Introduction to Floquet
- Lecture notes -

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Preface

These notes are based on a short course on the topic given at SISSA and at GGI in Florence.

Contents

1 Floquet theory: a classical warmup

The term "Floquet" is associated to periodicity in time. More specifically, in a classical context the Floquet theory was introduced to describe the behaviour of a set of *linear* differential equations with a time-periodic coefficients, which was in turn originating from the problem of the stability of periodic orbits in classical mechanics. In the quantum world, where the linearity of the Schrödinger equation is guaranteed from the start, the Floquet theory applies whenever the Hamiltonian governing the system is time-periodic, $\widehat{H}(t) = \widehat{H}(t + \mathrm{T})$, where T is the *period*.

Roughly speaking, one would need distinguish two main regimes of interest:

1) a regime of *slow driving*, often connected to some form of *adiabatic limit*, where the driving frequency $\Omega = \frac{2\pi}{\mathrm{T}}$ is suitably small, formally $\Omega \to 0$;

2) a regime of *fast driving*, when Ω is larger than the proper frequencies of the system that is driven.

This is nowadays a very intense field of research, with many interesting topics, ranging from quantum pumping (in the adiabatic regime) to Floquet engineering (tayloring non-trivial topological properties).

Among the many interesting phenomena, I will start my discussion with some classical physics, related to the following two classes of phenomena:

Parametric resonance: the example of the familiar *swing* (a planar pendulum) illustrate this phenomenon. You remember that a way to increase the amplitude of the swing oscillations is to perform a characteristic "up-and-down" movement of the center-of-mass of your body as the swing oscillates. You do that *twice* every period: going forward, and also backward. This corresponds to a driving period $\mathrm{T} = \mathrm{T}_0/2$, where T_0 is the natural period $\mathrm{T}_0 = 2\pi/\omega_0$, hence a frequency $\Omega_1 = 2\omega_0$. But when you were a little boy, your father would push you, usually *once* every period, hence with $\Omega_2 = \omega_0$. More generally, you could drive the system by "pushing" only once every n half-periods, hence with a driving frequency:

$$\Omega_n = \frac{2\omega_0}{n} \quad \text{with} \quad n = 1, 2, 3, \cdots \tag{1.1}$$

These are the "resonant" driving frequencies of the ordinary pendulum, where ω_0 is its natural frequency.

Dynamical stabilisation: An inverted pendulum is not stable. Nevertheless, as pointed out by Kapitza in 1951, if you oscillate vertically the suspension point at a sufficiently fast frequency Ω, you can stabilise it. The following video illustrates this. A similar phenomenon occurs in a *rotating saddle* (see video), and is also used to create radio-frequency *ionic traps*.

I will illustrate both phenomena with the example of the periodically driven pendulum. More precisely, a pendulum in which you oscillate the suspension point *vertically*. In one shot, we will be able to capture both phenomena in the same simple model.

Next, we will move to the quantum world. As an application, probably the simplest one, we will consider the problem of NMR or of a two-level atom under laser irradiation, close to resonance. We will discuss, using this example, the Shirley-Floquet approach that essentially promotes the Fourier index to a new extra dimension.

Useful references on different aspects of the story are: a tutorial review by a leading expert in the field, M. Holthaus [1], a very nice review on the issue of high-frequency expansions by the group of A. Polkovnikov [2], and two older excellent reviews [3, 4], the latter including also some mathematical aspects concerning the nature of the Floquet spectrum and other delicacies.

Periodically driven pendulum

Let us start considering a very familiar one-dimensional system: a planar pendulum made of a massless rod of length l ending with a point mass m. In the familiar swing, the driving occurs in different ways: if you "drive" the swing yourself, you do it by effectively modifying the position of your "center-of-mass", hence the effective length $l(t)$ of the "pendulum". If you are pushed by someone else, then you have a pendulum with a "periodic external force". We will drive the pendulum in a third (different) way, by oscillating *vertically* its suspension point, which has the advantage that we can describe in the same framework also the inverted pendulum stabilisation If $q = \theta$ denotes the angle formed with the vertical ($\theta = 0$ being the

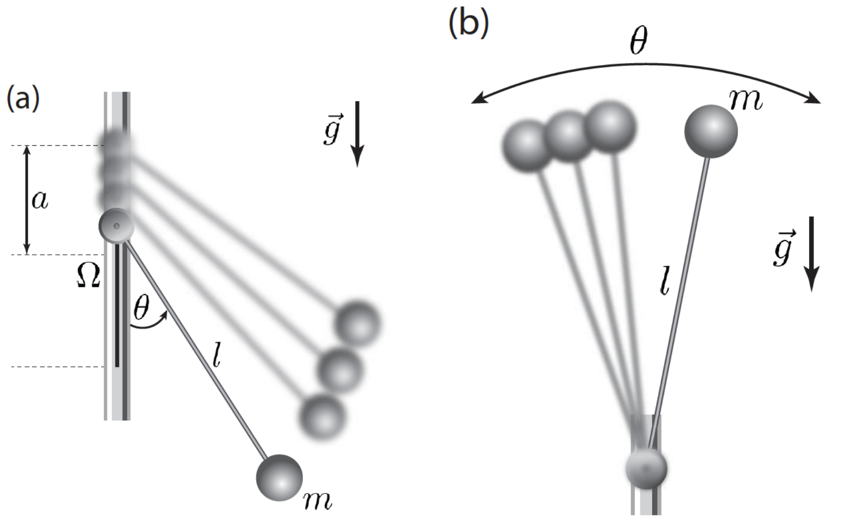

Figure 1.1: (a) Periodically shaken pendulum and (b) Kapitza pendulum. Figure taken from Ref. [2].

downward position), and $y_0(t)$ denotes the position of its suspension point, we can derive the equations of motion from the Lagrangian formalism. In a short while we will assume that $y_0(t) = A\cos\Omega t$, where A is the amplitude of the driving and Ω the driving frequency, but for the time being, let us proceed by keeping $y_0(t)$ to be general. In a system of reference with the y-axis oriented upwards and the x-axis horizontally, the position $x(t)$ and $y(t)$ of the mass m is:

$$\begin{cases} x(t) = l\sin\theta(t) \\ y(t) = y_0(t) - l\cos\theta(t) \end{cases} \longrightarrow \begin{cases} \dot{x}(t) = l\dot{\theta}\cos\theta \\ y(t) = \dot{y}_0 + l\dot{\theta}\sin\theta \end{cases} \tag{1.2}$$

The Lagrangean $\mathcal{L}(\theta, \dot{\theta}, t)$ is therefore given by:

$$\begin{aligned} \mathcal{L}(\theta, \dot{\theta}, t) &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy \\ &= \frac{ml^2\dot{\theta}^2}{2} + ml\dot{y}_0\dot{\theta}\sin\theta + mgl\cos\theta + \cancel{\left(\frac{m}{2}\dot{y}_0^2 - mgy_0\right)} , \end{aligned} \tag{1.3}$$

where we drop the last terms, which are simply functions of time which would not enter in the Euler-Lagrange equations. The associated *momentum* is given by:

$$p_\theta = \frac{\partial\mathcal{L}}{\partial\theta} = ml^2\dot{\theta} + ml\dot{y}_0\sin\theta \qquad \longrightarrow \qquad \dot{\theta} = \frac{p_\theta}{ml^2} - \frac{\dot{y}_0}{l}\sin\theta \ . \tag{1.4}$$

A simple calculation will give us the Hamiltonian $H(\theta, p_\theta, t)$, which we denote by H_{lab} because it is the Hamiltonian in the laboratory reference frame where you observe the suspension point to oscillate:

$$\begin{aligned} H_{\text{lab}}(\theta, p_\theta, t) &= p_\theta\dot{\theta} - \mathcal{L} \\ &= \frac{p_\theta^2}{2ml^2} - p_\theta\frac{\dot{y}_0}{l}\sin\theta + \frac{m}{2}\dot{y}_0^2\sin^2\theta - mgl\cos\theta \\ &= \frac{(p_\theta - ml\dot{y}_0\sin\theta)^2}{2ml^2} - mgl\cos\theta \ . \end{aligned} \tag{1.5}$$

Observe that the laboratory Hamiltonian contains a non-standard kinetic term. But we should be able to describe the same phenomenon in a reference frame that moves together with the suspension point: simply imagine that that pendulum, and the suspension point, are located into an elevator from which you would not see the outside world. In that non-intertial system, the mass would feel a *non-inertial force* due to $\ddot{y}_0$, hence experiencing an effectively time-dependent acceleration of gravity:

$$g \longrightarrow g(t) = g + \ddot{y}_0 = g - A\Omega^2\cos\Omega t \ . \tag{1.6}$$

In essence, by the *equivalence principle*, you would anticipate a moving-frame Hamiltonian with a standard kinetic term and a modified $g(t)$:

$$H_{\text{mov}}(\theta, p_\theta, t) = \frac{p_\theta^2}{2ml^2} - m\,(g + \ddot{y}_0)\,l\cos\theta \tag{1.7}$$

To make sense of the equivalence of the previous two descriptions, one should perform a *canonical transformation* of variables in the Hamiltonian formalism. If you are a bit rusty about canonical transformations in classical Hamiltonian dynamics, here is a nice detour which performs this transformation in a *quantum framework*, which I personally find much more transparent. [1] So, let us promote our Hamiltonians to be quantum, writing:

$$\widehat{H}_{\rm lab}(t) = \frac{(\hat{\rm p}_\theta - ml\dot{y}_0 \sin\theta)^2}{2ml^2} - mgl\cos\theta \tag{1.8}$$

where $\hat{\rm p}_\theta$ is the canonical momentum, with

$$\hat{\rm p}_\theta = -i\hbar\frac{\partial}{\partial\theta} \equiv \widehat{\rm L}_z \; . \tag{1.9}$$

Notice that $\hat{\rm p}_\theta$ is the *angular momentum* around the z-axis. The quantum problem is set in the Hilbert space of *periodic functions* $\psi(\theta) = \psi(\theta+2\pi)$, and the time-dependent Schrödinger equation would read:

$$i\hbar\frac{\partial}{\partial t}\psi(\theta,t) = \widehat{H}_{\rm lab}(t)\psi(\theta,t) \; . \tag{1.10}$$

Consider now a time-dependent *unitary transformation* performed with an operator

$$\widehat{U}_t = {\rm e}^{if(\theta,t)} \; , \tag{1.11}$$

where $f(\theta,t)$ is periodic in θ, and should be suitably chosen so that the transformed kinetic energy is standard. Let us see how the momentum is transformed:

$$\hat{\rm p}_\theta \to \widehat{U}_t^\dagger \hat{\rm p}_\theta \widehat{U}_t = \hat{\rm p}_\theta + \hbar f'(\theta,t) \; . \tag{1.12}$$

where $f' = \partial_\theta f$. The transformation of the Hamiltonian is therefore:

$$\widehat{H}_{\rm lab}(t) \to \widehat{U}_t^\dagger \widehat{H}_{\rm lab}(t)\widehat{U}_t = \frac{(\hat{\rm p}_\theta + \hbar f'(\theta,t) - ml\dot{y}_0\sin\theta)^2}{2ml^2} - mgl\cos\theta \; , \tag{1.13}$$

where we see that the kinetic term becomes standard provided

$$\hbar f'(\theta,t) = ml\dot{y}_0\sin\theta \qquad \longrightarrow \qquad \hbar f(\theta,t) = -ml\dot{y}_0\cos\theta \; . \tag{1.14}$$

But you should refrain from thinking that this transformed Hamiltonian, which in essence becomes that of the standard un-driven pendulum, governs the motion. The unitary transformation, being time-dependent, adds an extra term in the Schrödinger dynamics. This fact is so general, that we formulate it in a more abstract ket-notation, without reference to the specific problem at hand. The result is the following. If $|\psi(t)\rangle = \widehat{U}_t|\widetilde{\psi}(t)\rangle$, then the Schrödinger equation for $|\widetilde{\psi}(t)\rangle$ reads:

$$i\hbar\frac{\partial}{\partial t}|\widetilde{\psi}(t)\rangle = \left[\widehat{U}_t^{-1}\widehat{H}_{\rm lab}(t)\widehat{U}_t - i\hbar\widehat{U}_t^{-1}\frac{\rm d}{{\rm d}t}\widehat{U}_t\right]|\widetilde{\psi}(t)\rangle \equiv \widetilde{H}(t)|\widetilde{\psi}(t)\rangle \; , \tag{1.15}$$

where $\widehat{U}_t^{-1} = \widehat{U}_t^\dagger$ for a unitary transformation, and the transformed Hamiltonian governing the dynamics contains a characteristic *extra term*:

$$\widetilde{H}(t) = \widehat{U}_t^\dagger\widehat{H}_{\rm lab}(t)\widehat{U}_t - i\hbar\widehat{U}_t^\dagger\dot{\widehat{U}}_t \; . \tag{1.16}$$

In the specific case we are studying, $\widehat{U}_t = {\rm e}^{if(\theta,t)}$, the extra term reads:

$$-i\hbar\widehat{U}_t^\dagger\dot{\widehat{U}}_t = \hbar\partial_t f(\theta,t) = -ml\ddot{y}_0\cos\theta \; , \tag{1.17}$$

[1] Sometimes, the quantum route is "simpler" (in the sense that we got more school training to it) that the corresponding classical one. Think of Kubo linear-response theory, as another example.

where we used the choice of $\hbar f(\theta,t) = -ml\dot{y}_0 \cos\theta$ which simplifies the kinetic term. Hence, we get:

$$\widetilde{H}(t) = \widehat{U}_t^\dagger \widehat{H}_{\rm lab}(t)\widehat{U}_t - ml\ddot{y}_0\cos\theta = \frac{\hat{\mathrm{p}}_\theta^2}{2ml^2} - m(g+\ddot{y}_0)l\cos\theta \equiv \widehat{H}_{\rm mov}(t)\ . \tag{1.18}$$

So, the transformed Hamiltonian is precisely the moving-frame Hamiltonian we had guessed on the basis of the equivalence principle.

Let us now return to classical mechanics. From now on we derive our Hamilton's equations from the moving-frame Hamiltonian, which we simply denote by H. The Hamilton's equations read:

$$\begin{cases} \dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{ml^2} \\ \dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = -m\left[g - A\Omega^2\cos(\Omega t)\right]\sin\theta \end{cases} . \tag{1.19}$$

Hence, transforming it into a second-order equation:

$$\frac{d^2\theta}{dt^2} = -\left[\omega_0^2 - \frac{\Omega^2 A}{l}\cos(\Omega t)\right]\sin\theta\ , \tag{1.20}$$

where $\omega_0 = \sqrt{g/l}$ is the frequency of the unperturbed pendulum in the linear regime. This time-dependent *non-linear* equation needs to be studied numerically, in general, and displays chaos in certain regions of parameter space To proceed, one might think of exploring the region of validity of the linear regime at small θ, by studying the stability of the linear-equation solutions. Two obvious fixed points solutions around which to linearise are the ordinary pendulum $\theta(t) = 0$ and the inverted pendulum $\theta(t) = \pi$. Linearising the equation, with the usual substitution $\sin\theta \approx \theta$, in the ordinary pendulum case we get:

$$\frac{d^2\theta}{dt^2} + \left[\omega^2 - \frac{\Omega^2 A}{l}\cos(\Omega t)\right]\theta(t) = 0\ . \tag{1.21}$$

In the inverted pendulum case, posing $\theta = \pi + \phi$ and expanding for small ϕ using $\sin\theta = -\sin\phi \approx -\phi$, we get:

$$\frac{d^2\phi}{dt^2} + \left[-\omega^2 + \frac{\Omega^2 A}{l}\cos(\Omega t)\right]\phi(t) = 0\ . \tag{1.22}$$

Now, let us make the equation dimensionless by measuring time in units of $2/\Omega$, i.e., defining a dimensionless time $t' = \Omega t/2$. Omitting for simplicity the prime, both equations in dimensionless form become:

$$\ddot{\theta} + \left[\epsilon - 2h\cos(2t)\right]\theta(t) = 0\ , \tag{1.23}$$

where we have introduced the two parameters:

$$\text{Ordinary pendulum:} \begin{cases} \epsilon = \left(\dfrac{2\omega_0}{\Omega}\right)^2 \\ h = \dfrac{2A}{l} \end{cases} \qquad \text{Inverted pendulum:} \begin{cases} \epsilon = -\left(\dfrac{2\omega_0}{\Omega}\right)^2 \\ h = -\dfrac{2A}{l} \end{cases} \tag{1.24}$$

In this form, the linearised equation is known as *Mathieu's equation.*

The region of stability of the linearised equation in the $\epsilon - h$ plane is shown in the right part of Fig. 1.2. Shortly, we will show the connection between this problem and the apparently unrelated Schrödinger problem of an electron moving in a periodic potential $V(x) = V_0 \cos(2\pi x/a)$: the stability regions of the Mathieu's equation coincide with the energy regions where allowed energy bands for the Bloch problem are possible. Conversely, regions of instability of the Mathieu equation coincide with the *spectral gap regions* of the Bloch problem.

But, before entering into this, let us make two final comments concerning the problem of a pendulum where the suspension point is oscillated *horizontally*.

Exercise 1.1. Show that the moving-frame Hamiltonian for a pendulum whose suspension point oscillates horizontally $x_0 = A\cos(\Omega t)$ is:

$$H_{\text{mov}}(\theta, p_\theta, t) = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta - mlA\Omega^2\cos(\Omega t)\,\sin\theta\,. \tag{1.25}$$

Show that the second-order equation of motion is:

$$\ddot{\theta} = -\omega_0^2 \sin\theta + \frac{A}{l}\Omega^2\cos(\Omega t)\,\cos\theta \tag{1.26}$$

Deduce that the linearized inverted pendulum now satisfies, with $\theta = \pi + \phi$:

$$\ddot{\phi} = +\omega_0^2\phi - \frac{A}{l}\Omega^2\cos(\Omega t)\,, \tag{1.27}$$

hence is always unstable. Deduce finally that for the ordinary pendulum:

$$\ddot{\theta} = -\omega_0^2\theta + \frac{A}{l}\Omega^2\cos(\Omega t)\,, \tag{1.28}$$

which describes an harmonic oscillator subject to an external horizontal forcing. Notice that this has a resonance at $\Omega = \omega_0$.

A second comment has to do with *feedback control* of dynamical systems. As detailed in this nice MIT lecture, engineers study a lot the possible feedbacks that you can add to a system to stabilise it. Needless to say, the inverted pendulum is a problem that can be very easily stabilsed by a feedback which controls horizontally the suspension point. But the crucial point is that you have to *observe* the value of $\theta(t)$ and $\dot{\theta}(t)$, and provide a horizontal acceleration which (linearly) depends on the observed values. This is not the problem we have dealt with.

Bloch electron in a periodic potential

Consider the quantum Schrödinger eigenvalue (SE) problem for a particle moving in one-dimension in a periodic potential

$$V(x) = V_0\cos\left(\frac{2\pi x}{a}\right) = V_0\cos(Gx)\,, \tag{1.29}$$

where $G = \frac{2\pi}{a}$ is a reciprocal lattice point. The time-independent Schrödinger problem reads:

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V(x)\psi(x) = E\psi(x)\ . \tag{1.30}$$

In absence of potential, the solutions are simple plane waves $\psi_k(x) = \mathrm{e}^{ikx}$ with eigenvalues $E_k^{(0)} = \hbar^2 k^2/(2m)$. In presence of the potential, a relevant energy scale is the unperturbed energy at the border of the first Brillouin-Zone (BZ), i.e., at $k = G/2 = \pi/a$, known in the optical lattice literature as *recoil energy*:

$$E_{\mathrm{R}} = \frac{\hbar^2}{2m}\frac{G^2}{4} = \frac{\hbar^2\pi^2}{2ma^2}\ . \tag{1.31}$$

Introducing the dimensionless variable $\tilde{x} = Gx/2$, you can rewrite the SE as:

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\tilde{x}^2} + \left[\frac{E}{E_{\mathrm{R}}} - \frac{V_0}{E_{\mathrm{R}}}\cos(2\tilde{x})\right]\psi(\tilde{x}) = 0\ . \tag{1.32}$$

Again, omitting the tilde, we get:

$$\psi''(x) + [\epsilon - 2h\cos(2x)]\,\psi(x) = 0\ , \tag{1.33}$$

where

$$\text{Bloch electron:} \quad \begin{cases} \epsilon = \dfrac{E}{E_{\mathrm{R}}} \\ 2h = \dfrac{V_0}{E_{\mathrm{R}}} \end{cases} . \tag{1.34}$$

So, once again, the Mathieu equation emerges. One important aspect of this resemblance has to do with boundary conditions: the solutions to such periodic linear differential equations are themselves *not* strictly periodic, but only periodic "*up-to-a-phase*". In the Schrödinger case, this amounts to the familiar Bloch theorem: the solutions $\psi_k(x)$ with energy E_k can be written as $\psi_k(x) = e^{ikx}u_k(x)$ where k is a quasi-momentum and $u_k(x)$ is a periodic function $u_k(x+a) = u_k(x)$. The wave-vector k can either run over all real axis (extended zone scheme) or be restricted to the first BZ, at the price of introducing an extra band-index n, writing $\psi_{n,k}(x) = e^{ikx}u_{n,k}(x)$, and $E_{n,k}$. Notice that, for $V_0 \neq 0$ the energies $E_{n,k}$ show *spectral gaps* at the boundaries of the BZ, $k = \pm\pi/a$, and at the BZ center $k = 0$, and this happens for all the bands, although the gaps rapidly decrease for increasing n. The lowest gap, obtained from degenerate first-order perturbation theory between the two degenerate solutions $e^{\pm i\pi x/a}$, coupled by the perturbation, is linear in $|V_0|$, while higher gaps are smaller because they come from higher order perturbations. Details about this are given in the next section.

The corresponding picture for the Mathieu's equation is, clearly, identical. By spanning the natural frequency ω_0 one encounters spectral gaps around the unperturbed boundary and Zone-center points, which happen to be at $(2\omega_0/\Omega)^2 = n^2$ with $n = 1, 2, \cdots$. The solutions can be written as

$$\theta_{n,\nu}(t) = \mathrm{e}^{i\nu t}u_{n,\nu}(t)\ , \tag{1.35}$$

where n is a band index, $\nu \in [-\frac{\Omega}{2}, \frac{\Omega}{2}]$ is called Floquet quasi-energy, and $u_{n,\mu}(t)$ is a periodic function, with period $\mathrm{T} = 2\pi/\Omega$. In this form, this results is known as *Floquet theorem.* [2]

[2] A word of caution is in order here. You should not think that the Schrödinger problem in Eq. (1.32) is the

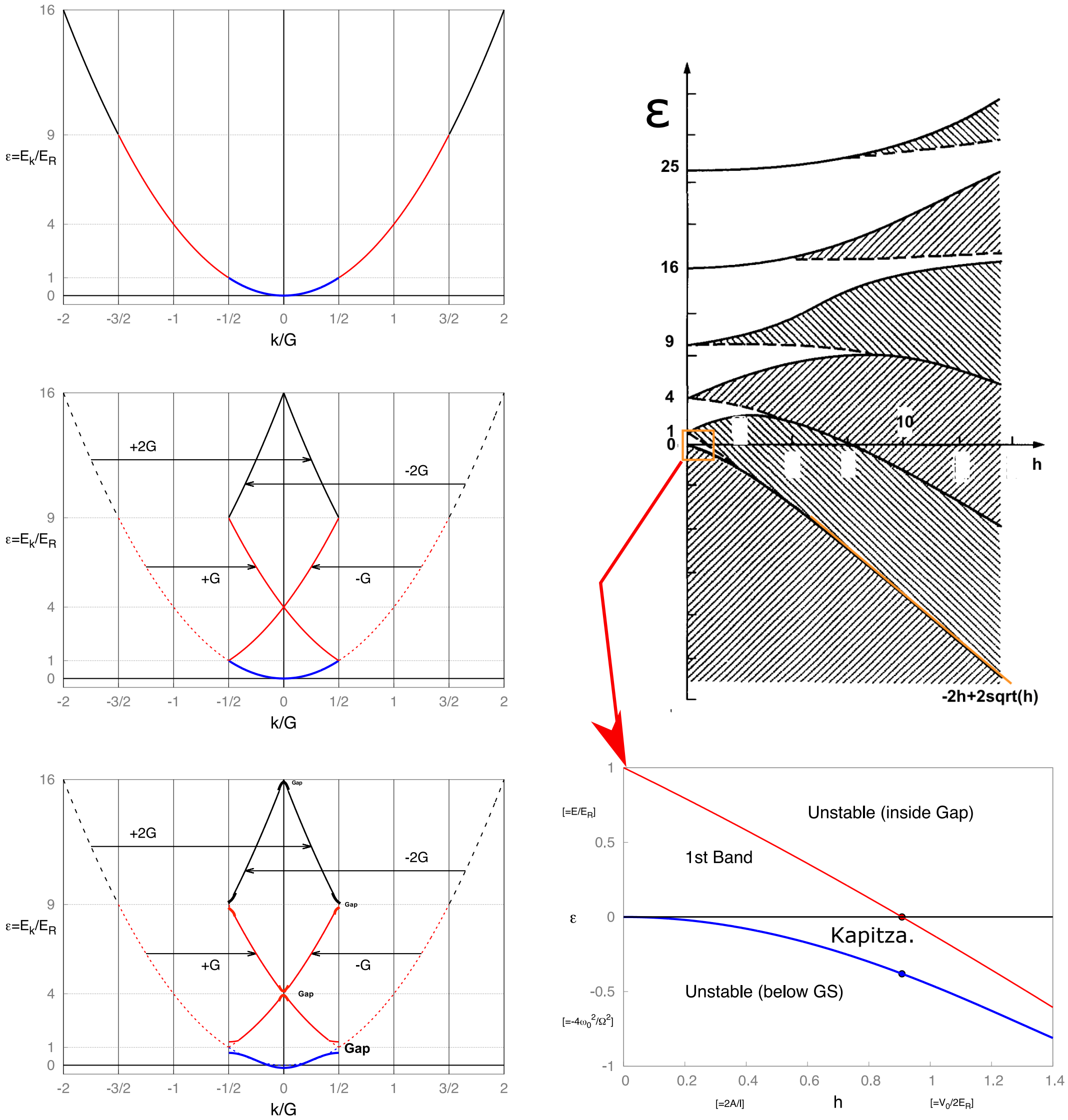

Figure 1.2: Top right: stability diagram for the Mathieu equation; in the y-axis you have $(\omega/\Omega)^2$, in the x-axis the perturbation h. Shaded regions are (Lyapunov) unstable for the linear equation, where the full non-linear equation should to be studied. Bottom right: zoom of the stability diagram. Left: Usual diagram for opening of gaps in the free-electron dispersion, when a cosine potential is turned on. In the Bloch case solutions inside the spectral gaps are simply discarded because they are not associated to allowed wave-functions.

A perturbative approach to the periodic Bloch function

The present section is in some sense a diversion. It shows a general transformation which well be later invoked in the Floquet case, which leads to studying the *periodic part* of the SE $u_{\mathbf{k}}(\mathbf{x})$. Next, we see how one can develop a "perturbation theory" calculation for the Bloch bands and spectral gaps.

The first step is again a general *gauge transformation* that you can apply to the SE for a particle in a periodic potential:

$$\widehat{H}\psi_{\mathbf{k}}(\mathbf{x}) = \left[\frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{x})\right]\psi_{\mathbf{k}}(\mathbf{x}) = E_{\mathbf{k}}\psi_{\mathbf{k}}(\mathbf{x}) \; . \tag{1.38}$$

The transformation is quite general, and we will formulate it somewhat general terms. The crucial aspect is that:

$$\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\,\hat{\mathbf{p}}\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} = \hat{\mathbf{p}} + \hbar\mathbf{k} \; . \tag{1.39}$$

As a consequence, from the Bloch theorem form $\psi_{\mathbf{k}}(\mathbf{x}) = \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\,u_{\mathbf{k}}(\mathbf{x})$, you immediately deduce that:

$$\widehat{H}_{\mathbf{k}}u_{\mathbf{k}}(\mathbf{x}) = \left[\frac{(\hat{\mathbf{p}}+\hbar\mathbf{k})^2}{2m} + V(\mathbf{x})\right]u_{\mathbf{k}}(\mathbf{x}) = E_{\mathbf{k}}u_{\mathbf{k}}(\mathbf{x}) \; . \tag{1.40}$$

where the Hamiltonian $\widehat{H}_{\mathbf{k}} = \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\,\widehat{H}\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}$ is explicitly $\mathbf{k}$-dependent. The advantage is that now $u_{\mathbf{k}}(\mathbf{x})$ is *cell periodic*, $u_{\mathbf{k}}(\mathbf{x}+\mathbf{a}) = u_{\mathbf{k}}(\mathbf{x})$, hence they leave in the same Hilbert space (those of periodic functions) for all $\mathbf{k}$.

Let us return to the original one-dimensional problem with $V(x) = V_0\cos Gx$. The periodic function $u_k(x+a) = u_k(x)$ can be expanded in a the standard Fourier orthonormal basis set of periodic functions in $[0,a]$:

$$\phi_n(x) = \frac{1}{\sqrt{a}}\mathrm{e}^{inGx} \qquad \text{with} \qquad \langle\phi_n|\phi_{n'}\rangle_{\text{cell}} = \int_0^a \mathrm{d}x\,\phi_n^*(x)\phi_{n'}(x) = \delta_{n,n'} \; . \tag{1.41}$$

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quantum problem that correspond to the classical Mathieu's pendulum: they are just the same problem, with different names for the variables. The classical-quantum correspondence is, at the level of the full non-linear problems, that between the $H(t)$ of the classical driven pendulum

$$H_{\mathrm{C}}(\theta, p_\theta, t) = \frac{p_\theta^2}{2ml^2} - mlg(t)\cos\theta \; , \tag{1.36}$$

where $g(t) = g - A\Omega^2\cos(\Omega t)$, and the corresponding equation for the quantum operator $\hat{H}$:

$$\hat{H}_{\mathrm{Q}}(t) = \frac{\hat{\mathrm{p}}_\theta^2}{2ml^2} - mlg(t)\cos\theta \; , \tag{1.37}$$

where now $\hat{\mathrm{p}}_\theta$ is the quantum angular momentum:

$$\hat{\mathrm{p}}_\theta = -i\hbar\frac{\partial}{\partial\theta} \; ,$$

The appropriate Hilbert space is that of normalizable periodic functions $\psi(\theta)$, such that $\psi(\theta+2\pi) = \psi(\theta)$. The eigenfunctions of angular momentum are $\phi_n(\theta) = e^{in\theta}/\sqrt{2\pi}$, but the cosine term couples them. The classical driven pendulum, being a non-linear dynamical system, must be dealt with an explicit integration of the classical Hamilton's equations. Quantum mechanically, on the contrary, we have a bonus: we can study the time-evolution through a Floquet approach by studying the evolution operator over just one period, but the price to pay is that we have an infinite-dimensional problem that we need to integrate over a period: some form of discretization/truncation is essential.

Expanding $u_k(x)$ we write:

$$u_k(x) = \sum_{n=-\infty}^{+\infty} C_n^{(k)} \phi_n(x) \qquad \text{with} \qquad C_n^{(k)} = \langle \phi_n | u_k \rangle_{\text{cell}} = \int_0^a \mathrm{d}x\, \phi_n^*(x) u_k(x) \,. \tag{1.42}$$

The SE then becomes a matrix equation for the Fourier coefficients $C_n^{(k)}$:

$$\frac{\hbar^2}{2m} (nG+k)^2 C_n^{(k)} + \sum_{n'} \langle \phi_n | V | \phi_{n'} \rangle_{\text{cell}} C_{n'}^{(k)} = E_k C_n^{(k)} \,. \tag{1.43}$$

For the specific case [3] of $V(x) = V_0 \cos Gx$, this reduces to a kind of *nearest-neighbohr tight-binding* problem in Fourier space:

$$4\left(n + \frac{k}{G}\right)^2 C_n^{(k)} + \frac{V_0}{2E_{\text{R}}} \left(C_{n+1}^{(k)} + C_{n-1}^{(k)} \right) = \frac{E_k}{E_{\text{R}}} C_n^{(k)} \,. \tag{1.44}$$

Quite clearly, we can set the problem in matrix form.

Let us consider the case $k = 0$, at the center of the BZ. Defining $V_0/E_{\text{R}} = v_0$, and omitting the superscript in the C_n coefficients, the matrix will have the form:

$$\begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 16 & v_0/2 & 0 & 0 & 0 & \cdots \\ \cdots & v_0/2 & 4 & v_0/2 & 0 & 0 & \cdots \\ \cdots & 0 & v_0/2 & 0 & v_0/2 & 0 & \cdots \\ \cdots & 0 & 0 & v_0/2 & 4 & v_0/2 & \cdots \\ \cdots & 0 & 0 & 0 & v_0/2 & 16 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ C_{-2} \\ C_{-1} \\ C_0 \\ C_{+1} \\ C_{+2} \\ \vdots \end{bmatrix} = \frac{E_0}{E_{\text{R}}} \begin{bmatrix} \vdots \\ C_{-2} \\ C_{-1} \\ C_0 \\ C_{+1} \\ C_{+2} \\ \vdots \end{bmatrix} \tag{1.45}$$

If you truncate the matrix to the central "0" (a 1×1 matrix), you get the lowest order approximation $E_0 = 0$. Indeed, a truncation to a 1×1 central block will give all the unperturbed eigenvalues at the center of the BZ: 0, 4 (doubly deg.), 16 (doubly deg.), etc. To the next order, you consider the 3×3 matrix that you see in the central part. The 3 eigenvalues are:

$$\lambda = \begin{cases} 2 + 2\sqrt{1 + v_0^2/8} \approx 4 + \frac{v_0^2}{8} \\ 4 \\ 2 - 2\sqrt{1 + v_0^2/8} \approx -\frac{v_0^2}{8} \end{cases} \tag{1.46}$$

This is, evidently, the result of a second-order perturbation theory calculation. And you might proceed further, by considering the 5×5 block you see above, which would give you a result up to 4^{th} order in perturbation theory.

[3] Use that:

$$\langle \phi_n | \mathrm{e}^{\pm iGx} | \phi_{n'} \rangle_{\text{cell}} = \delta_{n,n'\pm 1} \,.$$

Next, let us consider the border of the BZ, $k = \frac{G}{2}$.

$$\left[\begin{array}{c|c|c|c|c|c} & \vdots & \vdots & \vdots & \vdots & \\ \hline \cdots & 4\left(-\frac{3}{2}\right)^2 & v_0/2 & 0 & 0 & \cdots \\ \hline \cdots & v_0/2 & 4\left(-\frac{1}{2}\right)^2 & v_0/2 & 0 & \cdots \\ \hline \cdots & 0 & v_0/2 & 4\left(+\frac{1}{2}\right)^2 & v_0/2 & \cdots \\ \hline \cdots & 0 & 0 & v_0/2 & 4\left(+\frac{3}{2}\right)^2 & \cdots \\ \hline & \vdots & \vdots & \vdots & \vdots & \end{array}\right] \left[\begin{array}{c} \vdots \\ \hline C_{-2} \\ C_{-1} \\ C_0 \\ C_{+1} \\ \hline \vdots \end{array}\right] = \frac{E_{\frac{G}{2}}}{E_{\mathrm{R}}} \left[\begin{array}{c} \vdots \\ \hline C_{-2} \\ C_{-1} \\ C_0 \\ C_{+1} \\ \hline \vdots \end{array}\right] \qquad (1.47)$$

Here the lowest-order perturbation theory comes from the central 2×2 block: you recognize a *degenerate first-oder* result, with the two eigenvalues being

$$\lambda_{\pm} = 1 \pm \frac{v_0}{2} \ .$$

Once again, increasing the block size gives higher-order perturbation theory and higher bands at the border of the BZ.

Lyapunov exponents

Let us make a step back and return to a more general setting of a motion in classical phase space.

Let $\mathbf{X}^{(0)}(t) = (\mathbf{q}^{(0)}(t), \mathbf{p}^{(0)}(t))$ collectively denote the solution of a Newtonian mechanics flow (Hamiltonian or dissipative, doesn't matter) in the n-dimensional phase space:

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}, t) \ , \qquad (1.48)$$

starting from some initial condition at time t_0: $\mathbf{X}^{(0)}(t_0) = \mathbf{X}_0$. Consider now a different phase-trajectory $\mathbf{X}(t)$ starting at $t = t_0$ from a nearby point $\mathbf{X}_0 + \mathbf{w}_0$, and define $\mathbf{w}(t)$ to be the deviation $\mathbf{X}(t) - \mathbf{X}^{(0)}(t) = \mathbf{w}(t)$. In components, expanding the flow equation around the unperturbed trajectory we have:

$$\begin{aligned} \dot{X}_i = \cancel{\dot{X}_i^{(0)}} + \dot{w}_i &= F_i(\mathbf{X}^{(0)}(t) + \mathbf{w}(t), t) \\ &= \cancel{F_i(\mathbf{X}^{(0)}(t), t)} + \sum_j \frac{\partial F_i}{\partial X_j}(\mathbf{X}^{(0)}(t), t)\, w_j(t) + \cdots \ , \end{aligned} \qquad (1.49)$$

where the $\cdots$ indicate higher order terms in $\mathbf{w}$. Defining $\mathbf{J}(t)$ to be the Jacobian matrix

$$[\mathbf{J}]_{ij}(t) = \frac{\partial F_i}{\partial X_j}(\mathbf{X}^{(0)}(t), t) \ , \qquad (1.50)$$

and dropping higher-order terms we end-up with the linearized equations:

$$\dot{\mathbf{w}}(t) = \mathbf{J}(t) \cdot \mathbf{w}(t) \ . \qquad (1.51)$$

If $\mathbf{J}$ was *independent of* t, then the solution of the linearized problem would be an exponential $\mathbf{w}(t) = \mathrm{e}^{\mathbf{J}(t-t_0)} \cdot \mathbf{w}_0$, and we would analize its stability in terms of eigenvalues of $\mathbf{J}$ (stability is guaranteed if the real part of all eigenvalues is negative). Unfortunately, when $\mathbf{J}$ depends on t, the solution can only be given in terms of a "time-ordered exponential", roughly the same difficulty that you encounter when the quantum Hamiltonian is time-dependent. Nevertheless, the fact that $\mathbf{w}(t)$ must be *linearly related* to $\mathbf{w}(t_0)$ is simple to grasp: the matrix connecting $\mathbf{w}(t)$ to $\mathbf{w}(t_0)$ is the *propagator* $\mathbf{L}(t,t_0)$ in terms of which [4]

$$\mathbf{w}(t) = \mathbf{L}(t,t_0)\cdot\mathbf{w}(t_0)\ . \tag{1.52}$$

The mathematicians have been able to prove (within the theory of ergodic multiplicative processes) that the following limit exists:

$$\lambda_L = \lim_{t\to\infty}\frac{1}{2t}\log\Big(\mathrm{Tr}[\mathbf{L}^\dagger(t,t_0)\mathbf{L}(t,t_0)]\Big)\ . \tag{1.53}$$

λ_L is called the (maximum) Lyapunov exponent. Notice that in the time-independent case, this is just the maximum real part of the eigenvalues of $\mathbf{J}$. Whenever $\lambda_L > 0$, the solution $\mathbf{X}^{(0)}(t)$ is *unstable* and small deviations $\mathbf{w}_0$ of the initial condition are (generally) amplified in an exponential way. [5]

The previous theory generalises to the time-dependent case the "small oscillation" expansion around equilibrium points which should be familiar to you from elementary mechanics.

Classical Floquet-Lyapunov theory

Let us now consider the important case where the linearised problem has a Jacobean which is *periodic in time*:

$$\dot{\mathbf{w}}(t) = \mathbf{J}(t)\cdot\mathbf{w}(t) \quad \text{with} \quad \mathbf{J}(t+\mathrm{T}) = \mathbf{J}(t)\ , \tag{1.54}$$

where T is the period. This might occur in different circumstances, for instance:

1) to analyse the stability of a *time-periodic orbit* $\mathbf{X}^{(0)}(t+\mathrm{T}) = \mathbf{X}^{(0)}(t)$ in an autonomous system, where the flow itself $\mathbf{F}(\mathbf{X})$ does not depend on time explicitly: this includes, for instance, a "limiting cycle" in a dissipative system.

2) to analyse the linear stability of an *external periodic driving* around some (time-independent) fixed point solution $\mathbf{X}^{(0)}$: this is the case we encountered in the driven pendulum.

[4] Evidently, the propagator $\mathbf{L}(t,t_0)$ satisfies an entirely similar linear equation:

$$\frac{d}{dt}\mathbf{L}(t,t_0) = \mathbf{J}(t)\cdot\mathbf{L}(t,t_0)\ ,$$

whose (formal) solution is:

$$\mathbf{L}(t,t_0) = \mathrm{Texp}[\int_{t_0}^{t} dt'\mathbf{J}(t')] \stackrel{\mathrm{def}}{=} \sum_{n=0}^{\infty}\frac{1}{n!}\int_{t_0}^{t}dt_1\cdots\int_{t_0}^{t}dt_n\mathrm{T}[\mathbf{J}(t_1)\cdots\mathbf{J}(t_n)]\ ,$$

where, by definition, $\mathrm{T}[\cdots]$ orders the "operators" with the prescription "later times to the left". Among the properties of the propagator, it is worth mentioning that

$$\mathbf{L}(t_2,t_0) = \mathbf{L}(t_2,t_1)\cdot\mathbf{L}(t_1,t_0)$$

[5] Obviously, if the flow is Hamiltonian, since by Liouville's theorem the phase-space volume must be conserved, there must be "directions" that are exponentially shrinked, as $\mathrm{e}^{-\lambda_L t}$.

One should *not* think that a solution of the linear problem in Eq. (1.54) with a periodic $\mathbf{J}(t)$ should be periodic as well. As for the electronic bands in a crystalline solid, recall that the solutions of the problem, the Bloch states $\psi_k(x)$, are themselves not periodic: as assured by Bloch's theorem, they can be written as the product of a periodic function $u_k(x)$ times a phase factor e^{ikx}: $\psi_k(x) = \mathrm{e}^{ikx} u_k(x)$. What we are going to show is, essentially, a similar result for the solutions of the periodic linear problem in Eq. (1.54). Indeed, a theory due to Floquet and Lyapunov (see for instance F.R. Gantmacher, *Theory of Matrices*, Chap. XIV, Sec. 3) shows that n linearly independent solutions of the linear time-periodic problem can always be written as a product of a time-periodic part $\mathbf{u}_j(t)$ times a pure "exponential term" $\mathrm{e}^{\lambda_j(t-t_0)}$:

$$\mathbf{w}_j(t) = \mathrm{e}^{\lambda_j(t-t_0)} \mathbf{u}_j(t) \qquad \text{with } \; \mathbf{u}_j(t+\mathrm{T}) = \mathbf{u}_j(t) \; . \tag{1.55}$$

The eigenvalues λ_j can be deduced by studying the propagator over one period. They play a crucial role: if $\mathrm{Re}\,\lambda_j > 0$ the linear system will be *unstable*. I will give you the details of how this comes about — in a slightly adapted proof — after discussing the quantum version of the theorem. It suffices here to say that the crucial ingredient in the story is the one-period propagator $\mathbf{L}_0 = \mathbf{L}(t_0+\mathrm{T}, t_0)$, which can be obtained by integrating the matrix version of the flow equation $\dot{\mathbf{W}} = \mathbf{J}(t)\cdot\mathbf{W}(t)$, in general *numerically*, starting from initial condition $\mathbf{W}(t_0) = \mathbf{1}$:

$$\mathbf{W}(t_0+\mathrm{T}) = \mathbf{L}(t_0+\mathrm{T}, t_0)\cdot\mathbf{W}(t_0) \equiv \mathbf{L}(t_0+\mathrm{T}, t_0) \; . \tag{1.56}$$

Assume that $\mathbf{L}(t_0+\mathrm{T}, t_0)$ can be diagonalized. It means that n eigenvectors exist, denote them by $\mathbf{u}_j(t_0)$, such that:

$$\mathbf{L}_0\cdot\mathbf{u}_j(t_0) = \mu_j \mathbf{u}_j(t_0) \qquad \text{with } \; j = 1, \cdots, n \; , \tag{1.57}$$

where $\mu_j \in \mathbb{C}$, in general. We will see later on the important role played by these eigenvectors, known as *Floquet modes*. Here, it is enough to mention that the eigenvalues μ_j appearing in Eq. (1.57) are related to the λ_j discussed in Eq. (1.55) as follows:

$$\lambda_j = \frac{1}{\mathrm{T}} \log \mu_j \; , \tag{1.58}$$

Hence, the *stability* of the linear problem requires

$$\forall j : \qquad |\mu_j| \le 1 \qquad \Longleftrightarrow \qquad \mathrm{Re}\,\lambda_j \le 0 \; . \tag{1.59}$$

Application to the Mathieu equation

Let us apply the previous construction to studying the stability of the linear driven pendulum. First of all, we transform the Mathieu's equation into a linear $(n=2)$-dimensional problem by defining:

$$\mathbf{w}(t) = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} \qquad \Longrightarrow \qquad \dot{\mathbf{w}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 2h\cos(2t) - \epsilon & 0 \end{bmatrix}}_{\mathbf{J}(t)} \cdot \mathbf{w}(t) = \mathbf{J}(t)\cdot\mathbf{w}(t) \; . \tag{1.60}$$

where you should observe that $\mathrm{Tr}[\mathbf{J}(t)] = 0$. Next define the Floquet operator $\mathbf{L}_0 = \mathbf{L}(\mathrm{T}, 0)$ by solving the problem over one period $\mathrm{T} = \pi$ (for instance, by numerical integration) starting from the identity matrix $\mathbf{W}(0) = \mathbf{1}$:

$$\mathbf{W}(\mathrm{T}) = \mathbf{L}(\mathrm{T}, 0) \cdot \mathbf{W}(0) = \mathbf{L}(\mathrm{T}, 0) = \mathbf{L}_0 \; . \tag{1.61}$$

The general Jacobi identity (see later for a hint of a proof):

$$\det[\mathbf{W}(t)] = \det[\mathbf{W}(t_0)] \, \mathrm{e}^{\int_{t_0}^{t} \mathrm{d}t' \; \mathrm{Tr}[\mathbf{J}(t')]} \; . \tag{1.62}$$

together with the fact that $\mathrm{Tr}[\mathbf{J}(t)] = 0$ immediately implies that the 2×2 matrix $\mathbf{L}_0$ should have unit determinant: $\det \mathbf{L}_0 = 1$. Hence, in terms of the two eigenvalues $\mu_{1,2}$ of $\mathbf{L}_0$, we deduce that:

$$\det[\mathbf{L}_0] = \mu_1 \mu_2 = 1 \qquad \Longrightarrow \qquad |\mu_1||\mu_2| = 1 \; . \tag{1.63}$$

Hence, two possibilities are left:

stable) $\mu_1 = \mathrm{e}^{i\nu T} = \mu_2^*$: both eigenvalues are on the unit circle in the complex plane, and are complex conjugate. Hence $\mathrm{Re}\,\lambda_j = 0$, and the solutions are oscillatory.

unstable) $|\mu_1| < 1$, say, but then $|\mu_2| = |\mu_1|^{-1} > 1$, hence the system is unstable.

Exercise 1.2. It is an instructive (numerical) exercise to investigate the region of stability of the Mathieu equation in the $\epsilon - h$ plane by following the previous route. As discussed, this gives automatically the spectral gaps of the Bloch electron problem in a periodic potential. In the driven pendulum case you can try to see (again, numerically) how the stability is modified by adding a "viscous friction" force term $-\gamma\dot{\theta}$ in the Newton's equation.

2 Floquet theory in quantum mechanics

The term "Floquet" is associated to periodicity in time for periodically driven *linear* systems. In the quantum world, where the linearity of the Schrödinger equation is guaranteed from the start, the Floquet theory applies whenever $\widehat{H}(t) = \widehat{H}(t + \mathrm{T})$, where T is the *period*.

In this chapter we will first give a "proof" of the Floquet theorem in a Schrödinger setting, and then proceed showing how one can adapt such a proof to the classical case. We then discuss a simple periodically driven two-level system, relevant to NMR and to quantum optics, which can be solved exactly. We will also discuss the Shirley-Floquet approach which illustrates the emergence of "time as an extra dimension", in a regime of "slow driving".

The Floquet theorem

The following proof follows quite closely the approach of Ref. [1]. We start from the time-dependent Schrödinger equation (SE)

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \widehat{H}(t)|\psi(t)\rangle \ . \tag{2.1}$$

In principle, the whole dynamics is captured by the unitary time-evolution operator $\widehat{U}(t, t_0)$:

$$|\psi(t)\rangle = \widehat{U}(t, t_0)|\psi(t_0)\rangle \ , \tag{2.2}$$

which is in general a complicated time-ordered exponential. The Floquet theorem can be formulated into two equivalent way. On one hand, one can show for $t \in [t_0, t_0 + T]$:

$$\widehat{U}(t + n\mathrm{T}, t_0) \equiv \widehat{U}(t, t_0) \left[\widehat{U}(t_0 + \mathrm{T}, t_0)\right]^n \ . \tag{2.3}$$

Hence, the knowledge of $\widehat{U}(t, t_0)$ for $t \in [t_0, t_0 + T]$ is enough to write the evolution operator at any arbitrary time $t + n\mathrm{T}$. A particularly important role is played by the one-period evolution operator:

$$\widehat{F}_0 \stackrel{\mathrm{def}}{=} \widehat{U}(t_0 + \mathrm{T}, t_0) = \mathrm{e}^{-\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}\mathrm{T}} \ , \tag{2.4}$$

where $\widehat{H}_{\mathrm{F}_0}$ is in principle the Hermitean operator in terms of which one can always rewrite the unitary operator $\widehat{F}_0$. The second part of the story is that a diagonalization of $\widehat{F}_0$ provides in principle important states, the Floquet modes:

$$\widehat{F}_0|u_j(t_0)\rangle = \mathrm{e}^{-\frac{i}{\hbar}\epsilon_j \mathrm{T}}|u_j(t_0)\rangle \ , \tag{2.5}$$

which provide a *complete basis for the Hilbert space*. The associated ϵ_j are known as *quasi-energies*, and are determined only modulo $\hbar\Omega$. From here, with a simple step we arrive at constructing a *complete set of solutions of the time-dependent SE* which have the form:

$$|\psi_j(t)\rangle = \mathrm{e}^{-\frac{i}{\hbar}\epsilon_j(t-t_0)}\,|u_j(t)\rangle \qquad \text{with:} \qquad |u_j(t+\mathrm{T})\rangle = |u_j(t)\rangle\ . \tag{2.6}$$

Remarkably, given *any* initial state $|\psi(t_0)\rangle$, one can expand it in the basis of the Floquet modes:

$$|\psi(t_0)\rangle = \sum_j |u_j(t_0)\rangle\langle u_j(t_0)|\psi(t_0)\rangle = \sum_j C_j|u_j(t_0)\rangle\ , \tag{2.7}$$

with coefficients $C_j = \langle u_j(t_0)|\psi(t_0)\rangle$. At this point, its time-evolution can be simply expressed as:

$$|\psi(t) = \sum_j C_j \mathrm{e}^{-\frac{i}{\hbar}\epsilon_j(t-t_0)}\,|u_j(t)\rangle\ . \tag{2.8}$$

Let us see how to prove these statements.

Proof of the Floquet theorem

1) Evidently, the evolution operator $\widehat{U}(t,t_0)$, satisfies itself the SE, for any t_0. More generally, you can show that the following three general properties hold for the evolution operator:

$$\textbf{General}: \quad \begin{cases} [\mathbf{P1}] & i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}(t,t_0) = \widehat{H}(t)\widehat{U}(t,t_0) \\ [\mathbf{P2}] & \widehat{U}(t_0,t_0) = \mathbb{1} \\ [\mathbf{P3}] & \widehat{U}(t_2,t_0) = \widehat{U}(t_2,t_1)\widehat{U}(t_1,t_0) \end{cases} \ . \tag{2.9}$$

2) Let us now consider the particular case of a time-periodic Hamiltonian $\widehat{H}(t+\mathrm{T}) = \widehat{H}(t)$. Take the evolution operator $\widehat{U}(t+n\mathrm{T}, t_0+n\mathrm{T})$, which coincides with the identity for $t=t_0$, and write the SE for it. It reads:

$$\begin{aligned} i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}(t+n\mathrm{T},t_0+n\mathrm{T}) &= \widehat{H}(t+n\mathrm{T})\,\widehat{U}(t+n\mathrm{T},t_0+n\mathrm{T}) \\ &= \widehat{H}(t)\,\widehat{U}(t+n\mathrm{T},t_0+n\mathrm{T})\ , \end{aligned} \tag{2.10}$$

where in the last step we used the time-periodicity of $\widehat{H}(t)$. Hence we conclude that:

$$[\mathbf{P4}] \qquad \widehat{U}(t+n\mathrm{T},t_0+n\mathrm{T}) \equiv \widehat{U}(t,t_0)\ , \tag{2.11}$$

since these two propagators satisfy the same differential equation and the same initial value boundary condition at $t=t_0$. Property [**P4**] is the crucial property, peculiar to the time-periodic case, that the evolution operator satisfies in the time-periodic case: in words, you can cancel off an integer number of periods from the two times appearing in it.

3) Consider now $\widehat{U}(t_0+n\mathrm{T},t_0)$. Using [**P3**] repeatedly, and then [**P4**], you immediately conclude that:

$$\begin{aligned} \widehat{U}(t_0+n\mathrm{T},t_0) &\overset{[\mathbf{P3}]}{=} \widehat{U}(t_0+n\mathrm{T},t_0+(n-1)\mathrm{T})\cdots\widehat{U}(t_0+2\mathrm{T},t_0+\mathrm{T})\,\widehat{U}(t_0+\mathrm{T},t_0) \\ &\overset{[\mathbf{P4}]}{=} \widehat{U}(t_0+\mathrm{T},t_0)\cdots\widehat{U}(t_0+\mathrm{T},t_0)\,\widehat{U}(t_0+\mathrm{T},t_0) \\ &= \Big[\widehat{U}(t_0+\mathrm{T},t_0)\Big]^n \equiv \widehat{F}_0^n\ . \end{aligned} \tag{2.12}$$

In words, a propagation by an integer number of periods $n\mathrm{T}$ can be performed by applying n times the crucial *one-period*, or *Floquet*, propagator

$$[\mathbf{P5a}] \qquad \widehat{F}_0 \stackrel{\text{def}}{=} \widehat{U}(t_0+\mathrm{T},t_0) \qquad \Longrightarrow \qquad \widehat{U}(t_0+n\mathrm{T},t_0) = \widehat{F}_0^n \;, \tag{2.13}$$

where the subscript "0" in $\widehat{F}_0$ reminds us of the dependence on the initial time t_0.

4) Finally, consider $\widehat{U}(t+n\mathrm{T},t_0)$ where you can evidently restrict $t \in [t_0, t_0+\mathrm{T})$, since a larger value of t can be reabsorbed in n. Then, generalizing [**P5a**]:

$$[\mathbf{P5b}] \qquad \widehat{U}(t+n\mathrm{T},t_0) \stackrel{[\mathbf{P3}]}{=} \widehat{U}(t+n\mathrm{T},t_0+n\mathrm{T})\,\widehat{U}(t_0+n\mathrm{T},t_0) \stackrel{[\mathbf{P4}]}{=} \widehat{U}(t,t_0)\,\widehat{F}_0^n \;. \tag{2.14}$$

In conclusion, and quite remarkably, if you have access to $\widehat{U}(t,t_0)$ for $t \in [t_0, t_0+\mathrm{T}]$ you can write the propagator for an arbitrarily large time $t+n\mathrm{T}$.

5) The crucial ingredient in the story is the one-period propagator $\widehat{F}_0 = \widehat{U}(t_0+\mathrm{T},t_0)$, which can be obtained by integrating for one-period the SE equation, in general *numerically*, starting from initial condition $\mathbf{W}(t_0) = \mathbf{1}$.

$$\mathbf{W}(t_0+\mathrm{T}) = \widehat{U}(t_0+\mathrm{T},t_0)\,\mathbf{W}(t_0) \equiv \widehat{U}(t_0+\mathrm{T},t_0) \;. \tag{2.15}$$

Now, the Floquet operator, being *unitary* can be in principle diagonalised. [1] It means that a complete set of eigenvectors exist, let us indicate them by $|\mathbf{u}_j(t_0)\rangle$, such that

$$\widehat{F}_0|u_j(t_0)\rangle = \mathrm{e}^{-\frac{i}{\hbar}\epsilon_j \mathrm{T}}|u_j(t_0)\rangle \;, \tag{2.16}$$

where $\epsilon_j \in \mathbb{R}$, since the eigenvalues of a unitary operator must stay on the unit circle in the complex plane. We will soon see the crucial role played by these eigenvectors $|u_j(t_0)\rangle$, known as *Floquet modes*. Correspondingly, the phases ϵ_j are known as *quasi-energies*.

Observe that this implies that one can construct "$\log \widehat{F}_0$", or, equivalently, write:

$$\widehat{F}_0 = \mathrm{e}^{-\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}\mathrm{T}} \;, \tag{2.17}$$

where $\widehat{H}_{\mathrm{F}_0}$ is *Hermitean*. Using this, we can write:

$$\widehat{U}(t,t_0) = \underbrace{\widehat{U}(t,t_0)\mathrm{e}^{+\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}(t-t_0)}}_{\widehat{P}(t,t_0)}\,\mathrm{e}^{-\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}(t-t_0)} \equiv \widehat{P}(t,t_0)\,\mathrm{e}^{-\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}(t-t_0)} \;, \tag{2.18}$$

where $\widehat{P}(t,t_0)$ is given by:

$$\widehat{P}(t,t_0) \stackrel{\text{def}}{=} \widehat{U}(t,t_0)\,\mathrm{e}^{+\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}(t-t_0)} \;. \tag{2.19}$$

You can easily show that:

$$\begin{cases} \widehat{P}(t_0,t_0) = \widehat{U}(t_0,t_0) = \mathbb{1} \\ \widehat{P}(t_0+\mathrm{T},t_0) = \widehat{U}(t_0+\mathrm{T},t_0)\cdot \mathrm{e}^{+\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}\mathrm{T}} = \widehat{F}_0 \cdot \widehat{F}_0^\dagger = \mathbb{1} \end{cases} \tag{2.20}$$

[1] All normal operator, commuting with their Hermitean conjugate, can be in principle diagonalized by a unitary transformation.

and, more generally, that $\widehat{P}(t,t_0)$ is *time-periodic*:

$$\begin{aligned}\widehat{P}(t+\mathrm{T},t_0) &= \widehat{U}(t+\mathrm{T},t_0)\,\mathrm{e}^{+\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}(\mathrm{T}+t-t_0)} \\ &\stackrel{[\mathbf{P5b}]}{=} \widehat{U}(t,t_0)\,\widehat{F}_0\,\mathrm{e}^{+\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}\mathrm{T}}\,\mathrm{e}^{+\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}(t-t_0)} \equiv \widehat{P}(t,t_0)\;. \end{aligned} \tag{2.21}$$

The time-propagation of the Floquet modes $|u_j(t_0)\rangle$ is particularly noteworthy. It allows us to define a complete basis of Floquet states in the Hilbert space, which have the form:

$$\begin{aligned}|\psi_j(t)\rangle = \widehat{U}(t,t_0)|u_j(t_0)\rangle &= \widehat{P}(t,t_0)\,\mathrm{e}^{-\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}(t-t_0)}|u_j(t_0)\rangle \\ &= \mathrm{e}^{-\frac{i}{\hbar}\epsilon_j(t-t_0)}\;\widehat{P}(t,t_0)|u_j(t_0)\rangle \\ &= \mathrm{e}^{-\frac{i}{\hbar}\epsilon_j(t-t_0)}\;|u_j(t)\rangle\;, \end{aligned} \tag{2.22}$$

where

$$|u_j(t)\rangle \equiv \widehat{P}(t,t_0)|u_j(t_0)\rangle \qquad \text{is time-periodic: } |u_j(t+\mathrm{T})\rangle = |u_j(t)\rangle\;. \tag{2.23}$$

Summarizing, the Floquet theorem guarantees that there is a complete set of solutions of the time-dependent SE which have the form

$$|\psi_j(t)\rangle = \mathrm{e}^{-\frac{i}{\hbar}\epsilon_j(t-t_0)}\,|u_j(t)\rangle \qquad \text{with:} \qquad |u_j(t+\mathrm{T})\rangle = |u_j(t)\rangle\;. \tag{2.24}$$

Proof of the Floquet theorem for a classical linear system

In the following I have adapted the presentation given by Gantmacher to show its close analogy with the derivation give above for the time-periodic quantum Schrödinger problem. We will see that the essential difference is in the fact that the Floquet operator is no longer a unitary operator. The part that is different from the quantum case is in blue.

1) An n-dimensional linear problem admits n linearly independent solutions. Indeed, let $\mathbf{w}_1(t)$ be a solution starting from the initial value $\mathbf{w}_1(t_0)$, and $\mathbf{w}_2(t)$ a second solution starting from $\mathbf{w}_2(t_0)$: clearly, if α is a constant and $\mathbf{w}_2(t_0)=\alpha\mathbf{w}_1(t_0)$, then $\mathbf{w}_2(t)=\alpha\mathbf{w}_1(t)$; viceversa, if $\mathbf{w}_2(t_1)=\alpha\mathbf{w}_1(t_1)$ at a given value of $t=t_1$, then $\mathbf{w}_2(t)=\alpha\mathbf{w}_1(t)$ at all t, by the uniqueness of the Cauchy initial value problem. One can prove that this is general:[2] with $\mathbf{w}_1(t_0),\cdots,\mathbf{w}_n(t_0)$ are n linearly independent vectors, the $\mathbf{w}_1(t),\cdots,\mathbf{w}_n(t)$ will be n linearly independent solutions, which can be arranged as *columns* of an $n\times n$ matrix $\mathbf{W}(t)$ which is itself a matrix solution, known as *matrix integral*, of the linear equation

$$\dot{\mathbf{W}}(t) = \mathbf{J}(t)\cdot\mathbf{W}(t)\;, \tag{2.26}$$

with an associated propagator

$$\mathbf{W}(t) = \mathbf{L}(t,t_0)\cdot\mathbf{W}(t_0)\;. \tag{2.27}$$

[2]For a proof, see Gantmacher, Chap. XIV, Sec.1, where the following *Jacobi identity* is shown:

$$\det[\mathbf{W}(t)] = \det[\mathbf{W}(t_0)]\;\mathrm{e}^{\int_{t_0}^{t}\mathrm{d}t'\;\mathrm{Tr}[\mathbf{J}(t')]}\;. \tag{2.25}$$

The essential part of the proof is to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\det[\mathbf{W}(t)] = \mathrm{Tr}[\mathbf{J}(t)]\,\det[\mathbf{W}(t)]$$

which follows from differentiating $\det[\mathbf{W}(t)] = \sum_P(-1)^P W_{1P_1}W_{2P_2}\cdots W_{nP_n}$, using $\dot{W}_{ij} = \sum_k J_{ik}W_{kj}$.

Evidently, the propagator $\mathbf{L}(t,t_0)$, satisfies an identical linear equation, for any t_0. You can show that the following three general properties hold for the propagator:

$$\textbf{General}: \begin{cases} [\mathbf{P1}] & \dot{\mathbf{L}}(t,t_0) = \mathbf{J}(t)\cdot\mathbf{L}(t,t_0) \\ [\mathbf{P2}] & \mathbf{L}(t_0,t_0) = \mathbf{1} \\ [\mathbf{P3}] & \mathbf{L}(t_2,t_0) = \mathbf{L}(t_2,t_1)\cdot\mathbf{L}(t_1,t_0) \end{cases} . \tag{2.28}$$

2) Consider now the propagator $\mathbf{L}(t+n\mathrm{T}, t_0+n\mathrm{T})$. Evidently, $\mathbf{L}(t_0+n\mathrm{T}, t_0+n\mathrm{T}) = \mathbf{1}$. But on the other hand, if $\mathbf{J}(t)$ is time-periodic, $\mathbf{J}(t+n\mathrm{T}) = \mathbf{J}(t)$, then:

$$\begin{aligned} \dot{\mathbf{L}}(t+n\mathrm{T}, t_0+n\mathrm{T}) &= \mathbf{J}(t+n\mathrm{T})\cdot\mathbf{L}(t+n\mathrm{T}, t_0+n\mathrm{T}) \\ &= \mathbf{J}(t)\cdot\mathbf{L}(t+n\mathrm{T}, t_0+n\mathrm{T}) . \end{aligned} \tag{2.29}$$

Hence we conclude that:

$$[\mathbf{P4}] \qquad \mathbf{L}(t+n\mathrm{T}, t_0+n\mathrm{T}) \equiv \mathbf{L}(t,t_0) , \tag{2.30}$$

since these two propagators satisfy the same differential equation and the same initial value boundary condition at $t = t_0$. Property [**P4**] is the crucial property, peculiar to the time-periodic case, that a Floquet propagator satisfies: in words, you can cancel off an integer number of periods from the two times appearing in the propagator.

3) Consider now $\mathbf{L}(t_0+n\mathrm{T}, t_0)$. Using [**P3**] repeatedly, and then [**P4**], you immediately conclude that:

$$\begin{aligned} \mathbf{L}(t_0+n\mathrm{T}, t_0) &\overset{[\mathbf{P3}]}{=} \mathbf{L}(t_0+n\mathrm{T}, t_0+(n-1)\mathrm{T})\cdots\mathbf{L}(t_0+2\mathrm{T}, t_0+\mathrm{T})\cdot\mathbf{L}(t_0+\mathrm{T}, t_0) \\ &\overset{[\mathbf{P4}]}{=} \mathbf{L}(t_0+\mathrm{T}, t_0)\cdots\mathbf{L}(t_0+\mathrm{T}, t_0)\cdot\mathbf{L}(t_0+\mathrm{T}, t_0) \\ &= [\mathbf{L}(t_0+\mathrm{T}, t_0)]^n \equiv \mathbf{L}_0^n . \end{aligned} \tag{2.31}$$

In words, a propagation by an integer number of periods $n\mathrm{T}$ can be performed by applying n times the crucial *one-period*, or *Floquet*, propagator

$$[\mathbf{P5a}] \qquad \mathbf{L}_0 \overset{\text{def}}{=} \mathbf{L}(t_0+\mathrm{T}, t_0) \qquad \Longrightarrow \qquad \mathbf{L}(t_0+n\mathrm{T}, t_0) = \mathbf{L}_0^n , \tag{2.32}$$

where the subscript "0" in $\mathbf{L}_0$ reminds us of the dependence on the initial time t_0.

4) Finally, consider $\mathbf{L}(t+n\mathrm{T}, t_0)$ where you can evidently restrict $t \in [t_0, t_0+\mathrm{T})$, since a larger value of t can be reabsorbed in n. Then, generalizing [**P5a**]:

$$[\mathbf{P5b}] \qquad \mathbf{L}(t+n\mathrm{T}, t_0) \overset{[\mathbf{P3}]}{=} \mathbf{L}(t+n\mathrm{T}, t_0+n\mathrm{T})\cdot\mathbf{L}(t_0+n\mathrm{T}, t_0) \overset{[\mathbf{P4}]}{=} \mathbf{L}(t,t_0)\,\mathbf{L}_0^n . \tag{2.33}$$

In conclusion, and quite remarkably, if you have access to $\mathbf{L}(t,t_0)$ for $t \in [t_0, t_0+\mathrm{T}]$ you can write the propagator for an arbitrarily large time $t+n\mathrm{T}$.

5) The crucial ingredient in the story is the one-period propagator $\mathbf{L}_0 = \mathbf{L}(t_0+\mathrm{T}, t_0)$, which can be obtained by integrating the flow equation $\dot{\mathbf{W}} = \mathbf{J}(t)\cdot\mathbf{W}(t)$, in general *numerically*, starting from initial condition $\mathbf{W}(t_0) = \mathbf{1}$:

$$\mathbf{W}(t_0+\mathrm{T}) = \mathbf{L}(t_0+\mathrm{T}, t_0)\cdot\mathbf{W}(t_0) \equiv \mathbf{L}(t_0+\mathrm{T}, t_0) . \tag{2.34}$$

The question is: can $\mathbf{L}(t_0+\mathrm{T}, t_0)$ be *diagonalized*? Although the proof of this fact does not seem straightforward to me, it is essentially given for granted in Gantmacher's presentation. [3]

[3] Obviously, the diagonalization should be carried out in the complex field. Nevertheless, as far as I can see, the non-singular nature of $\mathbf{L}_0$ is not enough to guarantee that $\mathbf{L}_0$ can be diagonalized. But perhaps I am missing something.

So, let us assume that $\mathbf{L}(t_0+\mathrm{T},t_0)$ can be indeed diagonalized. It means that n eigenvectors exist, let us indicate them by $\mathbf{u}_j(t_0)$, such that

$$\mathbf{L}_0\cdot\mathbf{u}_j(t_0)=\mu_j\mathbf{u}_j(t_0) \qquad \text{with } \; j=1,\cdots,n\,, \tag{2.35}$$

where $\mu_j\in\mathbb{C}$, in general. We will soon see the crucial role played by these eigenvectors, known as *Floquet modes*. Observe that this implies that one can construct "$\log\mathbf{L}_0$", and therefore write:

$$\mathbf{L}(t,t_0)=\underbrace{\mathbf{L}(t,t_0)\cdot \mathrm{e}^{-\frac{(t-t_0)}{\mathrm{T}}\log\mathbf{L}_0}}_{\mathbf{P}(t,t_0)}\cdot \mathrm{e}^{\frac{(t-t_0)}{\mathrm{T}}\log\mathbf{L}_0}\equiv\mathbf{P}(t,t_0)\cdot\mathrm{e}^{\frac{(t-t_0)}{\mathrm{T}}\log\mathbf{L}_0}\,, \tag{2.36}$$

where $\mathbf{P}(t,t_0)$ is given by:

$$\mathbf{P}(t,t_0)\stackrel{\text{def}}{=}\mathbf{L}(t,t_0)\,\mathrm{e}^{-\frac{(t-t_0)}{\mathrm{T}}\log\mathbf{L}_0}\,. \tag{2.37}$$

You can easily show that:

$$\begin{cases}\mathbf{P}(t_0,t_0)=\mathbf{L}(t_0,t_0)=\mathbf{1}\\ \mathbf{P}(t_0+\mathrm{T},t_0)=\mathbf{L}(t_0+\mathrm{T},t_0)\cdot\mathrm{e}^{-\log\mathbf{L}_0}=\mathbf{L}_0\cdot\mathbf{L}_0^{-1}=\mathbf{1}\end{cases} \tag{2.38}$$

and, more generally, that $\mathbf{P}(t,t_0)$ is *time-periodic*:

$$\begin{aligned}\mathbf{P}(t+\mathrm{T},t_0) &= \mathbf{L}(t+\mathrm{T},t_0)\cdot\mathrm{e}^{-\frac{(t+\mathrm{T}-t_0)}{\mathrm{T}}\log\mathbf{L}_0}\\ &\stackrel{[\mathbf{P5b}]}{=} \mathbf{L}(t,t_0)\cdot\mathbf{L}_0\cdot\mathrm{e}^{-\left(1+\frac{(t-t_0)}{\mathrm{T}}\right)\log\mathbf{L}_0}\equiv\mathbf{P}(t,t_0)\,.\end{aligned} \tag{2.39}$$

The time-propagation of the n Floquet modes $\mathbf{u}_j(t_0)$ is particularly noteworthy. It allows us to define n Floquet solutions of the linear problem which have the form:

$$\begin{aligned}\mathbf{w}_j(t)=\mathbf{L}(t,t_0)\cdot\mathbf{u}_j(t_0) &= \mathbf{P}(t,t_0)\cdot\mathrm{e}^{\frac{(t-t_0)}{\mathrm{T}}\log\mathbf{L}_0}\cdot\mathbf{u}_j(t_0)\\ &= \mathrm{e}^{\frac{(t-t_0)}{\mathrm{T}}\log\mu_j}\,\mathbf{P}(t,t_0)\cdot\mathbf{u}_j(t_0)\\ &= \mathrm{e}^{\lambda_j(t-t_0)}\,\mathbf{u}_j(t)\,,\end{aligned} \tag{2.40}$$

where

$$\lambda_j=\frac{1}{\mathrm{T}}\log\mu_j\,, \tag{2.41}$$

and

$$\mathbf{u}_j(t)\equiv\mathbf{P}(t,t_0)\cdot\mathbf{u}_j(t_0) \qquad \text{is time-periodic: }\; \mathbf{u}_j(t+\mathrm{T})=\mathbf{u}_j(t)\,. \tag{2.42}$$

Clearly, the *stability* of the linear problem requires

$$\forall j: \qquad |\mu_j|\le 1 \qquad \Longleftrightarrow \qquad \mathrm{Re}\,\lambda_j\le 0\,. \tag{2.43}$$

The periodic moving frame and the extended Hilbert space

Let us return to the quantum case. Suppose that you have in some way constructed the unitary periodic part $\widehat{P}(t,t_0)$ of the evolution operator. Can we use it to move to a "rotating frame"? What would be the "effective Hamiltonian" governing the motion in such a rotating frame?

Recall that if $|\psi(t)\rangle=\widehat{P}(t,t_0)|\widetilde{\psi}(t)\rangle$, then the Schrödinger equation for $|\widetilde{\psi}(t)\rangle$ reads:

$$i\hbar\frac{\partial}{\partial t}|\widetilde{\psi}(t)\rangle=\left[\widehat{P}^{-1}\widehat{H}(t)\widehat{P}-i\hbar\widehat{P}^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{P}\right]|\widetilde{\psi}(t)\rangle\equiv\widetilde{H}(t)|\widetilde{\psi}(t)\rangle\,, \tag{2.44}$$

where $\widehat{P}^{-1} = \widehat{P}^\dagger$ for a unitary transformation, and the transformed Hamiltonian governing the dynamics contains a characteristic *extra term*:

$$\widetilde{H}(t) = \widehat{P}^\dagger \widehat{H}(t)\widehat{P} - i\hbar \widehat{P}^\dagger \dot{\widehat{P}} \,. \tag{2.45}$$

Recall that:

$$\widehat{P}(t,t_0) \stackrel{\text{def}}{=} \widehat{U}(t,t_0)\, \mathrm{e}^{+\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}(t-t_0)} \,. \tag{2.46}$$

From this you derive that:

$$\begin{aligned} i\hbar\dot{\widehat{P}} &= i\hbar\dot{\widehat{U}}\, \mathrm{e}^{+\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}(t-t_0)} - \widehat{U}\, \mathrm{e}^{+\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}(t-t_0)}\, \widehat{H}_{\mathrm{F}_0} \\ &= \widehat{H}(t)\widehat{U}\, \mathrm{e}^{+\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}(t-t_0)} - \widehat{P}(t,t_0)\, \widehat{H}_{\mathrm{F}_0} \\ &= \widehat{H}(t)\widehat{P}(t,t_0) - \widehat{P}(t,t_0)\, \widehat{H}_{\mathrm{F}_0} \end{aligned} \tag{2.47}$$

hence:

$$i\hbar \widehat{P}^\dagger \dot{\widehat{P}} = \widehat{P}^\dagger \widehat{H}(t)\widehat{P} - \widehat{H}_{\mathrm{F}_0} \,. \tag{2.48}$$

Therefore, the effective Hamiltonian in the moving frame is t-independent and coincides with $\widehat{H}_{\mathrm{F}_0}$:

$$\widetilde{H}(t) = \widehat{P}^\dagger \widehat{H}(t)\widehat{P} - i\hbar \widehat{P}^\dagger \dot{\widehat{P}} = \widehat{H}_{\mathrm{F}_0} \,. \tag{2.49}$$

This result should be taken with some care for several reasons [1]. First of all, the construction of $\widehat{P}$ requires a solution of the Floquet problem. Second, the Hamiltonian $\widehat{H}_{\mathrm{F}_0}$ is not really completely defined. [4] Recall that, by definition of Floquet modes as eigenvectors of $\widehat{F}_0 \equiv \widehat{U}(t_0+\mathrm{T},t_0) \equiv \mathrm{e}^{-\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}\mathrm{T}}$ you can write:

$$\mathrm{e}^{-\frac{i}{\hbar}\widehat{H}_{\mathrm{F}_0}\mathrm{T}}|u_j(t_0)\rangle = \mathrm{e}^{-\frac{i}{\hbar}\epsilon_j\mathrm{T}}|u_j(t_0)\rangle \,, \tag{2.50}$$

which suggests that the Floquet modes $|u_j(t_0)\rangle$ are indeed eigenvectors of $\widehat{H}_{\mathrm{F}_0}$ with eigenvalue ϵ_j:

$$\widehat{H}_{\mathrm{F}_0}|u_j(t_0)\rangle = \epsilon_j|u_j(t_0)\rangle \,, \tag{2.51}$$

But a moment's reflection shows that here you encounter exactly the same phase indeterminacy *modulo* 2π that you find when you take the logarithm of a complex number $z = \rho\mathrm{e}^{i\theta}$: as you remember $\log z = \log\rho + i(\theta + 2\pi m)$. Indeed, quite evidently, $\epsilon_j \to \epsilon_j + m\hbar\Omega$ leaves the phase factor invariant:

$$\mathrm{e}^{-\frac{i}{\hbar}\epsilon_j\mathrm{T}} \longrightarrow \mathrm{e}^{-\frac{i}{\hbar}(\epsilon_j+m\hbar\Omega)\mathrm{T}} \equiv \mathrm{e}^{-\frac{i}{\hbar}\epsilon_j\mathrm{T}} \,. \tag{2.52}$$

So, which of the infinite values $\epsilon_j + m\hbar\Omega$ should I associate to $|u_j\rangle$ in Eq. (2.51)? Let us see better this issue. Consider a Floquet state $|\psi_j(t)\rangle = \mathrm{e}^{-\frac{i}{\hbar}\epsilon_j(t-t_0)}\,|u_j(t)\rangle$, which as you remember satisfies the SE. Hence we can write:

$$\begin{aligned} i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-\frac{i}{\hbar}\epsilon_j(t-t_0)}\,|u_j(t)\rangle\right] &= \epsilon_j\mathrm{e}^{-\frac{i}{\hbar}\epsilon_j(t-t_0)}|u_j(t)\rangle + \mathrm{e}^{-\frac{i}{\hbar}\epsilon_j(t-t_0)}i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|u_j(t)\rangle \\ &= \widehat{H}(t)\left[\mathrm{e}^{-\frac{i}{\hbar}\epsilon_j(t-t_0)}\,|u_j(t)\rangle\right] \,. \end{aligned} \tag{2.53}$$

[4] Evidently, non-local terms should be in general expected.

Simplifying the phase factors you can also write the equation for the Floquet mode $|u_j(t)\rangle$ as follows:

$$\underbrace{\left[\widehat{H}(t) - i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right]}_{\widehat{K}} |u_j(t)\rangle \equiv \widehat{K}(t)|u_j(t)\rangle = \epsilon_j|u_j(t)\rangle \ . \tag{2.54}$$

The operator $\widehat{K}(t)$ is interesting: the Floquet modes $|u_j(t)\rangle$ are its eigenstates, with eigenvalue ϵ_j. Interestingly, if I consider the closely related Floquet mode $\mathrm{e}^{im\Omega(t-t_0)}|u_i(t)\rangle$ you immediately verify that:

$$\widehat{K}(t)\left[\mathrm{e}^{im\Omega(t-t_0)}|u_j(t)\rangle\right] = (\epsilon_j + m\hbar\Omega)\left[\mathrm{e}^{im\Omega(t-t_0)}|u_j(t)\rangle\right] \ . \tag{2.55}$$

So, multiplying the Floquet mode by $\mathrm{e}^{im\Omega(t-t_0)}$ leads to a *different* eigenvalue, $\epsilon_j + m\hbar\Omega$, for $\widehat{K}(t)$. The Floquet *state* associated to such modified Floquet mode and quasi-energy is precisely the same, as you immediately verify:

$$|\psi_j(t)\rangle = \mathrm{e}^{-\frac{i}{\hbar}(\epsilon_j + \cancel{m\hbar\Omega})(t-t_0)}\left[\mathrm{e}^{\cancel{im\Omega(t-t_0)}}|u_j(t)\rangle\right] \equiv \mathrm{e}^{-\frac{i}{\hbar}\epsilon_j(t-t_0)}|u_j(t)\rangle \ . \tag{2.56}$$

This suggests the following construction, originally due to J. Howland (Math. Ann. **207**, 315 (1974); Lecture Notes in Physics **130**, Springer, 1980). Consider the Hilbert space of periodic functions of time $t' \in [0, \mathrm{T}]$

$$\mathcal{L}_2[0,\mathrm{T}] = \{\varphi_m(t') = \mathrm{e}^{-im\Omega t'} \quad \text{with } m \in \mathbb{Z}\} \ , \tag{2.57}$$

where φ_m denotes the usual Fourier basis, with the usual scalar product:

$$\langle\varphi_{m'}|\varphi_m\rangle = \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \mathrm{d}t' \ \varphi^*_{m'}(t')\varphi_m(t') = \delta_{m',m} \ . \tag{2.58}$$

Next, take the Hilbert space $\mathcal{H}$ for the system under consideration, with a given basis $\mathcal{H} = \{|\phi_\alpha\rangle\}$, and consider the *tensor product*, with the natural product basis:

$$\mathcal{H}_{\mathrm{ext}} = \mathcal{L}_2[0,\mathrm{T}] \otimes \mathcal{H} = \{\varphi_m(t')|\phi_\alpha\rangle\} \ . \tag{2.59}$$

The most general element in $\mathcal{H}_{\mathrm{ext}}$ can be written as a linear combination with coefficient $u_{m,\alpha}$ of the basis elements:

$$\begin{aligned}
|u(t')\rangle &= \sum_\alpha \sum_{m=-\infty}^{+\infty} u_{m,\alpha}\,\varphi_m(t')|\phi_\alpha\rangle \\
&= \sum_\alpha C_\alpha(t')|\phi_\alpha\rangle \qquad\qquad \text{with } \ C_\alpha(t') = \sum_m u_{m,\alpha}\,\varphi_m(t') \\
&= \sum_{m=-\infty}^{+\infty} \varphi_m(t')|\Phi_m\rangle \qquad\qquad \text{with } \ |\Phi_m\rangle = \sum_\alpha u_{m,\alpha}|\phi_\alpha\rangle \ .
\end{aligned} \tag{2.60}$$

The second and third lines are two possible ways of rewriting the same general expression, which is by construction *time-periodic*: $|u(t'+\mathrm{T})\rangle = |u(t')\rangle$. Notice that in general $|u(t')\rangle$ is *entangled*, i.e., it cannot be factorised as a product of a periodic function of time times a state of $\mathcal{H}$, unless, for instance, the coefficients $u_{m,\alpha}$ can be factorized $u_{m,\alpha} = v_m z_\alpha$. In the extended Hilbert space $\mathcal{H}_{\mathrm{ext}}$ one can introduce a scalar product in a natural way:

$$\langle\langle u_1|u_2\rangle\rangle = \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \mathrm{d}t' \ \langle u_1(t')|u_2(t')\rangle \ . \tag{2.61}$$

The operator $\widehat{K}$ encountered previously can be naturally regarded as an operator in $\mathcal{H}_{\rm ext}$:

$$\widehat{K}(t') = \widehat{H}(t') - i\hbar\frac{\partial}{\partial t'} = \widehat{H}(t') + \hat{\rm p}_{t'} \qquad \text{with} \quad \hat{\rm p}_{t'} = -i\hbar\frac{\partial}{\partial t'} \;, \tag{2.62}$$

where we introduced the *canonical momentum* $\hat{\rm p}_{t'}$ associated to the *variable* t', which now acquires the same status as the position of a particle. Observe that the momentum $\hat{\rm p}_{t'}$ appears *linearly* in $\widehat{K}$, at variance with the usual non-relativistic quadratic behaviour. Observe also that t', promoted to being a variable, cannot be regarded as the usual "time parameter". A moment's reflection shows that, for instance, the Floquet state $|\psi_j(t)\rangle = {\rm e}^{-\frac{i}{\hbar}\epsilon_j(t-t_0)}|u_j(t)\rangle$ belongs to $\mathcal{H}_{\rm ext}$ only if you distinguish the *evolution time parameter* t, from the periodic time-variable t'. You can make sense of the whole construction in a framework that is known as $t-t'$ *formalism*, on which, however, we do not dwell.

The $t'-t$ formalism

Although not directly relevant to our discussion, let us spend a moment to explain the idea behind the $t-t'$ formalism [3,4]. The formalism is general, and applies also to a non-periodic time-dependence of $\widehat{H}(t)$. Consider at two-time state $|\Psi(t',t)\rangle$ which solves the t-*independent* SE associated to the Hamiltonian $\widehat{K}(t')$:

$$i\hbar\frac{\partial}{\partial t}|\Psi(t',t)\rangle = \widehat{K}(t')|\Psi(t',t)\rangle \;. \tag{2.63}$$

The solution can be formally written as:

$$|\Psi(t',t)\rangle = {\rm e}^{-\frac{i}{\hbar}\widehat{K}(t')(t-t_0)}|\Psi(t',t_0)\rangle \;. \tag{2.64}$$

The goal is to show that $|\psi(t)\rangle = |\Psi(t,t)\rangle$ is a solution of the time-*dependent* SE

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \widehat{H}(t)|\psi(t)\rangle \;. \tag{2.65}$$

Indeed, from Eq. (2.63) and the definition of $\widehat{K}(t')$ you immediately derive that:

$$i\hbar\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'}\right)|\Psi(t',t)\rangle = \widehat{H}(t')|\Psi(t',t)\rangle \;. \tag{2.66}$$

Consider now the t-derivative of $|\psi(t)\rangle = |\Psi(t,t)\rangle$:

$$\frac{\rm d}{{\rm d}t}|\psi(t)\rangle = \left(\frac{\partial}{\partial t} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}\right)|\Psi(t',t)\rangle\Big|_{t'=t} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'}\right)|\Psi(t',t)\rangle\Big|_{t'=t} \;, \tag{2.67}$$

where we used that $\partial t'/\partial t = 1$ on $t'=t$. Take now Eq. (2.66) and specialize it to $t'=t$:

$$i\hbar\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'}\right)|\Psi(t',t)\rangle\Big|_{t'=t} = \Big(\widehat{H}(t')|\Psi(t',t)\rangle\Big)_{t'=t} = \widehat{H}(t)|\Psi(t,t)\rangle \;. \tag{2.68}$$

Eq. (2.65) follows directly from Eqs. (2.67)-(2.68).

As a direct application of Eq. (2.64) to the Floquet case, consider $|\Psi_j(t',t_0)\rangle \equiv |u_j(t')\rangle$, with boundary condition $|\Psi_j(t_0,t_0)\rangle \equiv |u_j(t_0)\rangle$. Then:

$$|\Psi_j(t',t)\rangle = {\rm e}^{-\frac{i}{\hbar}\widehat{K}(t')(t-t_0)}|u_j(t')\rangle = {\rm e}^{-\frac{i}{\hbar}\epsilon_j(t-t_0)}|u_j(t')\rangle \;. \tag{2.69}$$

Setting $t' = t$ you deduce that:

$$|\Psi_j(t,t)\rangle = \mathrm{e}^{-\frac{i}{\hbar}\epsilon_j(t-t_0)}|u_j(t)\rangle \equiv |\psi_j(t)\rangle \ . \tag{2.70}$$

Hence, the correct Floquet state is recovered. Notice that the periodicity in t' does not imply that $|\Psi(t,t)\rangle$ is periodic.

Magnetic spin resonance and two-level atom

Consider now, as an application, the simplest two-dimensional quantum problem: a spin-1/2 in a time-periodic magnetic field. You can view it as the prototypical calculation behind an NMR setup. The spin Hamiltonian has a Zeeman coupling with a large field B_z and a smaller time-dependent part on the plane:

$$\widehat{H}(t) = \mu_B B_z \hat{\sigma}^z + \mu_B B_x(t)\hat{\sigma}^x + \mu_B B_y(t)\hat{\sigma}^y \tag{2.71}$$

There are two variants of this problem: one in which the time-dependent part is *circularly polarised*:

$$B_x(t) = B_\perp \cos(\Omega t) \qquad\qquad B_y(t) = B_\perp \sin(\Omega t) \ . \tag{2.72}$$

The second variant has a linearly polarised oscillating field: $B_x(t) = B_\perp \cos(\Omega t)$, $B_y = 0$.

There is a second incarnation of this problem which is quite standard in quantum optics: it is the so-called *two-level atom*. If you shine a laser of frequency Ω which is nearly resonant with a dipole-allowed electronic transition between the ground-state $|\mathrm{gs}\rangle$ and some excited state $|\mathrm{ex}\rangle$, with $\Delta E = \hbar\omega_0$. The natural identification with $\hat{\sigma}^z$ eigenstates is: $|\mathrm{gs}\rangle = |-\rangle$ and $|\mathrm{ex}\rangle = |+\rangle$. The interaction Hamiltonian can be written in two equivalent ways: as $\frac{e}{mc}\mathbf{A}(t)\cdot\hat{\mathbf{p}}$, where $\mathbf{A}(t)$ is the vector potential, or $\mathbf{E}(t)\cdot\hat{\mathbf{d}}$, where $\mathbf{E}(t)$ is the electric field and $\hat{\mathbf{d}}$ the dipole-moment operator. Either way, dipole selection rules lead to $\langle+|\hat{\mathbf{p}}|+\rangle = 0$ and $\langle-|\hat{\mathbf{p}}|-\rangle = 0$, hence the interaction enters only through $\hat{\sigma}^{x,y}$. Again the polarization of the laser light gives two possible couplings. We write the circularly polarised case as:

$$\widehat{H}_{\mathrm{CP}}(t) = \frac{\hbar\omega_0}{2}\hat{\sigma}^z + \frac{V_\perp}{2}\left(\hat{\sigma}^x\cos(\Omega t) + \hat{\sigma}^y\sin(\Omega t)\right) \tag{2.73}$$

where $V_\perp = \mu F$ denotes the dipole matrix element and the electric field, and we added a convenient factor 2 in the denominator. The linearly polarised case is written as:

$$\widehat{H}_{\mathrm{LP}}(t) = \frac{\hbar\omega_0}{2}\hat{\sigma}^z + V_\perp\hat{\sigma}^x\cos(\Omega t) \ . \tag{2.74}$$

With the identifications $2\mu_B B_z = \hbar\omega_0$ and $2\mu_B B_\perp = V_\perp$ the same Hamiltonians describe the NMR problem discussed above. These are the two periodically-driven spin-1/2 models we are going to study.

Exact solution for the circularly polarised case

The idea comes from the form of the circularly polarised interaction, which looks like a spin rotating around the z-axis at a frequency Ω. Let us consider the unitary operator

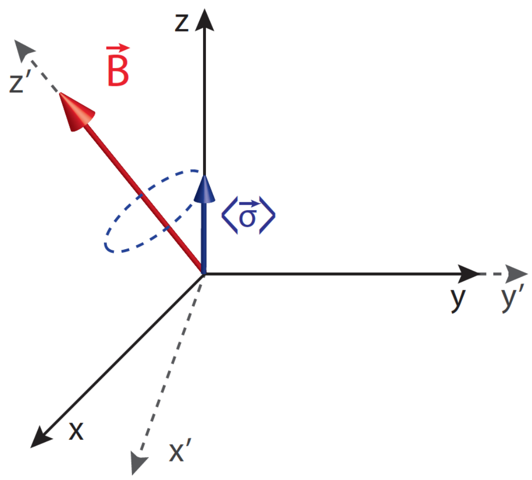

Figure 2.1: The effective magnetic field in the rotating frame for the circularly polarised case. Figure taken from Ref. [2]

implementing such a rotation in spin-space: [5]

$$\widehat{U}_t = \mathrm{e}^{-i\frac{\Omega t}{2}\hat{\sigma}^z} \ . \tag{2.75}$$

Notice that $\widehat{U}_0 = \mathbb{1}$ but $\widehat{U}_{\mathrm{T}} = \mathrm{e}^{-i\pi\hat{\sigma}^z} = -\mathbb{1}$. To make the transformation periodic, it is enough to multiply it by $\mathrm{e}^{i\frac{\Omega t}{2}\mathbb{1}}$, defining:

$$\widehat{P}(t,0) = \mathrm{e}^{i\frac{\Omega t}{2}(\mathbb{1}-\hat{\sigma}^z)} \ . \tag{2.76}$$

With such a candidate $\widehat{P}$, we are ready to set up our moving frame transformation. Define $|\psi(t)\rangle = \widehat{P}(t,0)|\widetilde{\psi}(t)\rangle$. The transformed Hamiltonian, including the extra term, is:

$$\widetilde{H}_{\mathrm{CP}}(t) = \widehat{P}^\dagger \widehat{H}_{\mathrm{CP}}(t)\widehat{P} - i\hbar\widehat{P}^\dagger\dot{\widehat{P}}_t = \widehat{P}^\dagger\widehat{K}(t)\widehat{P} \ . \tag{2.77}$$

It is straightforward to calculate the extra term:

$$-i\hbar\widehat{P}^\dagger\dot{\widehat{P}} = \frac{\hbar\Omega}{2}(\mathbb{1}-\hat{\sigma}^z) \tag{2.78}$$

To perform the transformation, we have to calculate $\widehat{P}^\dagger\hat{\sigma}^\alpha\widehat{P}$. Either by writing a differential equation, in the spirit of the Heisenberg equations of motion, or by a direct calculation [6] one can easily show that:

$$\begin{cases} \widehat{P}^\dagger(t,0)\,\hat{\sigma}^x\,\widehat{P}(t,0) = \hat{\sigma}^x\cos(\Omega t) - \hat{\sigma}^y\sin(\Omega t) \\ \widehat{P}^\dagger(t,0)\,\hat{\sigma}^y\,\widehat{P}(t,0) = \hat{\sigma}^y\cos(\Omega t) + \hat{\sigma}^x\sin(\Omega t) \\ \widehat{P}^\dagger(t,0)\,\hat{\sigma}^z\,\widehat{P}(t,0) = \hat{\sigma}^z \end{cases} \tag{2.79}$$

[5] Recall that a general rotation by an angle θ around the axis $\hat{\mathbf{n}}$ reads: $\mathrm{e}^{-\frac{i}{\hbar}\theta\hat{\mathbf{n}}\cdot\widehat{\mathbf{J}}}$, where $\widehat{\mathbf{J}}$ is the angular momentum.

[6] Using that

$$\mathrm{e}^{\pm i\frac{\Omega t}{2}\hat{\sigma}^z} = \cos\frac{\Omega t}{2} \pm i\sin\frac{\Omega t}{2}\hat{\sigma}^z$$

and $[\hat{\sigma}^\alpha, \hat{\sigma}^\beta] = 2i\epsilon^{\alpha\beta\gamma}\hat{\sigma}^\gamma$.

As a result, we deduce that:

$$\widetilde{H}_{\rm CP} = \frac{\hbar\Omega}{2}\mathbb{1} + \frac{\hbar(\omega_0-\Omega)}{2}\hat{\sigma}^z + \frac{V_\perp}{2}\hat{\sigma}^x \equiv \widehat{H}_{\rm F_0} \; . \tag{2.80}$$

Notice that the transformed Hamiltonian is time-independent: it is the desired Floquet effective Hamiltonian $\widehat{H}_{\rm F_0}$. It corresponds to an effective magnetic field given by:

$$\mu_B \mathbf{B}^{\rm eff} = \left(\frac{V_\perp}{2}, 0, \frac{\hbar(\omega_0-\Omega)}{2}\right) = \frac{\hbar\Omega_{\rm R}}{2}\hat{\mathbf{n}} \; , \tag{2.81}$$

where:

$$\left\{ \begin{array}{l} \hbar\Omega_{\rm R} = \sqrt{\hbar^2(\omega_0-\Omega)^2 + V_\perp^2} \\ \hat{\mathbf{n}} = \dfrac{V_\perp}{\hbar\Omega_{\rm R}}\hat{\mathbf{x}} + \dfrac{\omega_0-\Omega}{\Omega_{\rm R}}\hat{\mathbf{z}} = \sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}} \end{array} \right. \; , \tag{2.82}$$

θ denoting the azimuthal angle measured from the north pole, as usual when working with spherical coordinates. The effective magnetic field is illustrated in Fig. 2.1. The spectrum of this Hamiltonian is now immediate. The two quasi-energies are:

$$\epsilon_\pm = \frac{\hbar\Omega}{2} \pm \frac{\hbar\Omega_{\rm R}}{2} \quad (\mathrm{mod}\ \ \hbar\Omega) \; . \tag{2.83}$$

The associated Floquet modes are simply the two spin eigenstates $|\pm\hat{\mathbf{n}}\rangle$ in the direction $\hat{\mathbf{n}}$:

$$\left\{ \begin{array}{l} |u_+(0)\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle \\ |u_-(0)\rangle = \sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}|-\rangle \end{array} \right. \; . \tag{2.84}$$

The periodic Floquet modes are obtained by applying $\widehat{P}(t,0) = \mathrm{e}^{i\frac{\Omega t}{2}(\mathbb{1}-\hat{\sigma}^z)}$:

$$\left\{ \begin{array}{l} |u_+(t)\rangle = \widehat{P}(t,0)|u_+(0)\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}\mathrm{e}^{i\Omega t}|-\rangle \\ |u_-(t)\rangle = \widehat{P}(t,0)|u_-(0)\rangle = \sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}\mathrm{e}^{i\Omega t}|-\rangle \end{array} \right. \; . \tag{2.85}$$

Suppose that the initial state is $|\psi(0)\rangle = |-\rangle$, the ground state in absence of time-dependent field. The dynamics is interesting. You can view it as a *spin precession* around the direction $\hat{\mathbf{n}}$, as alluded at in Fig. 2.1. To proceed, we need to expand $|\psi(0)\rangle$ in Floquet modes:

$$|\psi(0)\rangle = C_+|u_+(0)\rangle + C_-|u_-(0)\rangle \qquad \text{with} \; \left\{ \begin{array}{l} C_+ = \sin\frac{\theta}{2} \\ C_- = -\cos\frac{\theta}{2} \end{array} \right. \; . \tag{2.86}$$

The state $|\psi(t)\rangle$ can be explicitly written as:

$$\begin{aligned} |\psi(t)\rangle &= C_+\mathrm{e}^{-i\frac{(\Omega+\Omega_{\rm R})}{2}t}|u_+(t)\rangle + C_-\mathrm{e}^{-i\frac{(\Omega-\Omega_{\rm R})}{2}t}|u_-(t)\rangle \\ &= \mathrm{e}^{-i\frac{\Omega}{2}t}\left(-i\sin\theta\,\sin\frac{\Omega_{\rm R}t}{2}\right)|+\rangle + \mathrm{e}^{+i\frac{\Omega}{2}t}\left(\cos\frac{\Omega_{\rm R}t}{2} + i\cos\theta\,\sin\frac{\Omega_{\rm R}t}{2}\right)|-\rangle \; . \end{aligned} \tag{2.87}$$

You can now calculate the probability $\mathrm{Prob}_+(t)$ that the spin has flipped, to the state $|+\rangle$, at time t:

$$\mathrm{Prob}_+(t) = \Big|\langle +|\psi(t)\rangle\Big|^2 = \sin^2\theta\,\sin^2\frac{\Omega_{\rm R}t}{2} = \frac{V_\perp^2}{V_\perp^2 + \hbar^2(\omega_0-\Omega)^2}\,\sin^2\frac{\Omega_{\rm R}t}{2} \; . \tag{2.88}$$

Notice the characteristic *resonance* at $\Omega = \omega_0$, with a period of oscillation $\mathrm{T_R} = \frac{2\pi}{\Omega_\mathrm{R}}$ which is different from the driving period. Indeed, exactly at resonance the so-called *Rabi frequency* Ω_R is given by:

$$\text{At resonance } (\omega_0 = \Omega) \quad \Longrightarrow \quad \hbar\Omega_\mathrm{R} = |V_\perp| \,. \tag{2.89}$$

This is on the reasons [7] why the coupling itself, $V_\perp/\hbar$, is often called "Rabi frequency" in the quantum optics literature. Notice, finally, that *at resonance*, the transformation to the moving frame induced by $\widehat{P}(t,0)$ is essentially equivalent to switching to the *interaction representation*, since the free evolution operator is:

$$\widehat{U}_0 = \mathrm{e}^{-\frac{i}{\hbar}\widehat{H}_0 t} = \mathrm{e}^{-i\frac{\omega_0 t}{2}\hat{\sigma}^z} \,. \tag{2.90}$$

Obviously, away from resonance the two approaches differ.

The rotating wave approximation for the linearly polarised case

The same transformation applied to $\widehat{H}_\mathrm{LP}(t)$ leads to: [8]

$$\widetilde{H}_\mathrm{LP}(t) = \underbrace{\frac{\hbar\Omega}{2}\mathbb{1} + \frac{\hbar(\omega_0-\Omega)}{2}\hat{\sigma}^z + \frac{V_\perp}{2}\hat{\sigma}^x}_{\text{Rotating Wave Approximation}} + \frac{V_\perp}{2}\Big(\hat{\sigma}^x\cos(2\Omega t) - \hat{\sigma}^y\sin(2\Omega t)\Big) \,. \tag{2.91}$$

Quite evidently, $\widehat{H}_\mathrm{LP}(t)$ is not solved exactly by the transformation to the moving frame, as the resulting transformed Hamiltonian is still time-dependent! Nevertheless, you observe that the time-dependence occurs now at frequency 2Ω, and this might lead to small effects. Taking the non-oscillating part of $\widetilde{H}_\mathrm{LP}(t)$ is known as *Rotating Wave Approximation* (RWA). Its regime of validity should be carefully checked.

Obviously, you could find the Floquet modes and quasi-energies for $\widehat{H}_\mathrm{LP}(t)$, since it is a periodically driven model. The approach has to resort to some numerics, however. Here is an exercise that guides you towards the exact numerical solution.

Exercise 2.1. You should have a numerical integration tool at your disposal.

1) Calculate the Floquet operator $\widehat{F}_0 \equiv \widehat{U}(\mathrm{T},0)$ for $\widehat{H}_\mathrm{LP}(t)$ by evolving numerically from time $t=0$ to time $t=\mathrm{T}$ the two standard basis spin vectors:

$$|+\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad\qquad |-\rangle = \begin{pmatrix}0\\1\end{pmatrix} \,.$$

The resulting states will form the two columns of the 2×2 matrix $\widehat{F}_0$. By monitoring the numerical evolution at $t\in[0,\mathrm{T}]$, you can in principle construct the two columns of $\widehat{U}(t,0)$.

[7] Alternatively, in absence of the main field along $\hat{\mathbf{z}}$, $V_\perp$ alone would provide a Rabi oscillation frequency as well, often denoted by ω_R to distinguish it from the full Ω_R.

[8] Essentially, you can write:

$$V_\perp\hat{\sigma}^x\cos(\Omega t) = \frac{V_\perp}{2}\left(\hat{\sigma}^x\cos(\Omega t) + \hat{\sigma}^y\sin(\Omega t)\right) + \frac{V_\perp}{2}\left(\hat{\sigma}^x\cos(\Omega t) - \hat{\sigma}^y\sin(\Omega t)\right) \,,$$

and observe that the second term rotates in the *opposite direction*.

2) Calculate the two eigenvectors $|u_{\pm}(0)\rangle$ and eigenvalues $\mathrm{e}^{-\frac{i}{\hbar}\epsilon_{\pm}\mathrm{T}}$ of $\widehat{F}_0$.

3) Expand the initial state $|\psi(0)\rangle = |-\rangle$ as

$$|\psi(0)\rangle = C_+|u_+(0)\rangle + C_-|u_-(0)\rangle$$

4) Calculate the state $|\psi(n\mathrm{T})\rangle$ at stroboscopic times $t_n = n\mathrm{T}$, according to:

$$|\psi(n\mathrm{T})\rangle = C_+\mathrm{e}^{-\frac{i}{\hbar}\epsilon_+ n\mathrm{T}}|u_+(0)\rangle + C_-\mathrm{e}^{-\frac{i}{\hbar}\epsilon_- n\mathrm{T}}|u_-(0)\rangle$$

5) Evaluate

$$\mathrm{Prob}_+(n\mathrm{T}) = \Big|\langle +|\psi(n\mathrm{T})\rangle\Big|^2 \;,$$

6) Compare your numerical results with those provided by the RWA, in different regimes of Ω.

7) What should you do to obtain information at all times t, and not just at stroboscopic times $n\mathrm{T}$?

The Shirley-Floquet approach

The Shirley-Floquet approach is based on Fourier transform. Take the eigenvalue problem in the extended Hilbert space:

$$\widehat{K}(t)|u^{(j)}(t)\rangle = \left[\widehat{H}(t) - i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right]|u^{(j)}(t)\rangle = \epsilon_j|u^{(j)}(t)\rangle \;, \tag{2.92}$$

where we have moved the label j to a superscript, and expand the periodic Floquet mode in Fourier modes:

$$|u^{(j)}(t)\rangle = \sum_m \varphi_m(t)|u^{(j)}_m\rangle \;,$$

where $\varphi_m(t) = \mathrm{e}^{-im\Omega t}$. Observe that the time-derivative term, when acting on $\varphi_m(t)$ gives:

$$-i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\varphi_m(t) = -m\hbar\Omega\,\varphi_m(t) \;.$$

Introducing the Hamiltonian Fourier components:

$$\widehat{H}^{(m)} = \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \mathrm{d}t'\;\mathrm{e}^{im\Omega t}\widehat{H}(t) \;, \qquad \Longrightarrow \qquad \widehat{H}(t) = \sum_m \widehat{H}^{(m)}\,\mathrm{e}^{-im\Omega t} \;, \tag{2.93}$$

you can rewrite the SE in the Shirley-Floquet form:

$$\sum_{m'} \widehat{H}^{(m')}|u^{(j)}_{m-m'}\rangle - (m\hbar\Omega)\,|u^{(j)}_m\rangle = \epsilon_j|u^{(j)}_m\rangle \;. \tag{2.94}$$

Notice the tight-binding form, in Fourier space, that we have reached.

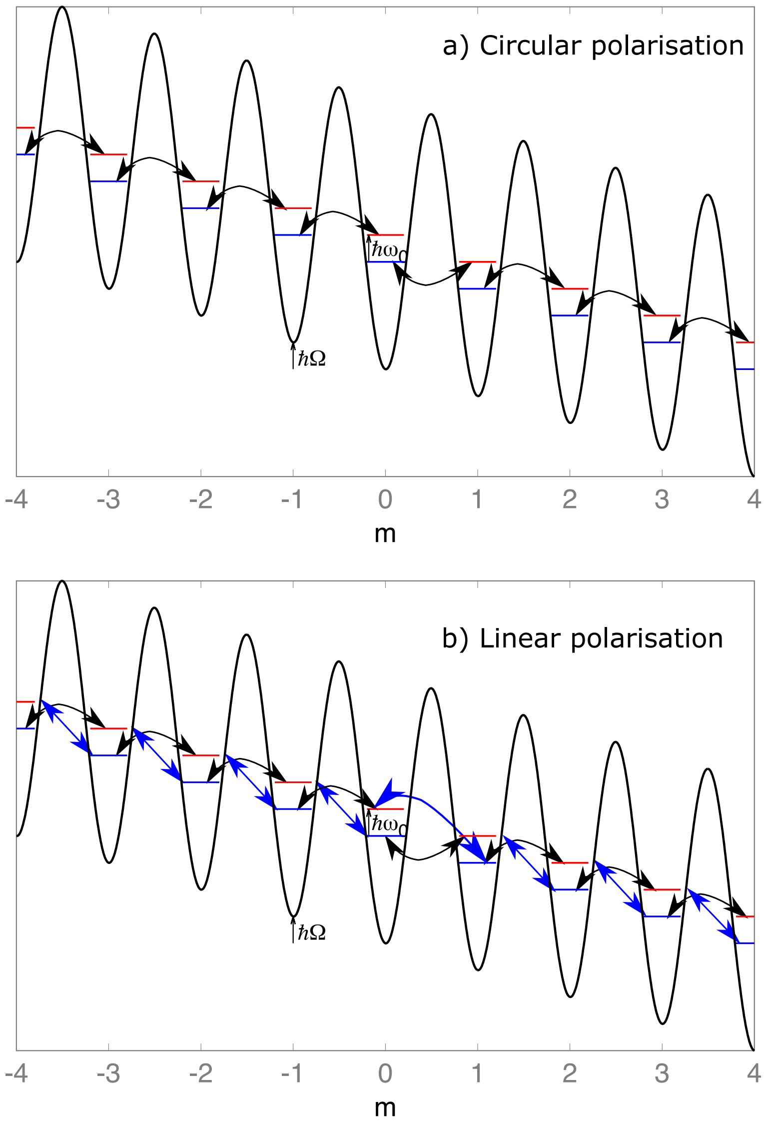

Figure 2.2: A sketch of the Shirley-Floquet eigenvalue problem. The Fourier index m acts as an extra tight-binding site with an associated "uniform electric field" set by the driving frequency Ω. In (a), corresponding to $\widehat{H}_{\rm CP}$, the system is split into a collection of "diatomic molecules", hence an exact solution follows. In (b), corresponding to $\widehat{H}_{\rm LP}$, the sites are collectively coupled, and no exact solution is possible.

Let us see what the happens when we apply this formalism to $\widehat{H}_{\mathrm{LP}}$ and $\widehat{H}_{\mathrm{CP}}$. The Hamiltonian matrix elements for $\widehat{H}_{\mathrm{LP}}$ are: [9]

$$\mathbf{LP}: \qquad \widehat{H}^{(m)} = \frac{\hbar\omega_0}{2}\hat{\sigma}^z\ \delta_{m,0} + \frac{V_\perp}{2}\hat{\sigma}^x\ \left(\delta_{m,+1} + \delta_{m,-1}\right)\ . \tag{2.95}$$

Figure 2.2(b) sketches the matrix elements of $\widehat{H}^{(m)}$ for the linear polarisation case. Notice that the drawing alludes to an extra spatial coordinate, associated to the Fourier index m, where the driving frequency Ω acts as a *uniform electric field* would act for an electron in one-dimension. The two lines (red and blue) allude to the two levels, separated by a splitting $\hbar\omega_0$. In the electric-field analogy, these would be *two different orbitals* localised at each minimum of the tilted "cosine" potential. The arrows allude to the matrix element coupling "opposite spins", induced by $\hat{\sigma}^x$: their common coupling constant is $V_\perp$. In the analogy suggested, the problem is almost identical to what Wannier had studied in 1960 for a band electron in a uniform electric field, originating a *Wannier-Stark ladder*. Essentially, Wannier predicted a ladder of possible states differing by a multiple of the full quantum of energy $\hbar\Omega$: this is nothing but the indeterminacy of $m\hbar\Omega$ in the quasi-energies.

Notice how the resonance condition $\omega_0 = \Omega$ predicts that a $|+\rangle$ state at m has precisely the same "energy" as $|-\rangle$ at $m-1$. For $\Omega \gg \omega_0$ one can develop a perturbation theory in $1/\Omega$ to deduce an effective Floquet Hamiltonian acting on each local (two-dimensional, in the present case) Hilbert space [2]. On the contrary, for $\Omega \to 0$, you get an "electric field" that is going to 0, and hence a *new dimensionality* emerges, associated to the Fourier index. More about this in the discussion below.

One might wonder why the circularly polarised case $\widehat{H}_{\mathrm{CP}}$ is exactly solvable, when viewed in such a Shirley-Floquet framework. The answer is simple. For $\widehat{H}_{\mathrm{CP}}$ we get: [10]

$$\mathbf{CP}: \qquad \widehat{H}^{(m)} = \frac{\hbar\omega_0}{2}\hat{\sigma}^z\ \delta_{m,0} + \frac{V_\perp}{2}\left(\hat{\sigma}^-\ \delta_{m,-1} + \hat{\sigma}^+\ \delta_{m,+1}\right)\ . \tag{2.96}$$

which implies that, in Fig. 2.2(a) the blue arrows are missing, and only the black ones survive. This means that a spin can be flipped upward, $|-\rangle \to |+\rangle$, if m is changed to $m' = m+1$, and vice-versa you can flip a spin downward, $|+\rangle \to |-\rangle$, by changing m to $m' = m-1$: these are two Hermitean-conjugate processes. The other two processes present for $\widehat{H}_{\mathrm{LP}}$ (the blue arrows) are here *absent*: this results in the fact that the system can be viewed as a collections of *diatomic molecules*, each $|+rangle$ at m being "connected" to $|-\rangle$ at $m-1$, and so on, hence exactly solvable.

One might wonder what happens to models having a larger local Hilbert space. For instance, if you consider a *periodically driven harmonic oscillator* the local Hilbert space is infinite-dimensional, with an equi-spaced tower of levels at each Fourier "site" m. Interestingly, such a model can also be solved. But you realise immediately that the magic ends

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[9] Use that:

$$\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} dt'\ e^{im\Omega t}\frac{e^{i\Omega t} + e^{-i\Omega t}}{2} = \frac{1}{2}\left(\delta_{m,+1} + \delta_{m,-1}\right)\ .$$

[10] Observe that:

$$\hat{\sigma}^x\cos(\Omega t) + \hat{\sigma}^y\sin(\Omega t) = e^{i\Omega t}\hat{\sigma}^- + e^{-i\Omega t}\hat{\sigma}^+\ .$$

immediately into a nightmare when the tower of levels extends up to infinity but is not equi-spaced: take just an infinite square well Schrödinger problem of width L, with its quadratic tower of states $\varepsilon_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}$, add an oscillating electric field $\mathcal{E}x\cos\Omega t$, and the nightmare is served [1]. In general, models on the continuum tend to show complicated spectral behaviour. If you want to learn more about such delicacies, look at Ref. [4].

Increasing the dimensionality of the system

This section is based on the paper by Martin, Refael &Halperin (arXiv 1612.02143). It suggests to use the Shirley-Floquet-Fourier idea *beyond* the purely periodic framework, to extend further the dimensionality of the system. Suppose you have a Hamiltonian which depends periodically on two or more phase-factors $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots)$, i.e., $\widehat{H}(\boldsymbol{\varphi})$, with $\varphi_i \in [0, 2\pi]$, and

$$\widehat{H}(\varphi_1, \cdots, \varphi_i + 2\pi, \cdots) = \widehat{H}(\varphi_1, \cdots, \varphi_i, \cdots) \ .$$

If the phases are made time-dependent, with *incommensurate frequencies* $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \cdots)$, $\boldsymbol{\varphi}(t) = \boldsymbol{\Omega} t$, the Hamiltonian becomes time-dependent but is *not periodic*: you often read about a *quasi-periodic* behaviour, $\widehat{H}(t) = \widehat{H}(\boldsymbol{\varphi}(t)) = \widehat{H}(\boldsymbol{\Omega} t)$. You can Fourier expand $\widehat{H}(\boldsymbol{\varphi})$:

$$\widehat{H}(\boldsymbol{\varphi}) = \sum_{\mathbf{m}} \widehat{H}^{(\mathbf{m})} \mathrm{e}^{-i\mathbf{m}\cdot\boldsymbol{\varphi}} \ . \tag{2.97}$$

Now you make the following *Ansatz* for the state:

$$|\psi(t)\rangle = \mathrm{e}^{-\frac{i}{\hbar}Et} \sum_{\mathbf{m}} \mathrm{e}^{-i\mathbf{m}\cdot\boldsymbol{\Omega}t} \, |\phi_{\mathbf{m}}\rangle \ . \tag{2.98}$$

where E and $|\phi_{\mathbf{m}}\rangle$ should be found. Substituting in the time-dependent SE

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = \widehat{H}(\boldsymbol{\Omega} t)|\psi(t)\rangle$$

you get the generalised Shirley-Floquet tight-binding form:

$$\sum_{\mathbf{m}'} \widehat{H}^{(\mathbf{m}')} |\phi_{\mathbf{m}-\mathbf{m}'}\rangle - \hbar\boldsymbol{\Omega}\cdot\mathbf{m} \, |\phi_{\mathbf{m}}\rangle = E|\phi_{\mathbf{m}}\rangle \ . \tag{2.99}$$

If you have two-incommensurate frequencies, the tight-binding (Fourier) lattice will be two-dimensional. Hence, even a single spin-1/2 can generate a two-dimensional problem provided you drive it with *small* incommensurate frequencies $\boldsymbol{\Omega}$.

Many interesting things can happen if you have a tight-binding two-dimensional problem. In particular, interesting topological aspects of the story can emerge. More about this in the original reference quoted above.

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