Florence 2019

Monday, 11 February 2019 09:20

Tentative plan of the lectures

I) Basic notions of what is a one-dimensional system:
- Introduction to bosonization
- Field theory description of 1D system: the Tomonaga-Luttinger Liquid (TLL)
- Experimental realizations in condensed matter and cold atomic gases

II) Beyond the TLL description:
- A Zest of numerical approach
- Effect of a periodic potential
- Notions of topology and Berezinskii-Kosterlitz Thouless transition
- Experimental realizations
- Double sine-Gordon model and topological phase transition
- Realizations with spin systems

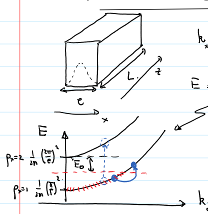
III) Disordered one dimensional systems:
- Basic concepts of Anderson localization
- Notions of Bose and Fermi glasses

IV) Coupled one dimensional systems:
- Case of ladders (spins, bosons and fermions)
- Effect of gauge fields
- Experimental realizations

V) Open issues.

I] Basic notions of 1D systems

1) What is a 1D system ?



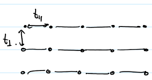
$k_x = \frac{\pi}{e} p_x \qquad k_z = \frac{\pi}{L} p_z$.

$E = \frac{k_x^2}{2m} + \frac{k_z^2}{2m}$. $\qquad$ L large, e small.

$\Delta E \sim U, T, \omega \dots$

$\Delta E \ll E_0$ cannot change p_x.

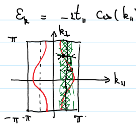
Make confinement strong enough to fix the transverse quantum number.



$$H = -t_\parallel \sum_{\langle ij \rangle} (c_i^\dagger c_j + h.c.) - t_\perp \sum_{\langle ij' \rangle} (c_i^\dagger c_{j'} + h.c)$$

$$= -\sum_k \varepsilon_k c_k^\dagger c_k.$$

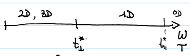
$\varepsilon_k = -2t_\parallel \cos(k_\parallel) - 2t_\perp \cos(k_\perp)$. $\qquad t_\perp \ll t_\parallel$.



$E_F = \varepsilon_k$.

$T \gg t_\perp$. ?

$E_{probe} \gg t_\perp$. ✓



dimensional crossover.

2) Models

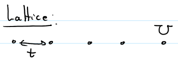
- Continuum.

$$H = \frac{1}{2m}\int dx \, \frac{\nabla\psi^\dagger \nabla\psi}{2m} + \frac{1}{2}\int dx\,dx' \, U(x-x')\rho(x)\rho(x')$$

$\rho(x) = \psi^\dagger(x)\psi(x)$. $\qquad - \mu \int dx \, \rho(x)$

bosons. $U(x-x') = U_0\,\delta(x-x')$ $\qquad$ Lieb-Lininger model.

- Lattice.



$$H = -t \sum_{\langle ij \rangle} (b_i^\dagger b_j + h.c) + \frac{U_0}{2}\sum_j \rho_j(\rho_j - 1).$$

$\rho_j = b_j^\dagger b_j$ $\qquad - \mu \sum_j \rho_j$

bose - Hubbard model.

\# Spins: $\quad$ spin 1/2.

XXZ model

$$H = J_{xy} \sum_{\langle ij \rangle} (S_i^x S_j^x + S_i^y S_j^y) + J_z \sum_{\langle ij \rangle} S_i^z S_j^z - h^z \sum_j S_j^z$$

$S^\alpha = \frac{1}{2}\sigma^\alpha$

\# Fermions (with spin 1/2)

Hubbard model

$$H = -t \sum_{\langle ij \rangle, \sigma} (c_{i\sigma}^\dagger c_{j\sigma} + h.c.) + U \sum_j \rho_{j\uparrow}\rho_{j\downarrow}$$

z eigenstates of spin. $\qquad - \mu \sum_j (\rho_{j\uparrow} + \rho_{j\downarrow})$

t-V model $\qquad H = -t \sum_{\langle ij \rangle} (c_i^\dagger c_j + h.c) + V \sum_j \rho_j \rho_{j+1} - \mu \sum_j \rho_j$

Exercise: What would be the model with $\frac{U}{2}\sum_j \rho_j(\rho_j - 1)$

$$\begin{cases} \rho_j = \rho_{j\uparrow} + \rho_{j\downarrow} \\ \rho_{j\sigma} = c_{j\sigma}^\dagger c_{j\sigma}. \end{cases}$$

II.] Bosonization

1) Basic problem / idea.

Lieb-Lininger model as an example.

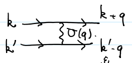
$$H = \int dx \, \frac{\nabla\psi^\dagger \nabla\psi}{2m} + \frac{1}{2}\int dx\,dx' \, U_0\,\delta(x-x')[\rho(x) - \rho_0][\rho(x') - \rho_0]$$

Lieb-Lininger model as an example.

$$H = \int dx\, \frac{\nabla\psi^+\nabla\psi}{2m} + \frac{1}{2}\int dx\,dx'\, U_0\,\delta(x-x')[\rho(x)-\rho_0][\rho(x')-\rho_0] - \mu\int dx\,[\rho(x)-\rho_0]$$

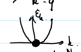
$$\to\ H = \sum_k \varepsilon_k b_k^+ b_k \qquad \varepsilon_k = \frac{k^2}{2m}$$

$$+ \sum_{k k' q} U(q)\, b^+_{k+q} b^+_{k'-q} b_{k'} b_k$$



→ perturbation. → divergent.

Free bosons.



all in $k=0$. $|\psi\rangle = (b_0^+)^N|\emptyset\rangle$

excitations $\varepsilon_k \sim \frac{k^2}{2m}$.

→ Mean field $\psi(x) = \psi_0 e^{i\theta}$.

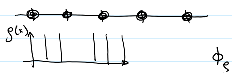
θ, ψ_0 numbers.

→ Something else ?

2) <u>bosonization:</u>

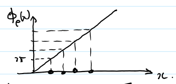
express this problem in term of collective excitations

F.D.M. Haldane PRL 47 1840 (81).

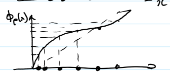


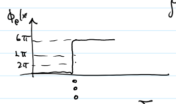
$$\hat\rho(x) = \sum_i \delta(x-\hat x_i).$$

$\phi_\ell(x)$ such that i get a particle when $\phi_\ell(x) = 2\pi n$. $n \in \mathbb{Z}$.



$\phi_\ell(x) = 2\pi\rho_0 x$. → particles separated by ρ_0^{-1}





$$\rho(x) = \sum_i \delta(x-x_i) \qquad \rho(x) = |\nabla\phi_\ell|\sum_n \delta(\phi_\ell(x) - 2\pi n).$$

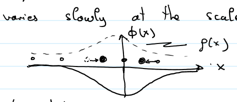
$$\rho(x) = |\nabla_x\phi_\ell(x)|\frac{1}{2\pi}\sum_p e^{ip[\phi_\ell(x)]}$$

$$\phi_\ell(x) = 2\pi\rho_0 x - 2\phi(x).$$

$$\rho(x) = |\rho_0 - \frac{1}{\pi}\nabla\phi(x)|\sum_p e^{i2p[\pi\rho_0 x - \phi(x)]}.$$

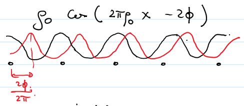
$p=0$ $\phi=0$

$\phi(x)$ varies slowly at the scale of ρ_0^{-1} (average distance between particles.



$p=1$. ϕ = cste

$$\rho = \rho_0 \cos(2\pi\rho_0 x - 2\phi)$$



$\psi(x) = [\rho(x)]^{1/2} \cdot e^{i\theta(x)}$. $\phi(x)$ $\theta(x)$ smooth collective variables.

$$[\psi(x), \psi^+(x')] = \delta(x-x').$$

$$[\phi(x), \frac{1}{\pi}\nabla\theta(x')] = i\,\delta(x-x') \qquad \frac{1}{\pi}\nabla\theta(x) = \Pi_\phi(x)$$

$$[\phi(x), \theta(x)] = 0.$$

$\phi(x) \leftrightarrow$ density fluctuations

$\theta(x) \leftrightarrow$ "superfluid" fluctuations.

θ number θ_0 $\langle\psi\rangle \simeq \rho_0 e^{i\theta_0}$ ← superfluid.

Generalization to higher dimensions TG + P. Le Doussal PRB (1995)

$$\psi^+(x) = [\rho(x)]^{1/2} e^{-i\theta(x)} \underset{\text{normalization.}}{\equiv} \text{"}\rho(x)\text{"}\, e^{-i\theta(x)}.$$

$$\text{"}\psi^+(x) \equiv (\rho_0 - \frac{1}{\pi}\nabla\phi)^{1/2} e^{-i\theta(x)} + \rho_0^{1/2}\sum_{p\neq 0} e^{i2p(\pi\rho_0 - \phi(x))}\, e^{-i\theta(x)}\text{"}$$

\# <u>Hamiltonian:</u>

$$H = \int dx\, \frac{\nabla\psi^+\nabla\psi}{2m}.$$

$$\psi(x) = [\rho(x)]^{1/2} e^{i\theta(x)}. \qquad \nabla\psi = \frac{\nabla\rho(x)}{2\rho(x)^{1/2}} e^{i\theta(x)} + \rho(x)^{1/2} e^{i\theta(x)}\, i\nabla\theta.$$

$$\nabla\psi^+\nabla\psi = \frac{(\nabla\rho)^2}{4\rho(x)} + \rho(x)(\nabla\theta)^2.$$

$$\int dx. \sum_p e^{i2p(\pi\rho_0 x - \phi(x))} \quad (\nabla\phi \,..\, 2\pi \,...)$$

$$\nabla\rho = \nabla[\rho_0 - \frac{1}{\pi}\nabla\phi] \qquad -\frac{1}{\pi}\nabla^2\phi.$$

$$H_{kin} = \frac{1}{2m}\int dx \left[\rho_0(\nabla\theta)^2 + \frac{1}{4\rho_0\pi^2}(\nabla^2\phi)^2\right]$$

$$\frac{1}{\pi}\nabla\theta \equiv \Pi_\phi.$$

$$H_{kin} = \frac{1}{2m}\int dx \left[\rho_0(\pi\Pi_\phi)^2 + \frac{1}{4\rho_0\pi^2}(\nabla^2\phi)^2\right]$$

$$S = i\int \pi\dot\phi - H[\Pi, \phi].$$

$$S = \int dx\,d\tau \left[\rho_0\pi^2\, \Pi^2 + i\,\Pi\,\dot\phi + \dots\right]$$

$$S = i\int \pi\dot\phi - H[\Pi,\phi]$$

$$S = \int dx\,d\tau \left[\frac{\rho_0\pi^2}{2m}\Pi^2 + i\Pi\dot\phi + \dots \right]$$

$$\frac{\rho_0\pi^2}{2m}\left[\Pi + \frac{i\dot\phi 2m}{2\rho_0\pi^2}\right]^2 + \frac{\dot\phi^2\rho_0\pi^2}{4\rho_0^2\pi^4}$$

$$S = \int dx\,d\tau\, \dot\phi^2 \frac{2m}{4\pi^2\rho_0} + \frac{1}{(2m)4\rho_0\pi^2}(\nabla^2\phi)^2$$

$$= \frac{1}{4\pi^2\rho_0}\int dx\,d\tau \left[2m\,\dot\phi^2 + \frac{1}{2m}(\nabla^2\phi)^2 \right]$$

$$= \frac{1}{4\pi^2\rho_0}\int dk\,d\omega_n \left[2m\,\omega_n^2\phi^*_{k\omega}\phi_{k\omega} + k^4\phi^*_{k\omega}\phi_{k\omega} \right]$$

$$S'_{Kin} = \frac{1}{4\pi^2\rho_0}\int dk\,d\omega_n \left[2m\,\omega_n^2 + \frac{1}{2m}k^4 \right]\phi^*_{k\omega}\phi_{k\omega},$$

$$\omega^2 = \frac{k^4}{(2m)^2} \quad\to\quad \omega \sim \frac{k^2}{2m}$$

$$S_{int} = \frac{U}{2}\int dx\,d\tau\, [\rho(x,\tau)-\rho_0][\rho(x,\tau)-\rho_0]$$

$$\rho = \rho_0 - \frac{1}{\pi}\nabla\phi \qquad\qquad S_{int} = \frac{U}{2}\int dx\,\frac{1}{\pi^2}(\nabla\phi)^2$$

$$S = \frac{1}{4\pi^2\rho_0}\int dx\,d\tau\left[2m\,\dot\phi^2 + \frac{1}{2m}(\nabla^2\phi)^2\right] + \int dx\,d\tau\,\frac{U}{2\pi^2}(\nabla\phi)^2$$

Spectrum $\quad -\omega^2 + \alpha k^4 + \text{cte}\, U k^2$

$$\omega = \sqrt{\alpha k^4 + c U k^2}$$

Low energy properties:

→ keep only the most relevant operator $U(\nabla\phi)^2$. $\cancel{(\nabla^2\phi)}$

→ remember: need a cutoff on k. Λ

$$H = \frac{1}{2\pi}\int dx\left[(u K)(\pi\Pi_\phi)^2 + \frac{u}{K}(\nabla\phi)^2\right]$$

$$\begin{cases} \omega = u k. & u: \text{ sound velocity} \\ K & \text{dimensionless parameter} \end{cases}$$

Tomanaga-Luttinger parameters

$$\begin{cases} (u K)\frac{1}{2\pi} = \frac{\rho_0}{2m} \\ \left(\frac{u}{K}\right)\frac{1}{2\pi} = \frac{U}{2\pi^2} \end{cases} \qquad \begin{cases} K \sim 1/\sqrt{U} \\ u = \sqrt{U} \end{cases}$$

$$S = \frac{1}{2\pi K}\int dk\,d\omega_n \left[\frac{1}{u}\omega_n^2 + u k^2\right]\phi^*_{k\omega_n}\phi_{k\omega_n}$$

Correlation functions

$$\langle \psi(x,\tau)\rangle =$$

$$\langle \psi(x,\tau)\psi^\dagger(0,0)\rangle =$$

$$\langle [\rho(x,\tau)-\rho_0][\rho(0,0)-\rho_0]\rangle =$$

$$\langle[\phi(x,\tau)-\phi(0,0)]^2\rangle = \left\langle\left[\sum_{q\omega}\left(e^{i\vec{k}\cdot\vec{r}} - 1\right)\phi_k\right]^2\right\rangle$$

$k = \begin{vmatrix} q \\ \omega_n \end{vmatrix}$

$$= \sum_{k_1k_2}\left(e^{i\vec{k}_1\vec{r}}-1\right)\left(e^{i\vec{k}_2\vec{r}}-1\right)\langle\phi^*_{k_1}\phi_{k_2}\rangle \longleftarrow \frac{K}{\omega_n^2+q^2}\delta_{k_1k_2}$$

$$= K\sum_{k_1}\left(e^{i\vec{k}_1\vec{r}}-1\right)\left(e^{-i\vec{k}_1\vec{r}}-1\right)\frac{1}{\omega_n^2+q^2} \qquad \Leftarrow \|k\|^2$$

$$K\sum_{q\omega}2[1-\cos(qx+\omega_n\tau)]\frac{1}{\omega_n^2+q^2}\, e^{-q/\Lambda}$$

$$\frac{K}{2}\operatorname{Log}\left[\frac{x^2+(|\tau|+\alpha)^2}{\alpha^2}\right] \qquad \alpha \sim 1/\Lambda$$

$$\langle e^{i\phi(x,\tau)}e^{-i\phi(0,0)}\rangle = e^{-\frac{1}{2}\langle[\phi(x,\tau)-\phi(0,0)]^2\rangle} = e^{-\frac{K}{2}\ln[\;]}$$

$$= \left(\frac{\alpha}{\sqrt{x^2+y^2}}\right)^{K/2} \qquad y = |\tau| + \alpha.$$

Power law correlation; Controlled by K.
Prefactor highly "non universal"
(depends on the cutoff procedure).

$$\rho(x,\tau) - \rho_0 = -\frac{1}{\pi}\nabla\phi + \rho_0 e^{i2\pi\rho_0 x - 2\phi(x)} + \rho_0 e^{i4\pi\rho_0 x - 4\phi(x)} + \dots$$

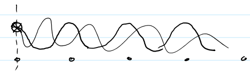
$$\langle[\delta\rho]_{x\tau}[\delta\rho]_{00}\rangle = \frac{1}{\pi^2}\langle\nabla\phi\,\nabla\phi\rangle + \rho_0^2 e^{i2\pi\rho_0 x}\langle e^{-i2\phi(x,\tau)}e^{i2\phi(0,0)}\rangle$$
$$+ \rho_0^2 e^{i4\pi\rho_0 x}\langle e^{-i4\phi(x,\tau)}e^{i4\phi(0,0)}\rangle$$

$$\langle e^{i2\phi}e^{i4\phi}\rangle = e^{-\text{cte}\sum_{q\omega}\frac{1}{q^2+\omega^2}} = e^{-\infty} = 0.$$

$$\langle[\delta\rho]_{x\tau}[\delta\rho]_{00}\rangle = \nabla_{x_1}\nabla_{x_2}\langle\phi(x_1\tau_1)\phi(x_2\tau_2)\rangle = -\frac{1}{2}\nabla_{x_1}\nabla_{x_2}\langle[\phi_{x_1\tau_1}-\phi_{x_2\tau_2}]^2\rangle$$

$$\langle[\delta\rho]_{x\tau}[\delta\rho]_{00}\rangle = \frac{K}{2\pi^2}\frac{(u\tau)^2-x^2}{((u\tau)^2+x^2)^2} + A^{(1)}_\Lambda\left(\frac{\alpha}{\sqrt{x^2+(u\tau)^2}}\right)^{2K}\cos(2\pi\rho_0 x)$$
$$+ A^{(2)}_\Lambda\left(\frac{\alpha}{\sqrt{x^2+(u\tau)^2}}\right)^{8K}\cos(4\pi\rho_0 x)$$

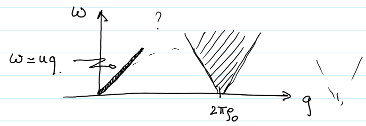
$$\langle [\delta\rho]_{x\tau}[\delta\rho]_{00}\rangle = \frac{K}{2\pi^2}\frac{(u\tau)^2-x^2}{((u\tau)^2+x^2)^2} + A_\Lambda^{(1)}\left(\frac{\alpha}{\sqrt{x^2+(u\tau)^2}}\right)^{2K}\cos(2\pi\rho_0 x) + A_\Lambda^{(2)}\left(\frac{\alpha}{\sqrt{x^2+(u\tau)^2}}\right)^{8K}\cos(4\pi\rho_0 x) + \dots$$



$$\left(\tfrac{1}{x}\right)^{2K}$$

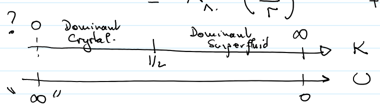
$$\chi(x,\tau) \to \chi(q,\omega_n) \to \chi^{\text{ret}}(q,\omega) \qquad \operatorname{Im}\chi(q,\omega) = A(q,\omega).$$

$$\operatorname{Im}\chi^{\text{ret}}(q,\omega) \equiv \sum_{n,m} e^{-\beta E_n}\,|\langle m|\delta\rho|n\rangle|^2\,\delta(\omega+E_n-E_m)\,(1-e^{-\beta\omega}).$$



$$\langle\psi(x)\rangle \equiv \rho_0^{1/2}\langle e^{i\theta(x)}\rangle = \rho_0^{1/2}\, e^{-\sum_{q\omega}\frac{1}{q^2+\omega^2}} = e^{-\infty} = 0.$$

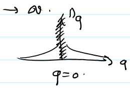
$$\langle\psi(x,\tau)\,\psi^\dagger(0,0)\rangle = \rho_0\langle e^{i\theta(x,\tau)}e^{-i\theta(0,0)}\rangle = \rho_0\, e^{-\frac{1}{2K}\ln[\sqrt{x^2+\tau^2}]}$$
$$= A_\Lambda^\theta\left(\frac{\alpha}{r}\right)^{1/2K} + \dots\dots$$



$$n_q = \int dx\,\langle\psi^\dagger(x)\psi(0)\rangle\, e^{iqx} \qquad x^{1-\frac{1}{2K}}$$
$$(1/x)^{1/2K}$$

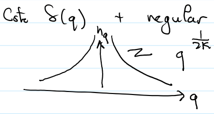
$$n_q \sim q^{\frac{1}{2K}-1}$$

true Superfluid $\langle\psi^\dagger(x)\psi(0)\rangle \to$ Cte $\quad x\to\infty.$



$$n_q \sim \text{Cte}\,\delta(q) + \text{regular}$$

n_q



- Low energy properties of interacting 1D bosonic systems

$$\int dq\, U(q)\,\rho_q\rho_{-q} \to \int U(q)\, q^2\,\phi_q^*\phi_q$$

- Unknown amplitudes
- thrown a lot of terms in H.

3) Concept of TLL.

Exact ! provided that one uses the exact values for u and K.

Find a way to | exactly compute u and K
| extract u and K from experiments. etc.

Specific heat $\quad C_v \sim \frac{T}{u}.$

Finite size dependence of energy $\quad \frac{E(L)-E(\infty)}{L^2} \propto \underset{1.}{\overset{\|}{c}}\,\boxed{u}.$

Response to a twist in b.c. $\quad \psi(L) \sim e^{i\nu}\psi(0) \quad \frac{\partial^2 E_0}{\partial\nu^2} \propto u.K.$

Compressibility $\quad \kappa^{-1} = \frac{u}{K}.$

Bethe Ansatz or Numerics.

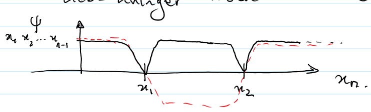
Amplitudes: $\quad \langle\psi(x,0)\,\psi^\dagger(0,0)\rangle \overset{\text{BA}}{\underset{\text{numerically}}{}} A_1\left(\frac{1}{x}\right)^{\nu}$

$$\langle \quad \rangle_\Lambda^{\text{FT}} \quad A_\Lambda\left(\frac{\alpha_\Lambda}{x}\right)^{\nu}.$$

$$\psi = A\, e^{i\theta} \qquad \langle\psi\psi^\dagger\rangle = A^2_{\alpha}\left(\frac{\alpha}{r}\right)^{1/2K}$$

Example of calculation of K and u.

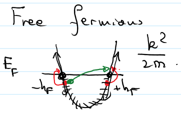
Lieb-Liniger model. $\quad U = \infty.$



$$H_{\text{boson}}^{\text{hard core}} \to H_{\text{free spinless fermion.}}$$

⚠ $\quad \psi_b \neq \psi_F.$

Free fermions $\qquad \rho_b = \rho_F.$



$$\langle\rho\,\rho\rangle \underset{\text{Free fermions}}{\overset{x\text{ large}}{=}} \frac{1}{x^2} + \cos(2k_F x)\,\frac{1}{x^2}.$$

$$\langle\rho\,\rho\rangle_{\text{TLL}} = \frac{1}{x^2} + \cos(2\pi\rho_0 x)\left(\frac{1}{x}\right)^{2K}.$$

$$2k_F = 2\pi\rho_0 \qquad K = 1. \qquad u = \frac{k_F}{}$$

$$\langle \rho\, \rho \rangle_{TLL} = 0\,\frac{1}{x^2} + \cos(2\pi\rho_0 x)\left(\frac{1}{x}\right)^{2K}.$$

$2k_F = 2\pi\rho_0$. $\quad K = 1.\quad u = \frac{k_F}{m}.$

Tonks - Girardeau gas / limit.

$$\langle \psi(x)\,\psi^\dagger(0)\rangle_{bosons} \sim \left(\frac{1}{x}\right)^{1/2K} = \frac{1}{\sqrt{x}}.$$

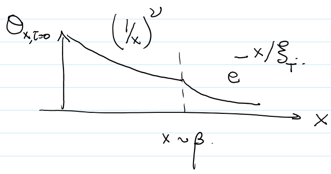
Effect of finite temperature

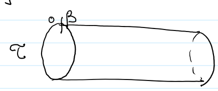
- $$\sum_{q\,\omega} \frac{1}{\omega^2+q^2}\left[1-\cos(qx+\omega\tau)\right]$$

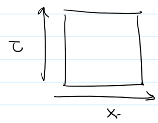
Matsubara

$$\sum_{\omega_n\, q} \frac{1}{\omega_n^2+q^2}\left[1-\cos(qx+\omega_n\tau)\right]$$

- theory is conformally invariant

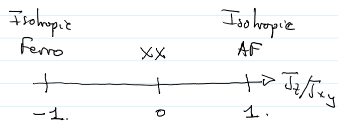






$\xi_T \propto \beta$

3) Spin systems.



$$H = J_{xy}\sum_{\langle ij\rangle}\left(S_i^x S_j^x + S_i^y S_j^y\right) + J_z\sum_{\langle ij\rangle} S_i^z S_j^z - h\sum_j S_j^z$$

$j = i+1$

Spin 1/2	$\lvert\downarrow\rangle$	$\lvert\uparrow\rangle$	S^z	S^+	S^-
hard core boson.	$\lvert 0\rangle$	$\lvert 1\rangle$	$S^z=-1/2\quad n_b=0$ $S^z=1/2\quad n_b=1$	$S^z_j = n_j^b - 1/2.$	

$S^x_j \to \tilde{S}^x_j(-1)^j$
$S^y_j \to \tilde{S}^y_j(-1)^j$
$S^z_j \to \tilde{S}^z_j$

$b^\dagger|0\rangle = |1\rangle \qquad [S^\alpha, S^\beta] = S^\gamma$

$S^+|\downarrow\rangle = |\uparrow\rangle.$

$$H = J_{xy}\sum_j\left(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right) + J_z\sum_j S_j^z S_{j+1}^z - h\sum_j S_j^z$$

$$= J_{xy}\sum_j\left(b_j^\dagger b_{j+1} + b_j b_{j+1}^\dagger\right) + J_z\sum_j (n_j - 1/2)(n_{j+1}-1/2)$$
$$- h\sum_j (n_j - 1/2)$$
$$\left[+\underbrace{\frac{U}{2}}_{\infty}\sum_j n_j(n_j-1)\right]$$

$b_j = (-1)^j\,\tilde{b}_j$

$$H = -J_{xy}\sum_j\left(\tilde{b}_j^\dagger \tilde{b}_{j+1} + h.c.\right) + J_z\sum_j(\tilde{n}_j - 1/2)(n_j - 1/2)$$
$$- h\sum_j(\tilde{n}_j - 1/2)$$
$$+ \text{ hard core.}$$

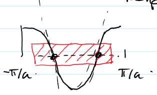
$$H = \sum_k \varepsilon_k\, b_k^\dagger b_k$$
$$\varepsilon_k = -2J_{xy}\cos(k)$$

$$H_{spin} \longrightarrow \frac{1}{2\pi}\int dx\; uK(\pi\Pi)^2 + \frac{u}{K}(\nabla\phi)^2 + J_z\sum_j(n_j - 1/2)(n_{j+1}-1/2).$$

$K = 1$ infinite repulsion.

$u = \left.\frac{\partial\varepsilon_k}{\partial k}\right|_{k=k_F}. \qquad h=0 \qquad \langle S^z_j\rangle = 0 \qquad \langle n_b - 1/2\rangle = 0$

$\langle n_b\rangle = 1/2.$

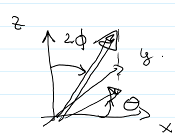


$k_F = \pi/2.$

$$\begin{cases} S^+ \to b_j^\dagger = (-1)^j\,\tilde{b}_j^\dagger = (-1)^j\left[\rho_0^{1/2}e^{-i\theta} + \rho_0^{1/2}e^{i2\pi\rho_0 x - 2\phi}\,e^{-i\theta}\right] \\ S^z = n_j - 1/2 = -\frac{1}{\pi}\nabla\phi + e^{i2\pi\rho_0 x - 2\phi} + h.c. \end{cases}$$

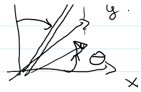
$\uparrow$
n^0

$k_F = \pi/2$



$$\begin{cases} S^z = n_j - \tfrac{1}{2} = -\frac{1}{\pi}\nabla\phi + e^{i2\pi\rho_0 x - 2\phi} + h.c. \end{cases}$$

$\overset{\uparrow}{n_j^0}$ $\qquad k_F = \pi/2.$

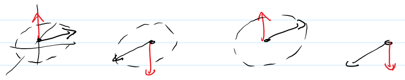


$$\begin{cases} S_j^+ = (-1)^j\, e^{-i\theta(r_j)} + e^{-i\theta(r_j)}\, e^{-i2\phi(r_j)}. \\ S_j^z = -\frac{1}{\pi}\nabla\phi\,(r_j) + (-1)^j\, e^{-i2\phi(r_j)} + h.c. \end{cases}$$

$$\langle S_{r_j}^z S_0^z\rangle = +\frac{1}{\pi^2}\langle\nabla\phi\,\nabla\phi\rangle + (-1)^{r_j}\langle e^{i2\phi(r_j)}\, e^{-i2\phi(0)}\rangle$$

$$= \cdots \tfrac{1}{x^2} + (-1)^x \left(\frac{1}{x}\right)^{2K'}$$

$$\langle S^+ S^-\rangle = \underline{(-1)^x\left(\frac{1}{x}\right)^{\frac{1}{2K'}}} + \left(\frac{1}{x}\right)^{\frac{1}{2K'}+2K'}$$



$J_z = 0 \qquad K = 1$

$$\begin{cases} \langle S_x^+ S_0^-\rangle = (-1)^j \frac{1}{\sqrt{x}} + \left(\frac{1}{x}\right)^{5/2} \\ \langle S_x^z S_0^z\rangle = (-1)^j\left(\frac{1}{x}\right)^2 + \frac{1}{x^2}. \end{cases} \qquad \boxed{J_z = 0}$$

$$\sum_j (n_j - \tfrac{1}{2})(n_{j+1} - \tfrac{1}{2})$$

$$\Big[-\frac{1}{\pi}\nabla\phi_j + (-1)^j e^{i2\phi} + h.c.\Big]\Big[-\frac{1}{\pi}\nabla\phi_{j+1} + (-1)^{j+1} e^{i2\phi_{j+1}} + h.c\Big]$$

$$\int dx\, \frac{1}{\pi^2}(\nabla\phi_x)^2 - \int dx\, \cos(4\phi_x).$$

$$H = \frac{1}{2\pi}\int u\,(\pi\Pi)^2 + u\,(\nabla\phi)^2 + \frac{J_z}{\pi^2}\int dx\,(\nabla\phi)^2 - J_z\int dx \cos(4\phi).$$

$$= \frac{1}{\pi}\int u'K'(\pi\Pi)^2 + \frac{u'}{K'}(\nabla\phi)^2 - J_z\int dx\, \cos(4\phi).$$

$$\begin{cases} u'K' = u. \\ \frac{u'}{K} = u + \frac{J_z}{\pi^2}. \end{cases}$$

Sine-Gordon. Hamiltonian.

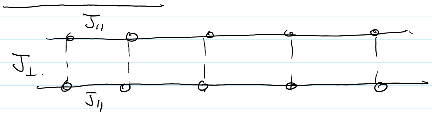
$$K' = \sqrt{\frac{1}{1+\frac{J_z}{\pi^2 u}}.} \qquad J_z \nearrow \qquad K \searrow$$

$$K\Big[\underbrace{J_z/J_{xy}}_{=1}\Big] = \tfrac{1}{2}.$$

Heisenberg point $\quad \langle S_r^z S_0^z\rangle \sim (-1)^r\left(\frac{1}{r}\right)$

$$\langle S_r^+ S_0^-\rangle \sim (-1)^r\left(\frac{1}{r}\right).$$

Problem 1



$$H_1 = H_{xxz}[\vec{S}_1]$$

$$H_2 = H_{xxz}[\vec{S}_2] \qquad \text{Ladder.}$$

$$H = H_1 + H_2 + J_\perp\sum_j \vec{S}_{1j}\cdot\vec{S}_{2j}$$

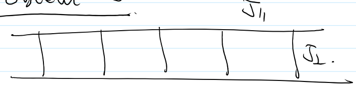
Problem 2

Chain Spin 1. $\qquad H^{(1)} = J\sum_{\langle ij\rangle}\vec{S}_i\cdot\vec{S}_j$

A. Luther $\qquad \vec{S}_{\text{spin }1} = \vec{S}_1 + \vec{S}_2.$

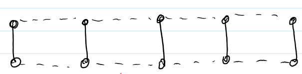
$$H = J\sum_{\langle ij\rangle}(\vec{S}_1 + \vec{S}_2)_j\cdot(\vec{S}_1 + \vec{S}_2)_{j+1}$$

Problem 3



$$H_1 + H_2 + J_\perp\sum_j \vec{S}_1\cdot\vec{S}_2 - h\sum_j(S_1^z + S_2^z).$$

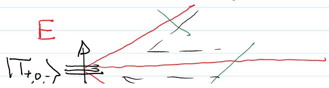
$J_\perp \gg J_\parallel.$



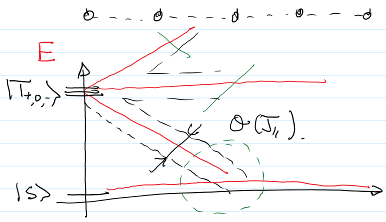
$J_\parallel = 0.$

$$\begin{cases} |T_+\rangle = |\uparrow\uparrow\rangle \\ |T_0\rangle = \frac{1}{\sqrt{2}}\big[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\big] \\ |T_-\rangle = |\downarrow\downarrow\rangle \end{cases} \qquad J/4.$$

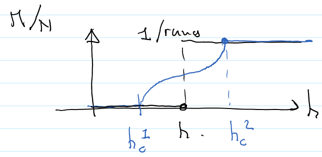
$$|S\rangle = \frac{1}{\sqrt{2}}\big[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big] = -\frac{3J}{4}.$$



H/.

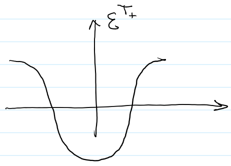


$$|S\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right] = -\frac{3J}{4}.$$



TG + A.M. Tsvelik. PRB (2000!).

$|S\rangle$ $|T_+\rangle$ rung.



$$H = J_\parallel \sum_j \left[\tilde{S}^x_j \tilde{S}^x_{j+1} + \tilde{S}^y_j \tilde{S}^y_{j+1} + \underbrace{\frac{1}{2}}\, \tilde{S}^z_j \tilde{S}^z_{j+1} \right]$$

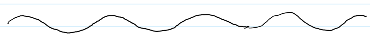
$$|\tilde{\uparrow}\rangle = |T\rangle$$
$$|\tilde{\downarrow}\rangle = |S\rangle.$$

III) Sine-Gordon equation.

1) Problems.

- $$H_{xxz} = \frac{1}{2\pi}\int dx \; uK(\pi\Pi)^2 + \frac{u}{K}(\nabla\phi)^2 - J_z \int dx \cos(4\phi).$$
- Lieb-Liniger $H = \frac{1}{2\pi}\iint [\_\_\_\_\_\_]$

+ periodic lattice.



$V(x) = V_0 \cos(Qx).$

$$H_{\text{quadratic}} + \int dx \; V(x)\,\rho(x)$$

$$V_0 \int dx \; \underline{\cos(Qx)} \left[\rho_0 - \frac{1}{\pi}\nabla\phi + \rho_0\, \underline{e^{i2\pi\rho_0 x - 2\phi}} + \cdots \right]$$

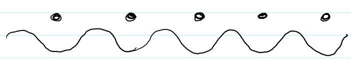
$Q = 2\pi\rho_0 \rightarrow$ no oscillations

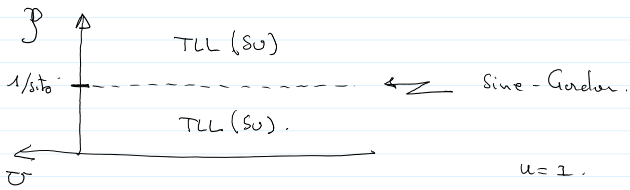
$$V_0 \int dx \; \cos(2\phi(x)).$$

$Q \neq 2\pi\rho_0$

$$V_0 \int dx \; \cos(2\phi(x) + \delta x)$$

$\uparrow$ $Q - 2\pi\rho_0$.

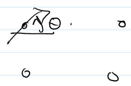




$u = 1.$

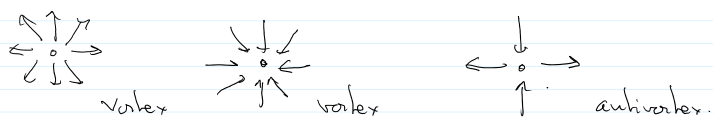
$$S = \frac{1}{2\pi K}\int dx\,d\tau.\left[(\partial_\tau\theta)^2 + (\partial_x\theta)^2 \right] - g\int dx \cos(2\phi(x)).$$

2D XY (classical) model



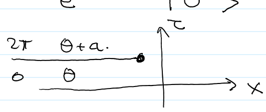
$$H = -J\sum_{\langle ij\rangle}\cos(\theta_i - \theta_j)$$
$$\approx \int dx\,dy \left[(\nabla_x\theta)^2 + (\nabla_y\theta)^2 \right]$$

Spin Wave approximation.
Kosterlitz - Thouless $\rightarrow$ vortices.

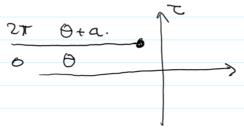


$|x\rangle \rightarrow |x+a\rangle.$ $e^{iaP}|x\rangle \rightarrow |x+a\rangle.$

$$e^{ia\Pi_\theta}\,|\theta\rangle \qquad e^{ia\int_{-\infty}^{x} dx' \underbrace{\Pi_\theta(x',\tau)}_{\frac{1}{\pi}\nabla\phi.}} |\theta\rangle$$



$i\,a\int^{x} dx' \ldots \qquad i\frac{a}{\pi}\,\phi(x,\tau)$



$$\overbrace{\quad\quad}^{}\ \frac{1}{\pi}\nabla\phi$$

$$e^{i\frac{a}{\pi}\int_{-\infty}^{x}dx'\,\nabla\phi} \equiv e^{i\frac{a}{\pi}\phi(x,\tau)}$$

Sine-Gordon theory $\longleftrightarrow$ Coulomb gas / 2D XY model.

*) Renormalization solution:

$$S = \frac{1}{2\pi K}\int dx d\tau\,[(\partial_\tau\phi)^2 + (\partial_x\phi)^2] - g\int dx d\tau \cos(p\phi(x))$$

$$\int \mathcal{D}\phi\ e^{-S_0}\left[1 - g\int dx d\tau \cos(p\phi) + \frac{g^2}{2}\int \underbrace{dr_1}_{dx d\tau} dr_2\ \cos(p\phi_{r_1})\cos(p\phi_{r_2})\right.$$

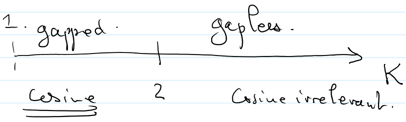
$$\int d^2r_1\, d^2r_2 \quad \langle \cos(p\phi_1)\cos(p\phi_2)\rangle_{S_0}.$$

$$e^{-\frac{p^2}{2}K\ln[r_1-r_2]}$$

$$L^4\ L^{-\frac{p^2}{2}K}.$$

transition (g infinitesimal). $4 = \frac{p^2}{2}K$.

$K = \frac{8}{p^2}$ $\quad p = 2 \quad$ $\underline{K = 2}$.

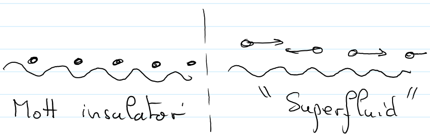


$$S = \int dq d\omega\ [\omega^2 + q^2]\,\phi^*_{q\omega}\phi_{q\omega} - g\int dx d\tau\left[1 - \frac{p^2\phi^2}{2}\right]$$

$$\frac{gp^2}{2}\int dq d\omega\cdot \phi^*_{q\omega}\phi_{q\omega},$$

$$= \int dq d\omega\cdot[\omega^2 + q^2 + m^2]\,\phi^*_{q\omega}\phi_{q\omega}.$$

$m^2 \propto g.$



best quadratic action $\quad S_{trial} = \int dq d\omega\ G^{-1}(q,\omega)\,\phi^*_{q\omega}\phi_{q\omega}.$

Minimize. $\quad F_{true} \leq F_{trial} + \langle S - S_{trial}\rangle_{S_{trial}}.$

\# Renormalization procedure.

theory cutoff Λ $\longrightarrow$ lower cutoff $\Lambda' < \Lambda$

K $\quad\quad K'$

$g.$ $\quad\quad g'$

physics$_1$ $=$ physics$_2$

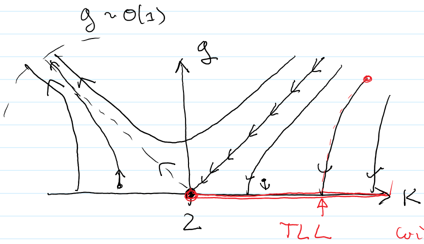
$$\Lambda(\ell) = \Lambda_0 e^{-\ell}.$$

$K(\ell)$ $\quad g(\ell)$.

$$\begin{cases} \dfrac{\partial K}{\partial \ell} = -g^2(\ell). \\[2mm] \dfrac{\partial g}{\partial \ell} = (2 - K(\ell))\,g(\ell). \end{cases}$$

$p = 2$

$g \sim O(1)$



TLL with renormalized values of K.

Quantum phase transition. $\quad$ // $\quad K_c = 2$

2 $\quad$ TLL $\quad$ with renormalized values of K.

Quantum phase transition.

Massless phase $\rightarrow$ TLL. $\qquad \langle \psi_x \psi_0^\dagger \rangle \sim \left(\frac{1}{x}\right)^{1/2K^*}$ $\quad K_c = 2$

\# <u>order in the massive phase</u> ?

$$- \int \cos(\phi) \qquad \int (\partial_x \phi)^2 + (\partial_\tau \phi)^2$$

$$\langle e^{i\frac{\phi}{2}} \rangle \neq 0$$

$\phi = 0 \qquad \langle \ \rangle = 1$

$\phi = 2\pi \qquad \langle \ \rangle = -1$.

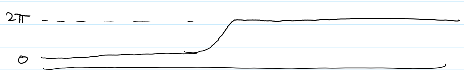
$$\rho(x) = \rho_0 - \frac{1}{\pi} \nabla \phi.$$

$$e^{\frac{i}{2}\phi(x)} = e^{\frac{i\pi}{2}\int_{-\infty}^{x} dy\, \rho(y)}.$$

Non local (String!) order parameter

1 boson / site

3') <u>topological excitations</u>:



$\frac{2\pi}{p}$. $\qquad \cos(p\phi)$. $\qquad \Delta S^z$.

$$\rho(x) = \rho_0 - \frac{1}{\pi}\nabla\phi.$$

$$\int_{-\infty}^{+\infty} [\rho(x) - \rho_0]\, dx = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \nabla\phi\, dy = -\frac{1}{\pi}\left[\phi(+\infty) - \phi(-\infty)\right].$$

Spin chain.

$\uparrow \ \downarrow \ \uparrow \ \downarrow \ \uparrow \ \downarrow \ \uparrow$ $\qquad$ ground state

$\uparrow \ \downarrow \ \uparrow \ \uparrow \ \uparrow \ \downarrow \ \uparrow$ $\qquad \Delta S^z = 1$.

2D - 3D $\quad$ Magnon. $\quad$ "particle" $\quad$ Energy ω $\quad$ Momentum q

$$\omega = \varepsilon(q).$$

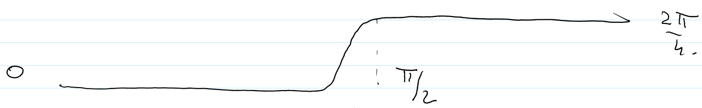
Spin chains $\qquad \cos(4\phi)$.

$$S^z = \rho - 1/2.$$

$$\phi_{-\infty} = 0 \longrightarrow \phi_{+\infty} = \frac{2\pi}{4}.$$

$$\Delta S^z = -\frac{1}{\pi}\left[\phi_{+\infty} - \phi_{-\infty}\right] = -\frac{1}{\pi}\left[\frac{2\pi}{4}\right] = -\frac{1}{2}$$

$$S^z = \rho - 1/2 = -\frac{1}{\pi}\nabla\phi + (-1)^j \cos(2\phi_{r_j})$$



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$\Delta S^z = 1/2$.

Spinon.

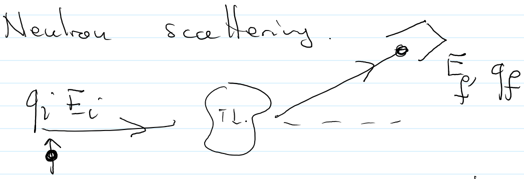
1 magnon $\rightarrow$ 2 spinons.

$$q = q_1 + q_2.$$

$$\vec{q} = \vec{q}_1 + \vec{q}_2 .$$

$$E = \varepsilon(q_1) + \varepsilon(q_2).$$ Continuum of excitation.

Neutron scattering.

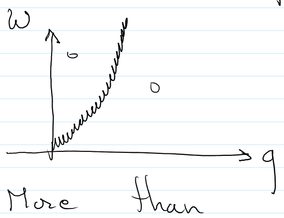


$$\sum_\nu |\langle \nu | S^\alpha | 0 \rangle|^2 \delta(\omega + E_0 - E_\nu).$$

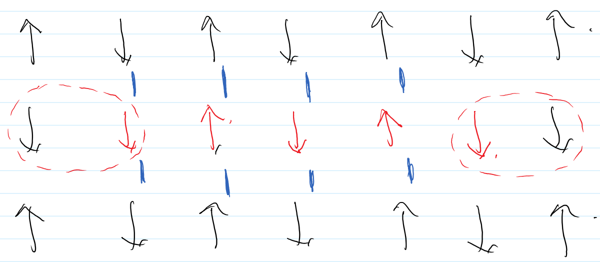
$$\omega = E_f - E_i$$

Fourier transform of the retarded spin-spin correlation.

$$\chi^{\alpha\beta}(q,\omega) = \int_{rt} e^{iqr+\omega t} \left[-i\,\Theta(t) \langle [S^\alpha(r,t), S^\beta(0,0)] \rangle \right.$$



More than 1D.



IV] Fermions:

$$\rho(x) = \rho_0 - \frac{1}{\pi}\nabla\phi + \rho_0 \sum_p e^{ip(2\pi\rho_0 x - 2\phi)}$$

$$\psi_B = [\rho(x)]^{1/2} e^{i\theta(x)}.$$

$$\{\psi(x), \psi^+(x')\}_+ = \delta(x-x').$$

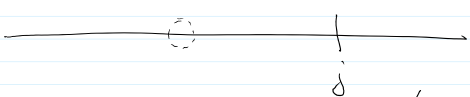
$$\psi_B(x)\, e^{i\frac{1}{2}\phi_\rho(x)} = \psi_F(x). \longrightarrow \{\psi_F(x), \psi_F^+(x')\}_+ = \delta(x-x').$$

$$[\psi_B(x), \psi_B^+(x')] = \delta(x-x').$$

Spins:

$$\begin{cases} S_j^+ = b_j^+ \\ S_j^z = b_j^+ b_j - 1/2 \end{cases}$$

$$\begin{cases} S_j^+ = f_j^+ e^{i\pi \sum_{k<j} f_k^+ f_k} \\ S^z = f^+ f - 1/2 \end{cases}$$

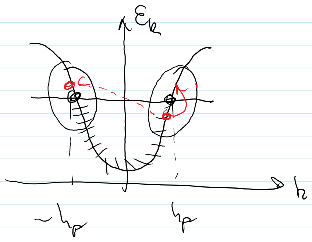


$$\psi_F(x) = \sum_p e^{i2p(\pi\rho_0 x - \phi(x))}\, e^{i(\pi\rho_0 x - \phi(x))}\, e^{i\theta(x)}$$

$$\psi_F(x) = \underbrace{e^{i(\pi\rho_0 x - \phi(x))} e^{i\theta(x)}}_{p=0} + e^{-i(\pi\rho_0 x - \phi(x))} e^{i\theta(x)}.$$

+ higher harmonics.

$+$ higher harmonics.



$$k = \frac{2\pi}{L} p. \qquad \Delta_k = \frac{2\pi}{L} N = 2k_F.$$

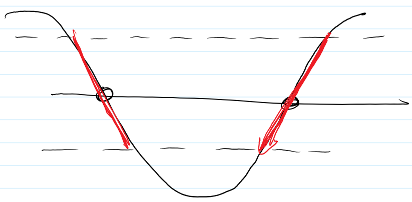
$$\pi \rho_0 = k_F.$$

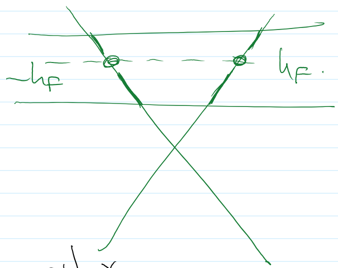
$$\psi_F(x) = e^{ik_F x} e^{i[-\phi(x) + \theta(x)]} + e^{-ik_F x} e^{i[\phi(x) + \theta(x)]}$$

$$\psi_F(x) = \sum_k e^{ikx} c_k = \sum_{k \approx k_F} e^{ikx} c_k^\dagger + \sum_{k \approx -k_F} e^{ikx} c_k.$$

$$= \sum_{q \approx 0} e^{i[k_F + q]x} c_{k_F + q} + \sum_{q \approx 0,} e^{i[-k_F + q]x} c_{-k_F + q}.$$

$$= e^{ik_F x} \underbrace{\sum_{q \approx 0} e^{iqx} c^\dagger_{k_F + q}}_{\psi_R(x)} + e^{-ik_F x} \psi_L(x).$$





$$\psi(x) = e^{ik_F x} \psi_R(x) + e^{-ik_F x} \psi_L(x) \qquad \psi_{R,L} = e^{i[\mp \phi(x) + \theta(x)]}$$

Hubbard model.

$$H = \sum_\sigma \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma}. + U \sum_j \rho_{i\uparrow} \rho_{i\downarrow}.$$

$$\varepsilon_k = -2t \cos(k)$$

$$H = \frac{1}{2\pi} \int dx \left[uK (\pi \Pi_\uparrow)^2 + \frac{u}{K} (\nabla \phi_\uparrow)^2 \right] + H^0_\downarrow$$

$u = v_F \quad K = 1 \quad$ (free fermions)

$$H = U \sum_j \delta\rho_\uparrow \delta\rho_\downarrow$$

$$\left[-\frac{1}{\pi} \nabla \phi_\uparrow + \rho_0 e^{i 2k_F x - 2\phi_\uparrow(x)} + h.c \right]$$

$$\left[-\frac{1}{\pi} \nabla \phi_\downarrow + \rho_0 e^{i 2k_F x - 2\phi_\downarrow(x)} + h.c \right].$$

$$U \left[\frac{1}{\pi^2} \nabla \phi_\uparrow \nabla \phi_\downarrow, + \rho_0^2 e^{i 4k_F x} e^{-i(2\phi_\uparrow + 2\phi_\downarrow)} + h.c \right.$$

$$\rho_0^2 e^{i(2\phi_\uparrow - 2\phi_\downarrow)} + h.c \Big]$$

$$+ \rho_0^2 \, e^{i(2\phi_\uparrow - 2\phi_\downarrow)} + h.c. \Big] .$$

$$H = \frac{v_F}{2\pi} \int \Big[(\pi \Pi_\uparrow)^2 + (\nabla\phi_\uparrow)^2 \Big] + H_\downarrow$$

$$+ \frac{U}{\pi^2} \int \nabla\phi_\uparrow \nabla\phi_\downarrow + U \int e^{i 4k_F x + (2\phi_\uparrow + 2\phi_\downarrow)} \; h.c.$$

$$+ U \int e^{i(2\phi_\uparrow - 2\phi_\downarrow)} + h.c.$$

$$\begin{cases} \phi_\rho = (\phi_\uparrow + \phi_\downarrow) \frac{1}{\sqrt{2}} & \Theta_\rho = \frac{1}{\sqrt{2}} (\Theta_\uparrow + \Theta_\downarrow) & [\phi_\rho, \nabla\Theta_\rho] \ldots \\ \phi_\sigma = (\phi_\uparrow - \phi_\downarrow) \frac{1}{\sqrt{2}} & \Theta_\sigma = \frac{1}{\sqrt{2}} (\Theta_\uparrow - \Theta_\downarrow) . & [\phi_\sigma, \nabla\Theta_\sigma] \ldots \end{cases}$$

$$H_\rho^0 + H_\sigma^0 + \frac{U}{\pi^2} \left((\nabla\phi_\rho)^2 - (\nabla\phi_\sigma)^2 \right) .$$

$$+ U \int dx \; e^{i 4k_F x + 2\sqrt{2}\phi_\rho} + h.c.$$

$$+ U \int dx \; e^{i 2\sqrt{2}\phi_\sigma} + h.c.$$

$H = H_\rho + H_\sigma .$ Spin charge separation !

$$H_\rho = \frac{v_F}{2\pi} \int dx \; (\pi\Pi_\rho)^2 + \left[1 + \frac{U}{\pi^2} \right] (\nabla\phi_\rho)^2 + \frac{U}{\pi^2} \int dx \cos(2\sqrt{2}\phi_\rho + 4k_F x)$$

$$H_\sigma = \frac{v_F}{2\pi} \int dx \; (\pi\Pi_\sigma)^2 + \left[1 - \frac{U}{\pi^2} \right] (\nabla\phi_\sigma)^2 + \frac{U}{\pi^2} \int dx \cos(2\sqrt{2}\phi_\sigma) .$$

$u_\sigma K_\sigma$ $\qquad \frac{u_\sigma}{K_\sigma}$.

$$\begin{cases} K_\sigma > 1 & U > 0 \\ K_\sigma < 1 & U < 0 . \end{cases}$$

$k_F = \pi/2 \to 1$ part / site $\quad \cos(2\sqrt{2}\phi_\rho)$

$k_F \neq \pi/2$ no cosine charge.