrelevant. But we see from (3.45) that the seemingly irrelevant operator (3.44) is in fact *asymptotically* constant. It means that to compute the asymptotic decay of the $4k_F$ part of the correlation function one should compute, in presence of the $g_{1\perp} \cos(\sqrt{8}\phi_\sigma)$ term in the Hamiltonian, the decay of

$$\langle e^{i(4k_F x - 2\sqrt{2}\phi_\rho(x))} e^{i(2\sqrt{2}\phi_\rho(0))} \rangle \sim \left(\frac{\alpha}{r}\right)^{4K_\rho} \quad (3.46)$$

This leads to a much smaller power law decay than naively anticipated. Thus, a more faithful expression for the density operator would be

$$\begin{aligned} \rho(x) = \rho_0 - \frac{\sqrt{2}}{\pi} \nabla \phi_\rho(x) + \rho_0 [e^{i(2k_F x - \sqrt{2}\phi_\rho(x))} \cos(\sqrt{2}\phi_\sigma(x)) + \text{h.c.}] \\ + \rho_0 [e^{i(4k_F x - 2\sqrt{2}\phi_\rho(x))} + \text{h.c.}] \quad (3.47) \end{aligned}$$

Remember that in the above sum, the precise coefficients in front of the cosines are meaningless since they are non-universal and depend on the ultraviolet cutoff of the theory.

The total density correlation function in a Luttinger liquid with spins is thus given by

$$\begin{aligned} \langle \rho(x, \tau) \rho(0) \rangle = & \rho_0^2 + \frac{K}{\pi^2} \frac{y_\alpha^2 - x^2}{(x^2 + y_\alpha^2)^2} + \rho_0^2 A_2 \cos(2\pi \rho_0 x) \left(\frac{\alpha}{r}\right)^{K_\rho + K_\sigma} \\ & + \rho_0^2 A_4 \cos(4\pi \rho_0 x) \left(\frac{\alpha}{r}\right)^{4K_\rho} + \dots \end{aligned} \quad (3.48)$$

The difference is important. In the naive answer one is always dominated by the $2k_F$ component of the density since the $4k_F$ one would be always less relevant. In fact, since the $4k_F$ component does not depend on the spin part, we see that (given that $K_\sigma = 1$ for spin isotropic repulsive interactions) for $K_\rho < 1/3$ the system is dominated by the $4k_F$ component of the density. This has important consequences to which I will come back in Chapters 4 and 7.

This example shows that one should be extremely careful when taking an operator to estimate its dimension. Contractions with other operators as in (3.44) can lead to a part that decays *more slowly* than each of the original operators. This is the well-known operator product expansion (Cardy, 1996). It is particularly crucial in one dimension since most of the operators are cosine like, and it is very easy to obtain squares of cosines as in (3.45). We will thus have many occasions to see this phenomenon again.

3.2 Semiclassical and physical interpretations

Many of the formulas that have been derived have simple physical interpretations. Let us examine some of them in this chapter and other interpretations will be apparent as we proceed with new examples.

First, the field ϕ is obviously related to the density of particles. Since the long wavelength part of the density is simply $\rho(x) = -\nabla\phi(x)/\pi$ we see immediately two facts, illustrated in Fig. 3.2. Let us add a single-particle. A particle at point x_0 would manifest itself as a kink in ϕ since $\rho(x) = \delta(x - x_0)$. Since

$$\int_{-\infty}^{x > x_0} dx' \rho(x') = 1 = \frac{-1}{\pi} [\phi(x) - \phi(-\infty)] \quad (3.49)$$

one sees that the step in ϕ is quantized and is a measure of the total charge added in the system. The converse is of course true. If we solve some model and find some kink-like excitations for ϕ then we can interpret these excitations as some charge carrying excitations whose charge (not necessarily integer) is given by (3.49). This gives a very simple interpretation for the single-particle operator. Since it is an operator creating a charge at point x it should create a π kink in ϕ for $x' \in]-\infty, x]$. In quantum mechanics the operator that translates some variable is the exponential of the conjugate variable. If $[X, P] = i\hbar$

$$X e^{-iaP} = e^{-iaP} (X + a\hbar) \quad (3.50)$$

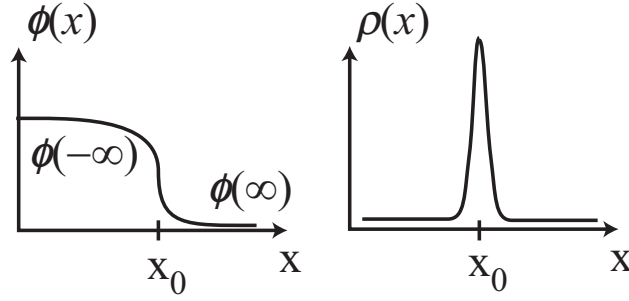


FIG. 3.2. A kink in ϕ at point x_0 corresponds to the creation of a particle at this point. The amplitude of the kink gives the charge of the particle. Thus, soliton excitations in ϕ define the quantum number of the ‘particles’ in the system.

Thus, the operator creating a charge 1, that is, a kink of amplitude π in ϕ is ($\hbar = 1$)

$$e^{-i \int_{-\infty}^x dy \pi \Pi(y)} = e^{-i[\theta(x) - \theta(x=-\infty)]} \quad (3.51)$$

For bosons this is all we need and we see that we recover our formula (3.10). The term $\theta(x = -\infty)$ is a simple choice of phase. It can safely be dropped in *physical* operators such as ψ_B^\dagger since this phase always appears multiplied by an integer number. It would not be true if operators such as $e^{i\nu\theta}$ with ν non-integer would be considered. Although the theory in terms of the fields ϕ and θ can be pathological due to the non-local nature of θ these pathologies disappear when one computes physical objects. For fermions one sees that such an operator cannot be a fermionic creation operator since two such operators at different points would commute and not anticommute. One has thus to multiply it by a phase factor that gives a minus sign when commuting two operators. Using the relation (A.8) and the fact that the commutator between ϕ and θ is the sign function (see (2.24)), it is easy to check that

$$\psi^\dagger(x) \propto e^{i(\pm\phi(x) - \theta(x))} \quad (3.52)$$

does the job. Of course, one has to introduce an operator that changes the *global* charge by one (this is the Klein factor of (2.30)). Since this operator spreads the charge in the whole space it cannot give any contribution in the thermodynamic limit (beyond a global sign). Then the bosonic operator (3.52) ensures that this charge is concentrated at a given point in space. It is thus the only important part when computing correlation functions.

The fields ϕ and θ are very convenient variables since they give rise to semiclassical interpretations. ϕ allows for a semiclassical interpretation of the density operator. If we take a constant ϕ , then the long wavelength part of the density ($\nabla\phi$) vanishes and only the modulated part remains. The density is

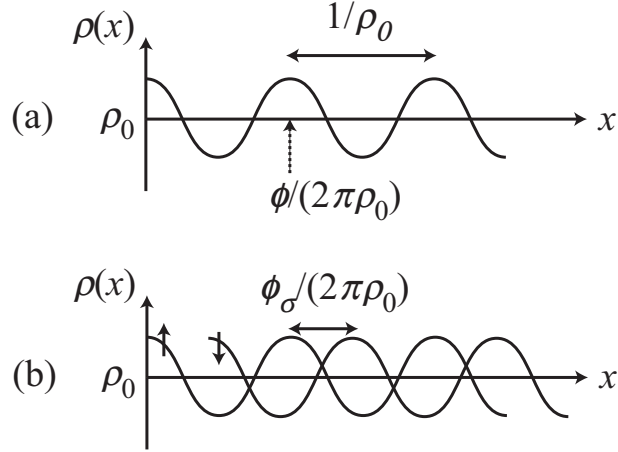


FIG. 3.3. A classical representation of the density. The field ϕ gives the position of the maxima of a wave of density. For systems with spins ϕ_ρ gives the maxima of charge, whereas ϕ_σ is the offset between the spin \uparrow and \downarrow density waves.

$$\rho(x) \propto \rho_0 \cos(2k_F x - 2\phi) \quad (3.53)$$

One can thus see the density as a density wave of period $2k_F$ or $2\pi\rho_0$ in general (see (3.8)). The value of ϕ gives the position of the maxima of the density, as shown in Fig. 3.3. For a system with spin the fields ϕ_ρ and ϕ_σ have a similar interpretation. One has for the density of each spin species in the approximation where ϕ is a constant:

$$\begin{aligned} \rho_\uparrow(x) &\propto \cos(2k_F x - \sqrt{2}\phi_\rho - \sqrt{2}\phi_\sigma) \\ \rho_\downarrow(x) &\propto \cos(2k_F x - \sqrt{2}\phi_\rho + \sqrt{2}\phi_\sigma) \end{aligned} \quad (3.54)$$

Thus, ϕ_ρ gives the phase of the charge density whereas ϕ_σ is the dephasing between the \uparrow and \downarrow density waves. As shown in Fig. 3.3, if $\phi_\sigma = 0$ the two waves are in phase. The spin density then is zero and the charge is modulated. On the contrary, if $\phi_\sigma = \pi/\sqrt{2}$, the two densities are in opposite phase and thus the total charge is constant whereas the spin is modulated. One thus recovers the expressions for the various CDW and SDW_z operators (2.113).

Of course, this perfect wave exists only if the field ϕ orders. We will see such cases in the subsequent chapters. For Hamiltonian (3.25) the field ϕ fluctuates, which means that the precise phase of the wave changes with space and time and thus the correlation between different points of the wave are lost. This is what gives rise to the power law decay of the various density correlations. This interpretation in terms of wave also allows to derive the current. One can use the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla j = 0 \quad (3.55)$$

which together with the expression for the long wavelength part of the density (3.9) gives $j = \partial_t \phi / \pi$. One can interpret this result by saying that for a wave

$$\rho(x) = \rho_0 + \rho_0 \cos(2\pi\rho_0 x - 2\phi) \quad (3.56)$$

which has the proper average density, a change of phase of ϕ by π corresponds to the motion of a charge ρ_0 by a distance $1/\rho_0$. Using that the current is $j = \rho_0 v$ and taking for v the phase velocity gives back the expression for the current. The field θ has similar interpretations in terms of the phase of the particles (3.10). It is analogous to the standard superconducting phase in a BCS wavefunction. Such an interpretation allows to also recover the current from the standard expression (see, e.g. in a superconductor, Tinkham 1975). The possibility to have classical representations in mind is one of the great advantages of the boson representation, and specially of the ϕ , θ field representation. Indeed, if these fields are ordered, they are identical to the familiar mean field expressions for either a superconductor (θ) or a density mean field theory (ϕ). The fact that these fields are conjugate shows that their fluctuations compete and Hamiltonian (3.25) reflects this competition. The possibility for these bosonic fields to order also drastically simplifies the analysis of some strong coupling limit of the Hamiltonian.

Let us complete our tour of the semiclassical interpretations by looking at the excitations of a system with spins. As we saw the Hamiltonian separates into two parts, a charge part and a spin part. As a result, single-particle fermionic excitations do not exist. The excitations are the ones described by the fields ϕ_ρ and ϕ_σ . Let us represent again our ground state by a state in which ϕ_ρ and ϕ_σ are constant. For repulsive interactions this leads to a SDW. If now we remove a fermion it means that at a single site we create a kink *both* in ϕ_ρ and ϕ_σ . These two excitations are free to propagate completely independently. In the fermion language let us see what happens if we let the hole propagate as shown in Fig. 3.4(a). In that case one reaches the state of Fig. 3.4(b). At one point there is a site where a charge is lacking but the spin environment is purely antiferromagnetic. This corresponds to an excitation that is a kink in ϕ_ρ but no disturbance in ϕ_σ . This excitation is known as a holon. In another part of the system there is a place where no charge is missing but we have two neighboring spin up. This is a spin excitation with a spin 1/2 compared to the ground state. This excitation is known as a spinon, and corresponds to a kink in ϕ_σ . In one dimension, we see that the spinon and holon are free to separate, and thus any single-particle excitation dissociates into these elementary excitations. In higher dimension if we reproduce our cartoon, we see that because of the presence of other chains the spinon and the holon would be separated by a string of frustrated bonds. Indeed, all the spins between the spinon and the holon are parallel to the ones of the neighboring chains. This costs the antiferromagnetic exchange, and thus the spinon and holon are held together by a string whose energy grows linearly with the distance. They are thus confined and their bound state is the single-particle fermionic excitation.

Finally, let me define for (3.25) the two fields (see (C.46))

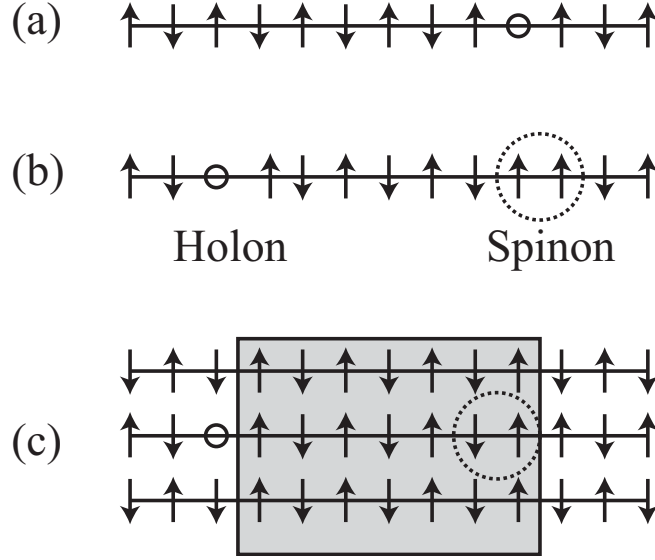


FIG. 3.4. In a one-dimensional system a single-particle excitation (a) is converted into an excitation that contains only charge degrees of freedom (holon) and spin degrees of freedom (spinon). This is represented schematically in (b). In (c) one sees that in a higher-dimensional system the spinon and holon would be held together by a string whose energy cost grows with distance. They thus form a bound that is the single-particle fermionic excitation.

$$\begin{aligned}\phi_R &= K\theta - \phi \\ \phi_L &= K\theta + \phi\end{aligned}\tag{3.57}$$

and the corresponding ‘densities’

$$\tilde{\rho}_{R,L} = \pm \frac{1}{2\pi} \partial_x \phi_{R,L}\tag{3.58}$$

It is easy to check from (2.28) that for the non-interacting case $K = 1$ the $\tilde{\rho}_{R,L}$ do coincide for fermions with the densities of right and left movers. This is of course not the case for the interacting case, but the fields $\phi_{R,L}$ have interesting properties. First, they commute between each other and satisfy (using (2.24))

$$[\phi_R(x), \phi_R(x')] = -[\phi_L(x), \phi_L(x')] = i\pi K \text{Sign}(x - x')\tag{3.59}$$

The total density $\tilde{\rho}_R + \tilde{\rho}_L$ is the true total density in the system (using (3.9)). These fields diagonalize Hamiltonian (3.25), which can be rewritten as

$$\frac{1}{4\pi K} \int dx [(\partial_x \phi_R)^2 + (\partial_x \phi_L)^2]\tag{3.60}$$

These fields thus satisfy the equation of motion

$$\frac{\partial \tilde{\rho}_{R,L}}{\partial t} = i[H, \tilde{\rho}_{R,L}] = \mp \partial_x \tilde{\rho}_{R,L} \quad (3.61)$$

Thus, the densities $\tilde{\rho}_{R,L}$ (and the fields $\phi_{R,L}$) are only functions of $x - ut$ and $x + ut$. These fields describe excitations that propagate only in the right (resp. left) direction. For this reason they are called chiral fields. They are the generalization to the interacting case of the right and left free fermions. In presence of interactions right and left fermions interact. But one can still find an excitation (that contains both right and left fermions) that propagates solely to the right or to the left. I will come back to these fields in Chapter 10.

3.3 Links with 2D statistical mechanics

There are many links between the one-dimensional Luttinger liquid problem and classical systems in $1 + 1$ dimension. This mapping between a quantum system in d dimensions and a classical system in $d + 1$, where the (imaginary) time direction plays the role of an extra spatial dimension, is of course very familiar. Path integral representation (see Appendix C) is a very convenient way to show such connections. It is of course a great advantage to be able to make such connections between different physical models since one can borrow the knowledge of one to solve the other. Let me examine here more precisely some of those links.

3.3.1 Elastic systems

Let us consider a classical crystal where we denote by R_i^0 the equilibrium position of the particles, and by u_i the displacements relative to their equilibrium positions. The interaction between the particles of the crystal can be approximated by an elastic Hamiltonian. Although it is possible in principle to stick to the lattice, it is much more convenient to go to the continuum. The Hamiltonian becomes

$$H = \frac{c}{2} \int d^d r (\nabla u(r))^2 \quad (3.62)$$

where c is the elastic constant. Of course, a true elastic Hamiltonian has more complicated (bulk, tilt, and shear) coefficients, but this isotropic elasticity will be enough for our purposes. The elastic approximation assumes that $|u_{i+1} - u_i| \ll a$. Let us take a scalar displacement and a two-dimensional system x, z . This represents a crystal of lines along the z direction as shown in Fig. 3.5. The Hamiltonian of our classical crystal is

$$H = \frac{c}{2} \int dx dz [(\nabla_z u(x, z))^2 + (\nabla_x u(x, z))^2] \quad (3.63)$$

It is exactly identical to the action (3.26) of the one-dimensional quantum problem (where the velocity has been absorbed in the change of variables $z = u\tau$).

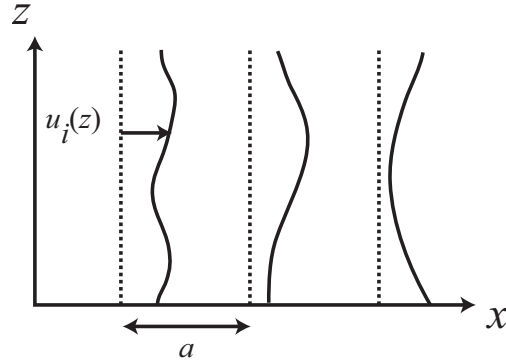


FIG. 3.5. A crystal of lines is defined by a scalar displacement $u_i(z)$ which in the continuum limit along x becomes the function $u(x, z)$. This classical system in its elastic limit is equivalent to a Luttinger liquid. The lines can be viewed as the space-time trajectories of the particles.

System	Quantum	Classical
Dimension	d	$d+1$
β	Temperature	$u\beta\hbar =$ Size of system in direction z
\hbar	quantum fluctuations	Temperature

Table 3.1 *Equivalence between a quantum system of dimension d and a classical system in dimension $d + 1$*

The lines of our elastic system can be viewed as the space-time trajectories of the quantum particles (Pollock and Ceperley, 1987). As usual $u\beta\hbar$ for the quantum problem is the size along z of the classical problem, which is defined on a torus to ensure the periodicity. The elastic constant c is equivalent to $1/(\pi K)$ for the quantum problem, whereas T the temperature of the classical problem plays the role of \hbar for the quantum problem. For a quantum system at zero temperature $\beta = \infty$ the role played by quantum fluctuations (whose amplitude is controlled by \hbar) is equivalent to the role played by the thermal fluctuations (controlled by T) in the two-dimensional classical system. Since we see that the combination $1/\hbar K$ (for quantum) or c/T (for classical) appears, changing K (or c for classical) is a way to control the amount of fluctuations in the system. The equivalence is recalled in Table 3.1. Since by changing the interactions in the system one changes the amount of quantum fluctuations, one-dimensional systems are a paradise for quantum phase transitions (Sachdev, 1998).

To go further let us express the density of the crystal in the continuum limit. The density is

$$\rho(x) = \sum_i \delta(x - R_i^0 - u_i) \quad (3.64)$$

To go to a continuum limit as we did for the elastic Hamiltonian is a little tricky. Indeed, one has to keep the discrete nature of u_i in (3.64) even if we performed the continuous limit in H . This is due to the fact that we want to have the density at scales comparable or smaller than the lattice spacing itself. This is relevant for systems such as vortices in type II superconductors (Tinkham, 1975) for which the lattice spacing is in fact quite large and thus many things (e.g. disorder) can vary at a lengthscale much shorter than the lattice spacing. It is thus crucial to remember the discrete nature of the lattice to compute the density at a given point r . On the other hand, we know (this is the elastic limit) that u itself varies very slowly at the scale of the lattice spacing a so we can use this fact to simplify (3.64). To derive the density for the classical crystal (Giamarchi and Le Doussal, 1995) we proceed exactly along the same lines and in the same spirit than the one we used to derive the phenomenological bosonization in the previous section. In order to take the continuum limit, one can introduce a smooth displacement field $u(r, z)$ by

$$u(r) = \int_{BZ} \frac{d^d q}{(2\pi)^d} e^{iqr} \sum_j e^{-iqR_j^0} u_j \quad (3.65)$$

such that $u(R_i) = u_i$ and which has no Fourier components outside of the Brillouin zone (BZ). In terms of the smooth field (3.65) one can introduce the relabelling field

$$\phi(r) = r - u(\phi(r)) \quad (3.66)$$

In the absence of dislocations there is a unique solution of (3.66) giving $u(r)$ as a function of $\phi(r)$. ϕ is the equivalent of ϕ_l of the quantum problem. At the location of the particles ϕ takes the integer value

$$\phi(R_i^0 + u_i) = R_i^0 \quad (3.67)$$

Substituting (3.66) in (3.64) one gets

$$\rho(r) = \sum_i \delta(R_i^0 - \phi(r)) \det[\partial_\alpha \phi_\beta(r)] \quad (3.68)$$

Using the integral representation of the δ function, (3.68) becomes

$$\rho(r) = \det[\partial_\alpha \phi_\beta] \int \frac{d^d q}{(2\pi)^d} \rho_0(q) e^{iq\phi(r)} \quad (3.69)$$

where

$$\rho_0(q) = \sum_i e^{iqR_i^0} \quad (3.70)$$

is the density of the associated perfect system. For the case of a perfect lattice $\rho_0(q)$ is


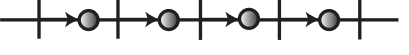
$\rho =$  $\nabla u(r)$
 $+$  $\cos(K_0(r - u(r)))$
 $+$...

FIG. 3.6. Equation (3.73) realizes a Fourier decomposition of the density relative to the wavevectors corresponding to the lattice spacing. It is easy to see physically that the $q \sim 0$ component is $-\rho_0 \nabla u$. A variation of the density at the scale of the lattice spacing can be decomposed in $\cos(K_n(r - u))$ where a constant (at the scale of the lattice spacing) u would just shift the peaks of the cosine. K_n are the vectors of the reciprocal lattice. Putting all harmonics together gives back (3.73).

$$\rho_0(q) = \rho_0 (2\pi)^d \sum_K \delta(q - K) \quad (3.71)$$

where K are the vectors of the reciprocal lattice. Using (3.71) in (3.69) one gets

$$\rho(r) = \rho_0 \det[\partial_\alpha \phi_\beta] \sum_K e^{iK \cdot \phi(r)} \quad (3.72)$$

Assuming that we are in the elastic limit $\partial_\alpha u_\beta \ll 1$ one can expand (3.72) to get

$$\rho(r) \simeq \rho_0 [1 - \partial_\alpha u_\alpha(\phi(r)) + \sum_{K \neq 0} e^{iK(r - u(\phi(r)))}] \quad (3.73)$$

In (3.73), one can replace $u(\phi(r))$ by $u(r)$ up to terms of order $\partial_\alpha u_\beta \ll 1$. Note that in doing so u has negligible Fourier components outside the Brillouin zone, and thus there is a complete decoupling between the gradient term and higher K terms. We have realized a sort of Fourier decomposition of the density, separating terms varying at lengthscales much larger than the lattice spacing, from Fourier components varying with harmonics of the periodicity of the lattice. This can be schematically seen in Fig. 3.6. This representation of the density for a classical crystal holds provided we can define the labelling field. Physically this means

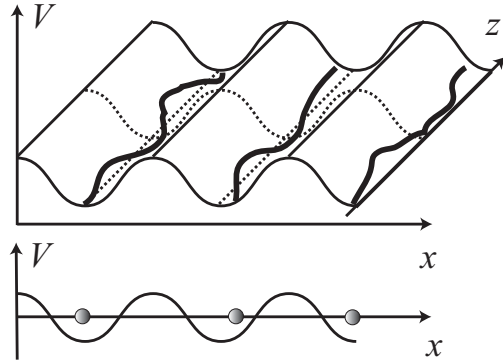


FIG. 3.7. A classical crystal in a potential periodic in the direction x (tin roof potential). This is analogous to a quantum problem on a lattice with a commensurate filling, and gives rise to the Mott transition.

that one can number in a unique way the atoms of the crystal. This is obviously possible if there are no defects such as dislocations in the crystal. For our two-dimensional crystal of lines this is always possible since one can always number the lines starting from (say) the left in a unique way. The formula for the density (3.73) is exactly identical to the one of our Luttinger liquid with $\phi/\pi = \rho_0 u$. One can thus carry the identity between the quantum problem and the crystal made of lines completely. For example, if one adds to the crystal a periodic potential $V(x, z) = V_0 \cos(K_0 x)$, which is shown in Fig. 3.7, this corresponds to a term in the classical Hamiltonian

$$H = \int dx dz V(x, z) \rho(x, z) \simeq V_0 \int dx dz \cos(K_0 u(x, z)) \quad (3.74)$$

using (3.73) and keeping only the terms that do not oscillate since the other terms vanish in the continuum limit. We recognize a sine-Gordon term. This problem is in fact analogous to the Mott transition that we will examine in detail in Section 4.2: the lines are pinned by the tin roof potential and would of course like to sit in the minima, but thermal fluctuations are shaking them. Which wins corresponds for the quantum problem to the Mott transition.

This equivalence is quite interesting. First, it provides a simple physical picture to help understanding the perhaps less intuitive quantum problem. It shows also clearly the spirit behind our representation procedure. We can define fields that vary slowly at the scale of the lattice, and the energy is a smooth function of these fields (displacements for the classical problem, field ϕ for the quantum one). We can thus retain *only* the long wavelength components of these fields (and thus take the continuum limit for them). However, the various operators can vary at a lengthscale comparable to the lattice (e.g. the density) and one cannot directly take the continuum limit for them. However, it is possible to express them as some involved functions of the slowly varying fields. We have

thus gained a direct continuum representation, and thus a faithful low-energy description of the system in terms of simple fields. The price to pay is that the *operators* themselves are complicated and hide in their form the complexity of the original (microscopic) problem.

Second and more important, the problem of a classical crystal submitted to various perturbations is of course a crucial problem of statistical mechanics. Fermion representation thus represent an interesting route to tackle such problems (and vice versa). For more details on various works on classical crystals I refer the reader to Blatter *et al.* (1994), Nattermann and Scheidl (2000) and Giamarchi and Bhattacharya (2002).

3.3.2 Coulomb gas and XY model

The equivalence between the Luttinger liquid and the crystal of lines was very direct. I examine now a more involved mapping based on the fact that the density of the system contains terms of the form $e^{i\phi}$. Let us consider a two-dimensional classical XY model, where on each site a classical spin is

$$\vec{S} = (\cos(\phi), \sin(\phi)) \quad (3.75)$$

The Hamiltonian of the system is

$$H = -\frac{J}{2} \sum_{i,a} \cos(\phi(i+a) - \phi(i)) \quad (3.76)$$

where a is a set of nearest neighbors vectors. The density–density correlations for the Luttinger liquid thus correspond to the spin–spin correlation of the XY model, if once again we assimilate one of the directions (say y) of space for the classical model to the imaginary time direction for the quantum problem and the field ϕ for the phase of the spin. What about the Hamiltonian? If ϕ was a smooth field one could expand

$$\phi(i+a) - \phi(i) = \vec{a} \cdot \vec{\nabla} \phi \quad (3.77)$$

and the XY Hamiltonian would become in the continuum limit (for a square lattice)

$$H_{XY} = \frac{J}{2} \int dx dy [(\nabla_x \phi)^2 + (\nabla_y \phi)^2] \quad (3.78)$$

that is, identical to our favorite elastic Hamiltonian in two dimensions. We know for the XY model that this expansion is valid provided that the temperature is low enough (Berezinskii, 1971; Kosterlitz and Thouless, 1973; Kosterlitz, 1974). At higher temperatures the XY model undergoes a transition (Berezinskii–Kosterlitz–Thouless or BKT) towards a disordered phase. For the low-temperature phase of the XY model we thus have equivalence with our LL Hamiltonian H_0 . We know that at higher temperatures, vortex configurations, where the angle ϕ is singular, start to be important. Such a configuration is shown in Fig. 3.8. We can easily

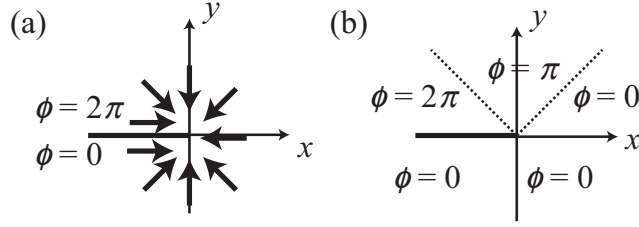


FIG. 3.8. (a) A vortex configuration for the classical XY model. (b) For the Luttinger liquid the operator $\cos(2\theta)$ creates such a vortex configuration.

see that such a configuration should exist. Indeed, if one considers $\int dr \nabla \phi$ over a closed loop it should give $2\pi n$ since ϕ is defined modulo 2π . If only spin waves (smooth configurations of ϕ) are allowed, this integral is zero. Vortices should thus be added to H_0 .

Let us now see how one could for the quantum problem create such a vortex. We start from a configuration where on each point (space, time) the field ϕ takes the value zero. This would correspond in the XY equivalence to a system where all spins point in the x direction. We saw in Section 3.2 that the operator $e^{i2\theta(x)}$ creates a kink of amplitude 2π on ϕ . Thus, if we apply $e^{i2\theta(x_0, \tau_0)}$ to our uniform configuration, it means that just after the time τ_0 , ϕ has been shifted by 2π on the left of the point x_0 . This leads to the configuration shown on Fig. 3.8(b). This configuration can be smoothly deformed to the more standard representation of a vortex (shown on Fig. 3.8(a)). The operator $e^{i2\theta(x)}$ (with the time ordering) is thus the vortex creation operator for the field ϕ . Since the Hamiltonian and commutation relations are symmetric the reverse is true and the operator $e^{i2\phi(x)}$ will create vortices in the field θ .

One can thus expect the analogy between the XY model and the Luttinger liquid to hold provided that these operators are included. To formalize this let us consider a Hamiltonian

$$H = H_0 + g_\phi \int dx \cos(\sqrt{8}\phi) + g_\theta \int dx \cos(\sqrt{8}\theta) \quad (3.79)$$

and compute the partition function. It is given by

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^2r_1 d^2r_2 \cdots d^2r_n \langle \left(\frac{g_\phi}{u} \cos(\sqrt{8}\phi(r_1)) + \frac{g_\theta}{u} \cos(\sqrt{8}\theta(r_2)) \right) \cdots \left(\frac{g_\phi}{u} \cos(\sqrt{8}\phi(r_n)) + \frac{g_\theta}{u} \cos(\sqrt{8}\theta(r_n)) \right) \rangle \quad (3.80)$$

a correlation function containing an arbitrary product of exponential factors of ϕ and θ is easily evaluated (see (C.38)). Let us take $g_\theta = 0$ to illustrate the result simply. Only even powers are non-zero, and if one expresses the cosine as $\sum_{\epsilon=\pm} e^{i\epsilon\sqrt{8}\phi}$ only ‘neutral’ configurations $\sum_i \epsilon_i = 0$ give a non-zero contribution.

In that case one can arbitrarily take $\epsilon_1, \dots, \epsilon_p = +1$ and $\epsilon_{p+1}, \dots, \epsilon_{2p} = -1$. In that case the partition function is simply given by

$$Z = \sum_{p=0}^{\infty} \frac{1}{(p!)^2} \int d^2r_1 d^2r_2 \cdots d^2r_{2p} g_\phi^{2p} e^{\sum_{i<j} [\epsilon_i \epsilon_j 4K] F_1(r_i - r_j)} \quad (3.81)$$

This is exactly the partition function of a gas of classical particles of fugacity g_ϕ (Chui and Lee, 1975). The exponential term can be identified with $e^{-E_{\text{class}}/T_{\text{class}}}$ where T_{class} and E_{class} are the temperature and the energy of the classical system. One can thus describe the classical system as a gas of classical charges that carry a charge ± 1 , are at the ‘temperature’ $1/T_{\text{class}} = 2K$, and interact with each other with a potential $V(r_i - r_j) = -2F_1(r_i - r_j) \sim -2 \log(r_i - r_j)$. Because of the form of the correlation functions in the LL only neutral configurations are allowed. This problem is exactly the well-known Coulomb gas problem in two dimensions. Indeed, a logarithmic potential satisfies the Poisson equation (see (2.126))

$$\Delta V(r) + 4\pi\delta(r) \quad (3.82)$$

The sine-Gordon model thus maps exactly to the Coulomb gas problem, which is known to be equivalent to the two-dimensional classical XY model (Kosterlitz, 1974). Our more complicated Hamiltonian (3.79) has also a simple Coulomb gas interpretation. We can still view the partition function as the partition function of a mixture of classical particles of two types with respective fugacities g_ϕ/u and g_θ/u . The first type are charged particles with charges $\pm\sqrt{K}$ interacting with the Coulomb potential $-2F_1$. The second type are magnetic monopoles with magnetic charges $\pm 1/\sqrt{K}$ interacting through the magnetic potential $-2F_1$. Indeed, taking Maxwell’s equation it is easy to check that if a density of magnetic monopoles exists it will generate a magnetic potential obeying the same Poisson’s equation than normal charges since

$$\nabla \cdot B(x) = \rho_m(x) \quad (3.83)$$

The $\phi\theta$ correlation F_2 describes the interaction between the charges and the magnetic monopoles. If a charge makes a circle around the origin then its phase must only change by a multiple of 2π due to the flux that its trajectory has enclosed. Computing this flux (Ahronov–Bohm flux) gives back F_2 . Our complete Hamiltonian (3.79) can thus be viewed as a mixture of charges and magnetic monopoles, each species needing to have only neutral configurations.

Given the amount of physics known for the XY model and Coulomb gases, such mappings are extremely useful. For example, we can directly borrow the results for the BKT phase transition in the XY model to infer the existence of a quantum phase transition for the sine-Gordon problem and extract the properties of the various phases. This is what we did in Chapter 2, directly on the sine-Gordon Hamiltonian. We will see other applications in Chapter 4.4. The trick works of course both ways and one can use the fermionic approach to tackle interesting questions for the classical problems (Lecheminant *et al.*, 2002).

3.4 Basics of conformal theory

We have seen that the correlation of a Luttinger liquid in the massless phase corresponds to the correlation functions of a classical two-dimensional system that sits exactly at criticality. For such systems it is known for a long time that the correlation functions are invariant by a large class of transformations. These include, for example, the continuous rotation (between x and τ direction), and scale transformations. In addition, the systems at criticality are invariant by a much broader class of transformations: the conformal transformations. These are the transformations that preserve the angles between a triplet of points. Such transformations obviously include the rotations, symmetries such as reflection etc., and the dilatations. In three dimensions, there are not many more operations in the conformal group than these simple transformations. The situation is however quite different in two dimensions. In that case the conformal group is very large, and the fact that the correlation functions are invariant by such transformations provides severe constraints and allows to extract a lot of relevant information without effort. Since the scope of the conformal theory exceeds largely the level of this book, I refer the reader to Cardy (1996) and Di Francesco *et al.* (1997) for more details. Here we will just illustrate the power of the method on a few examples.

The idea of the conformal method is to generalize the scale invariance of a theory that is at criticality. If one takes, for example, a classical system that is at a critical point, the correlation functions decay as power laws. As a result a critical theory is invariant under rescaling. If ψ_i denotes fields of the theory one has

$$\langle \psi_1(r_1) \psi_2(r_2) \cdots \psi_n(r_n) \rangle = b^{-\nu_1} b^{-\nu_2} \cdots b^{-\nu_n} \langle \psi_1(r'_1) \psi_2(r'_2) \cdots \psi_n(r'_n) \rangle \quad (3.84)$$

where $r'_i = b^{-1} r_i$. The quantities ν_i are the scaling dimensions of the operators ψ_i in the theory. For example, for a simple two-point correlation function

$$\langle \psi(r_1) \psi(r_2) \rangle \sim \left(\frac{1}{r_1 - r_2} \right)^{2\nu} \quad (3.85)$$

The rescaling is obviously satisfied with $\nu_1 = \nu_2 = \nu$. There are other transformations that can leave the correlation functions unchanged or trivially changed. This is, for example, the case of rotations or translations. In a critical theory one has thus simple transformations of the original coordinates for which we know exactly how to transform the correlation functions. The idea of the conformal theory is to generalize these transformations to a change of coordinates that are *locally* simple rescalings, rotations, or translations. If the Hamiltonian has only local interactions then one can expect that such a *local* transformation only affects the properties of the whole system locally and thus that one can still use the transformation formulas for the correlation functions if the points at which the correlations are computed are very far from each other. In other words, one can generalize formula (3.84) to obtain the asymptotic properties. The transformations that are locally identical to dilatations, rotations, translations are known

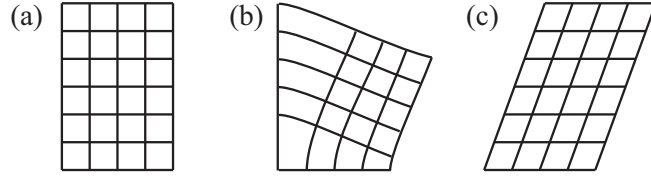


FIG. 3.9. (b) An example of a conformal transformation. It preserves locally the angles of the original lattice (a). (c) A forbidden transformation that causes shear in the coordinates.

as conformal transformations (they have to locally preserve angles). An example of such transformations is shown on Fig. 3.9. Such conformal transformations preserve locally the metric (length of an infinitesimal segment) of the system. Thus, if $r = (x_1, x_2, \dots, x_n)$

$$\sum_i dx'_i dx'_i = b(r)^{-2} \sum_i dx_i dx_i \quad (3.86)$$

The ‘number’ of such conformal transformations depends very much on whether one is in three or two dimensions. In three dimensions, the group of conformal transformations has a finite number of parameters, and thus is not much more useful than the simple original transformations. In two dimensions, however, the situation is quite different. It is very useful in two dimensions to parametrize a point $r = (x, y)$ using the complex number $z = x + iy$. Using this parametrization the metric becomes

$$(dx)^2 + (dy)^2 = dz d\bar{z} \quad (3.87)$$

Upon a general transformation one has

$$dz' d\bar{z}' = \left(\frac{\partial z'}{\partial z} dz + \frac{\partial z'}{\partial \bar{z}} d\bar{z} \right) \left(\frac{\partial \bar{z}'}{\partial z} dz + \frac{\partial \bar{z}'}{\partial \bar{z}} d\bar{z} \right) \quad (3.88)$$

This transformation satisfies (3.86) if and only if

$$\frac{\partial z'}{\partial \bar{z}} = \frac{\partial \bar{z}'}{\partial z} = 0 \quad (3.89)$$

Thus, in the transformation z' depends only on z and \bar{z}' only on \bar{z} . It means that *any* analytic function f of a complex variable z can generate a conformal transformation $z' = f(z)$. If one uses such a transformation the rescaling factor is, from (3.88) and (3.86)

$$dz' d\bar{z}' = |f'(z)|^2 dz d\bar{z} \quad (3.90)$$

and thus

$$b(r) = |f'(z)|^{-1} \quad (3.91)$$

The subgroup corresponding to rotations, dilatations, translations, and inversion is the so-called Möbius transformations

$$z' = \frac{Az + B}{Cz + D} \quad (3.92)$$

with $AD - BC \neq 0$. Of course, the fact that any analytic function can generate a conformal transformation gives an incredible number of possibilities. To write the transformation for the correlation functions we generalize (3.84) to a local transformation with (3.91)

$$\begin{aligned} \langle \psi_1(z_1, \bar{z}_1) \cdots \psi_n(z_n, \bar{z}_n) \rangle &= |f'(z_1)|^{\nu_1} \cdots |f'(z_n)|^{\nu_n} \\ &\times \langle \psi_1(z'_1 = f(z_1), \bar{z}'_1 = f(\bar{z}_1)) \cdots \psi_n(z'_n = f(z_n), \bar{z}'_n = f(\bar{z}_n)) \rangle \end{aligned} \quad (3.93)$$

Let us take a correlation function that can be written in terms of the complex variable z as

$$G(z_1, z_2) = \left(\frac{1}{z_1 - z_2} \right)^h \left(\frac{1}{\bar{z}_1 - \bar{z}_2} \right)^{\bar{h}} \quad (3.94)$$

Note¹⁷ that \bar{h} is not the complex conjugate of h but an independent real quantity. For example, for the correlation function (3.85) $h = \bar{h} = \nu$. If we now make a change of variables $z \rightarrow z'$, the correlation function transforms as

$$G(z'_1, z'_2) = \left[\frac{dz_1}{dz'_1} \right]^{h/2} \left[\frac{dz_2}{dz'_2} \right]^{h/2} \left[\frac{d\bar{z}_1}{d\bar{z}'_1} \right]^{\bar{h}/2} \left[\frac{d\bar{z}_2}{d\bar{z}'_2} \right]^{\bar{h}/2} G(z_1, z_2) \quad (3.95)$$

Formula (3.95) does not look much but is incredibly powerful. It allows us to obtain the correlation function in *any* restricted geometry that can be obtained by a conformal transformation of the plane. Here, let us just see how the use of conformal theory allows to get back the correlation functions at finite temperature. They can of course also be directly computed by standard methods (see Appendix C) but the conformal derivation is very general and can work for other geometries as well. At zero temperature the correlation function is simply

$$G(x, y = u\tau) = \left(\frac{1}{x^2 + y^2} \right)^\nu = \left(\frac{1}{z\bar{z}} \right)^\nu \quad (3.96)$$

G is thus a function of both z and \bar{z} . At $\beta = \infty$ the point (x, y) covers the whole complex plane. At finite temperatures the imaginary time is only between $\tau = 0$ and $\tau = \beta$ with periodic boundary conditions. We thus have to compute the correlation function on a torus, as shown in Fig. 3.10. We can make a conformal

¹⁷Notations are weird, but since they are the ones you find in the literature ... *vox populi, vox dei*.

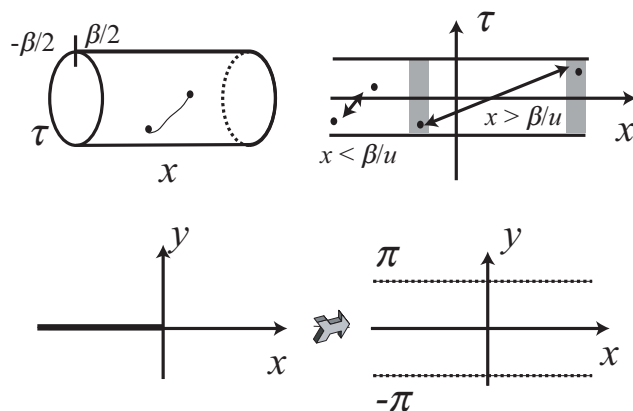


FIG. 3.10. Torus for finite-temperature correlation functions. When the distance between points is larger than β the system realizes that it is at finite temperature and becomes equivalent to a one-dimensional classical system. The bottom part shows that the complex plane with a cut on the negative real axis can be mapped on a torus by a conformal transformation. Thus, if we know the correlation functions at $\beta = \infty$ we also know them at any finite temperature.

transformation mapping the infinite plane into a torus. Such a transformation is obviously

$$z' = \frac{\beta}{2\pi} \log(z) \quad (3.97)$$

This imposes to put a cut in the complex plane. A possible choice is shown in Fig. 3.10. With this choice the imaginary part of the log is between $-\pi$ and $+\pi$. Transformation (3.97) thus maps the plane into a torus of size β along the imaginary axis. Let us call the correlation function on the torus

$$G(z'_1, z'_2, \bar{z}'_1, \bar{z}'_2) \quad (3.98)$$

Because of the translational invariance in space and time one can always take $z'_2 = \bar{z}'_2 = 0$. We can now use (3.95). We relate a point z' on the torus to a point z in the infinite plane by

$$z = e^{2\pi z'/\beta} \quad (3.99)$$

which is the inverse of (3.97). Equation (3.95) thus gives the correlation function on the torus in terms of the correlation function in the infinite plane whose expression (3.96) we know. Using

$$\frac{dz_1}{dz'_1} = \frac{2\pi}{\beta} e^{2\pi z'_1/\beta} \quad (3.100)$$

one obtains

$$G_{\text{torus}}(z'_1, z'_2 = 0, \bar{z}'_1, \bar{z}'_2 = 0) = \left[\frac{2\pi}{\beta} e^{2\pi z'_1/\beta} \right]^{\nu/2} \left[\frac{2\pi}{\beta} e^{2\pi \bar{z}'_1/\beta} \right]^{\nu/2} \left[\frac{2\pi}{\beta} \right]^\nu \\ \times G_{\text{plane}}(z_1(z'_1), z_2 = 1, \bar{z}_1(\bar{z}'_1), \bar{z}_2 = 1) \quad (3.101)$$

We have used that $z(z' = 0) = 1$ from (3.99). Using expression (3.96) for the correlation function in the plane one gets

$$G_{\text{torus}}(z'_1, z'_2 = 0, \bar{z}'_1, \bar{z}'_2 = 0) = \frac{(\pi/\beta)^{2\nu}}{(\sinh(\pi z'_1/\beta) \sinh(\pi \bar{z}'_1/\beta))^\nu} \quad (3.102)$$

Expression (3.102) is exactly the expression that we got in Appendix C doing the boson averages at finite temperatures.

Let us comment on this result. At zero temperature the correlation function decays with an exponent

$$G(r) \sim \left(\frac{1}{r} \right)^{2\nu} \quad (3.103)$$

where $r = \sqrt{x^2 + (u\tau)^2}$. At finite temperatures, as long as both x and the time τ are much smaller than β the system has no way to know that it is on a torus, and thus that it is not a truly *two* dimensional (classical) system. The correlation function thus still reflects the criticality of the infinite two-dimensional system. When x becomes much larger than β , the correlation function can ‘feel’ the finite size of the torus, as illustrated in Fig. 3.10. In that case the system is in fact like a classical one-dimensional system and its correlation function has to decay exponentially with a certain characteristic lengthscale ξ . It is obvious that the crossover occurs when $x \sim \beta$ and that $\xi \sim \beta$. The full behavior of the correlation function is illustrated in Fig. 3.11. Using (3.102) and expanding for large x , we find that

$$G(x \rightarrow \infty, \tau) \sim e^{-2|x|\nu/\beta} \quad (3.104)$$

leading to

$$\xi = \beta \frac{2\nu}{\pi} \quad (3.105)$$

We thus see that indeed the correlation length ξ is proportional to β as could be expected but the proportionality coefficient is *precisely* fixed by the exponent of the critical theory. Thus, computing exponential decay of the correlation function at finite β is also a way to obtain the exponent for the infinite system. Of course a similar analysis can be done to study finite size effects in space (see, e.g. Cazalilla, 2002).

The conformal method is extremely powerful since it allows trivially to take into account all possible geometries, and thus many possible boundary conditions. This is specially useful when coupled with an RG calculations. Indeed, when some coupling goes to strong coupling one can often view its effect as

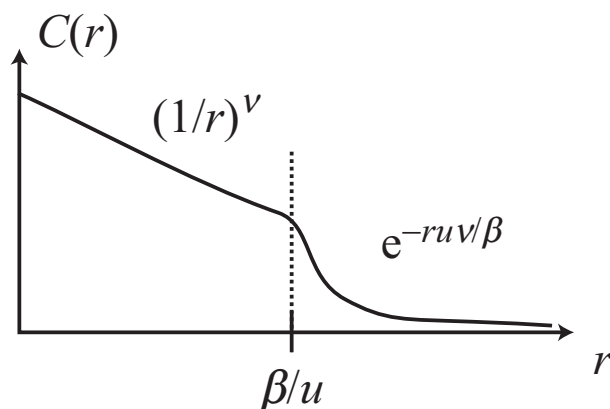


FIG. 3.11. Full behavior of the correlation functions at finite temperatures.

Beyond the thermal length $\xi = \beta u$ the decay of the correlations become exponential instead of a power law. A good approximation is thus to consider that all correlations are lost beyond this lengthscale and that the system is like at zero temperature below.

a simple constraint on the field ϕ . This constraint correspond to a change in boundary conditions that can be taken into account using the conformal theory. We will see some examples in Section 10.1 and Section 11.2.

Besides the simple correlation functions the conformal invariance allows to predict many properties. Of special importance is the size dependence of the energy of the ground state. For the boson model (3.25) this is again something that can be computed directly from the boson representation. But the result is much more general than for the quadratic Hamiltonian (3.25). I will not give here the demonstration of the formula for the energy since it is rather involved and refer the reader to Belavin *et al.* (1984), Cardy (1996) and Di Francesco *et al.* (1997) for more details. For a conformally invariant theory, the size dependence of the energy of the ground state is

$$\epsilon_0 - \frac{E(L)}{L} = \frac{c\pi u}{6L^2} \quad (3.106)$$

where ϵ_0 is the ground state energy per unit length in the thermodynamic limit, u is the velocity of the massless mode. c is a number that is only dependent on the type of conformal theory, that is, on the very nature of the model and is known as the central charge. It roughly gives the number of independent bosons in the theory. For the quadratic Hamiltonian of the Luttinger liquids the conformal charge is $c = 1$, as can be checked by a direct calculation of (3.106) from the boson Hamiltonian. Such a relation can be very useful to compute the Luttinger liquid parameter u . I will come back to the calculation of the Luttinger liquid parameters in Chapter 5.