An introduction to (Many-Body) Localization - Extra 1

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Ex. 1. Transport: Ballistic *vs* Diffusive motion. This is an exercise to familiarize with the Anderson model and with diffusive *vs* ballistic transport. You may refer to the lecture notes of A. Scardicchio and T. Thiery [arXiv:1710.01234] for further details.

Consider the Anderson Hamiltonian describing a particle in a $1d$ lattice $\mathbb{Z}$ with sites i, given by:

$$H = \sum_{i\in\mathbb{Z}} \epsilon_i |i\rangle\langle i| + V \sum_{i\in\mathbb{Z}} \left(|i\rangle\langle i+1| + |i+1\rangle\langle i|\right) = H_0 + V. \tag{1}$$

The on-site energies ϵ_i in the potential term H_0 are random with uniform distribution in the interval $\left[-\frac{W}{2}, \frac{W}{2}\right]$

(i) Compute the eigenvalues and eigenvectors of the free system with only the kinetic term ($H_0 = 0$), and determine the time-evolved wave function ψ_t of the particle initialized at the center site $i = 0$ at $t = 0$;
Hint. Consider first a finite-size lattice of length L, and then take $L \to \infty$. What happens to the spectrum and to the eigenstates in this limit?

(ii) Show that for $H_0 = 0$ the motion is ballistic, i.e., the mean-squared displacement of the particle behaves as:

$$\Delta x_t^2 = \langle \psi_t | X^2 | \psi_t \rangle - (\langle \psi_t | X | \psi_t \rangle)^2 \sim 2V^2 t^2$$

Hint. To compute the time-dependent quantity Δx_t^2 you may use the expansion $\Delta x_t^2 = \sum_{n=0}^{\infty} \frac{t^n}{n!} \partial_t \, \Delta x_t^2 |_{t=0}$ together with the equation of motion in Heisenberg picture, $\partial_t \langle \psi_t | X^2 | \psi_t \rangle = i \langle \psi_t | \left[H, X^2\right] |\psi_t\rangle$, iterated to get the higher-order powers.

Ballistic motion is typical of systems conserving momentum. When random impurities are added ($H_0 \neq 0$), a natural expectation is that transport becomes diffusive with a finite diffusion constant D:

$$\Delta x_t^2 \sim D\, t \qquad \text{where} \qquad D \sim v^2 \tau, \tag{2}$$

where v the typical velocity of the particle (which does not depend on the disorder), and τ the typical time separating scattering events. Let's make a simple estimate of how the diffusion constant D should behave as a function of the disorder strength W, by treating disorder perturbatively.

(iii) Assume that the system is in an eigenstate of the kinetic Hamiltonian $H_0 = 0$ with fixed momentum k and energy E_k, and with velocity $v = \frac{\partial E_k}{\partial k}$. To estimate the scattering time, we use Fermi Golden Rule as:

$$\frac{1}{\tau} = \Gamma_k = \pi \sum_{k'} |\langle k | H_0 | k' \rangle|^2 \delta\left(E_k - E_{k'}\right). \tag{3}$$

Show that, when averaged over the disorder, Fermi Golden Rule gives $\tau^{-1} \propto W^2 \nu(E_k)$, where $\nu(E_k)$ is the eigenvalue density of the Hamiltonian with $H_0 = 0$. Show that this implies that the diffusion constant is never zero for finite W.

(iv) When localization occurs, the diffusion constant D actually vanishes at a finite value of W: can you guess why the simple estimate based on Fermi Golden Rule is invalidated?

Question 1: What does out-of-equilibrium mean *in MBL*?

- **Localization in the lab: quenching in random potential.** The results of the experiment with the Anderson localized BEC condensate done at LENS in Florence can be found in [Roati et al, Nature, 453(7197):895–898, 2008], the very similar one done in Paris is discussed in [Billy et al, Nature, 453(7197):891–894, 2008]. For a review with details on these experiments, see [G. Modugno, Reports on progress in physics 73(10): 102401, 2010]. The imbalance decay has been first measured in a $1d$ cold atom experiment in [Schreiber et al, Science, 349(6250):842– 845, 2015]. Similar experiments with chains of molecules (trapped ions with dipolar interactions) are discussed in [Smith et al, Nat. Phys.12:907, 2016]. These type of experiments probing the coherent dynamics of interacting quantum systems usually involve artificial matter, that can be suitably isolated from the environment [at variance with electrons in a lattice, that couple to phonons] and for which interactions are tunable. There exist however solid state platforms for which the coupling to phonons is particularly weak, e.g. rare earths magnets [Silevitch et al, arXiv:1707.04952].

- ***"Permanently and robustly out-of-equilibrium"*: general reviews on MBL.**

  1. Abanin, Altman, Bloch and Serbyn, Colloquium: Many-body localization, thermalization, and entanglement, Reviews of Modern Physics 91 no. 2 (2019): 021001.
  2. Alet and Laflorencie, Many-body localization: An introduction and selected topics, Comptes Rendus Physique 19.6 (2018): 498.
  3. Abanin and Papic, Recent progress in many-body localization, Annalen der Physik 529.7 (2017): 1700169.
  4. Lectures by D. Huse at the non-equilibrium statistical physics 2015 program at ICTS:
     – https://www.youtube.com/watch?v=GBxtgyqvZz0&list=PL04QVxpjcnjjoN7xfc7Vy63uUSkDlOYiU&index=6&t=0s
     – https://www.youtube.com/watch?v=c-d6j4thp70&list=PL04QVxpjcnjjoN7xfc7Vy63uUSkDlOYiU&index=6

- ***"Because transport is suppressed".*** The field of localization begins with Anderson's seminal work [Phys. Rev. 109, 1492 (1958)], where localization is defined as absence of dissipative transport (hopping conductivity), as one reads from the abstract: "*diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place*". An analysis similar to Anderson's one (based on diagrammatics) has been extended to interacting fermions by Basko, Aleiner and Altshuler [Annals of Physics 321, 1126 (2006)], see also Gornyi, Mirlin and Polyakov [Phys. Rev. Lett. 95, 206603 (2005)].

  In the math literature, the absence of diffusive transport is referred to as "dynamical localization"; for non-interacting particles in finite d at strong disorder, it has been proven rigorously in [Frohlich and Spencer, Comm. Math. Phys., 88(1983):151], see also [Aizenman and Molchanov, Comm. Math. Phys., 157(1993):245]. The proofs rely on exponential bounds on the local matrix elements of the resolvent, or on their fractional moments, or on the eigenstates correlators. An account of mathematical results on localization can be found in the book [Aizenman and Warzel, Random Operators: Disorder Effects on Quantum Spectra and Dynamics, American Mathematical Society 2015].

- **On conductivity and linear response.** Let me integrate with some references the comments made in the lecture. First, linear response is discussed by Kubo in his 1957 work [R. Kubo, Journal of the Physical Society of Japan 12 (1957)]. For a pedagogical discussion of these results (included the derivation of the Kubo formula for the electric conductivity), you may look at section 5 of the lecture notes on "Statistical Field Theory" by T. Giamarchi available from his website [https://giamarchi.unige.ch/local/people/thierry.giamarchi/pdf/cours_sft.pdf]. Coming to disordered systems, numerical results on the d.c. conductivity based on the Kubo formula are discussed for example in [Berkelbach and Reichman, Physical Review B 81 (2010)] and in [Barišić, Kokalj, Balog and Prelovšek, Physical Review B 94 (2016)]. The adiabatic regime (as opposed to linear response) is looked at in [Khemani et al, Nature Physics 11 (2015)]. Subdiffusive transport in the ergodic (delocalized) phase has been discussed in plenty of works, you may have a look at this review paper on "The ergodic side of the many-body localization transition" by D. Luitz and Y. Bar Lev [Annalen der Physik 529 (2017)].

- ***"Dephasing without dissipation".*** The logarithmic growth of entanglement entropy in quantum quenches has been pointed out in [Bardarson, Pollmann and Moore, Phys. Rev. Lett. 109 (2012):017202], and had been previously observed in [De Chiara et al, Journal of Statistical Mechanics: Theory and Experiment, 2006(03):P03001]. Theoretical justifications of the logarithmic behavior based on the emergent integrability can be found in [Serbyn, Papic, Abanin, Physical review letters 110 (2013):260601]. The power-law relaxation of expectation values of local observables is discussed by the same authors in [Phys. Rev. B 90 (2014):174302]. For the emergence of the same behaviour in a toy model of the many-body Fock space (the Bethe lattice), see [Biroli and Tarzia, Phys. Rev. B 96 (2017) and Phys. Rev. B 102 (2020)].

Question 2: What would equilibrium even mean?

- **Thermalization of isolated quantum systems.** Understanding how thermodynamics emerges from unitary quantum dynamics is a longstanding, fundamental question that goes back at least to [von Neumann, Z. Phys. 57:30, 1929, see arXiv:1003.2133]. Here's a short list of rather recent review articles on this topic:

  1. Polkovnikov, Sengupta, Silva and Vengalattore, Non-equilibrium dynamics of closed interacting quantum systems, Rev. Mod. Phys. 83 (2011): 863.
  2. Nandkishore and Huse, Many-body localization and thermalization in quantum statistical mechanics, Annu. Rev. Condens. Matter Phys. 6.1 (2015): 15-38.
  3. Gogolin and Eisert, Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems, Reports on Progress in Physics 79.5 (2016): 056001.
  4. Yukalov, Equilibration and thermalization in finite quantum systems Laser Phys. Lett. 8 (2011): 485.
  5. Goold, Huber, Riera, Del Rio and Skrzypczyk, The role of quantum information in thermodynamics—a topical review, Journal of Physics A: Mathematical and Theoretical 49 (2016): 143001.

- **On dephasing and relaxation: an example and a counterexample.** The notion of equilibration described in the lecture requires an infinitely large system, in order for dephasing processes to be effective. A simple example of this mechanism (that can be worked out explicitly) is given by the transverse field Ising chain dynamics following a quantum quench. This is discussed in Sec. IV in [Essler and

Fagotti, Journal of Statistical Mechanics: Theory and Experiment 2016.6 (2016): 064002], see also the 2014 ICTP lecture by F. Essler available at http://video.ictp.it/WEB/2014/2014_07_14-smr2594/2014_07_14-09_00-smr2594.mp4.

An interesting example of long-lived coherent oscillations in many-body systems is given by the phenomenon of quantum scars, recently observed in interacting Rydberg atom chains [Bernien et al, Nature 551 (2017): 579]. In this case, the persistent oscillations are due to the fact that the initial state of the dynamics has a large overlap with only few special eigenstates of the Hamiltonian governing the unitary dynamics.