An introduction to (Many-Body) Localization - supplementary 2

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Ex. 2 : Resonances, Green functions and self-energies in a two level system. To introduce the basic mechanism behind localization, the suppression of (long-distance) resonances, we briefly discussed the two level system with Hamiltonian:

$$\hat{H} = \begin{pmatrix} \epsilon_1 & V \\ V & \epsilon_2 \end{pmatrix}. \tag{1}$$

Define $\bar{\epsilon} = (\epsilon_1 + \epsilon_2)/2$ and $\Delta = (\epsilon_2 - \epsilon_1)/2$. Consider also the local Green functions and self-energies, as introduced in the lecture:

$$G_{aa}(z) = \langle a| \frac{1}{z-H} |a\rangle = \frac{1}{z - \epsilon_a - S_a(z)}, \qquad a = 1,2. \tag{2}$$

(i) Show that the eigenvalues and eigenfunctions of this Hamiltonian take the form:

$$E_\pm = \bar{\epsilon} \pm \sqrt{\Delta^2 + V^2}, \qquad \phi_\pm = \left[V^2 + \left(\mp\Delta + \sqrt{\Delta^2+V^2} \right)^2 \right]^{-\frac{1}{2}} \begin{pmatrix} \mp\Delta + \sqrt{\Delta^2+V^2} \\ \pm V \end{pmatrix}. \tag{3}$$

Determine the behavior in the limits $V/|\Delta| \ll 1$ and $V/|\Delta| \gg 1$.

(ii) Show that the local Green function on, say, the first site takes the form:

$$G_{11}(z) = \frac{1}{z - \epsilon_1 - \frac{V^2}{z-\epsilon_2}}. \tag{4}$$

Hint. Compute the expansion of $G_{11}(z)$ over eigenstates. It might bee useful to use the parametrization $\phi_+ = (-\sin\theta, \cos\theta)$ and $\phi_- = (\cos\theta, \sin\theta)$ with

$$\sin(2\theta) = -\frac{V}{\sqrt{\Delta^2+V^2}}, \qquad \cos(2\theta) = \frac{\Delta}{\sqrt{\Delta^2+V^2}}. \tag{5}$$

(iii) Write down the formal perturbative series of (4) in powers of V, and show that it can be written as a sum over all paths $\mathcal{P}$ in the 2-sites lattice that start and end in site 1. Show that the weight of each path is given by a product of random terms (one for each site s in the path), as:

$$G_{11}(z) = \frac{1}{z-\epsilon_1} \sum_{\mathcal{P} \in \text{Paths}(1,1)} \prod_{s \in \mathcal{P}} \frac{V}{z - \epsilon_s}. \tag{6}$$

Ex. 3. Quantum coherence and dephasing. The first part of this exercise is just a very simple example of how information can be encoded in the relative phase between quantum states in a superposition. The second part is about dephasing in MBL systems, and it is based on the work [Serbyn, Papic and Abanin, Phys. Rev. B 90, 174302 (2014)].

(i) **A simple example.** Take a qbit (two level system) with states $|1\rangle = (1 \quad 0)^T$ and $|0\rangle = (0 \quad 1)^T$, and consider the Hadamard operation

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{7}$$

together with the classical random NOT operation, in which you flip the qbit with probability 1/2, and leave it unchanged with the same probability. For any possible initial state (either $|1\rangle$ or $|0\rangle$), determine the density matrix describing the system after applying either the Hadamard operation and the random NOT operation. Show that measurements of the state of the systems give the same outcome in both cases, despite the density matrices are different. In one of the two cases, however, it is possible to reconstruct the information on what was the initial state: how?
Hint. Use that the Hadamard is unitary

(ii) **Two LIOMS.** Consider an Hamiltonian with two LIOMS τ_1, τ_2 with spectrum ± 1, placed at distance r_{12} and interacting with Hamiltonian:

$$H = h_{12}\tau_1\tau_2, \quad h_{12} = h_0 e^{-\frac{r_{12}}{\xi}}. \tag{8}$$

Consider a generic initial state $|\psi_0\rangle$ having a product structure in the LIOM basis, $|\psi_0\rangle = (a\,|1\rangle_1 + b\,|-1\rangle_1) \otimes (c\,|1\rangle_2 + d\,|-1\rangle_2)$. Compute the reduced density matrix of the first LIOM, and show that the off-diagonal matrix elements vanish at times inversely proportional to h_{12} (choose $a = b = 1/\sqrt{2}$).

(iii) **Dephasing in MBL [more difficult!].** Consider the effective model for MBL systems, given in terms of LIOMS τ_i^z as:

$$H_{\rm MBL} = h_0 + \sum_i h_i \tau_i^z + \sum_{i,j} h_{ij}\, \tau_i^z\, \tau_j^z + \sum_{i,j,k} h_{ijk}\, \tau_i^z\, \tau_j^z\, \tau_k^z + \cdots, \tag{9}$$

with $h_I \sim e^{-\frac{d(I)}{\xi_{\rm MB}}}$. The eigenstates of this Hamiltonian can be labeled by the string $= (\tau_1, \cdots, \tau_L)$ of quantum numbers $\tau_k = \pm 1$ associated to the LIOMS, and can be denoted by $|\tau\rangle$.

- Compute the reduced density matrix $\rho_1(t)$ of the first LIOM τ_1^z for a system initialized in the superposition of simultaneous eigenstates of all LIOMS,

$$|\psi_0\rangle = \otimes_{k=1}^L \left(\alpha_k|1\rangle + \beta_k|0\rangle\right). \tag{10}$$

  In particular, show that the diagonal matrix elements can be written as

$$[\rho_1(t)]_{12} = \beta_1^* \alpha_1 \sum_{\tau'} P_{\tau'} e^{i\left[E_1(\tau') - E_0(\tau')\right]} \tag{11}$$

  where τ' labels one configurations of τ_i^z for $i = 2, \cdots, L$, $P_{\tau'}$ its probability, $E_1(\tau')$ is the eigenenergy of the state $|1\rangle \otimes |\tau'\rangle$ and $E_0(\tau')$ is that of the state $|0\rangle \otimes |\tau'\rangle$.

- Assuming that at time t the first LIOM has dephased with all LIOMS at distance smaller/equal to $d(t)$ defined by $tH_0 e^{-\frac{d(t)}{\xi_{\rm MB}}} \sim 1$ where H_0 is some typical energy scale, show that

$$[\rho_1(t)]_{12} \sim \left(\frac{1}{H_0 t}\right)^{C\xi_{\rm MB}}. \tag{12}$$

  Hint. Split the energies $E_{0/1}(\tau')$ into terms containing only degrees of freedom at distance $d < d(t)$ (that are fully dephased), and degrees of freedom at $d > d(t)$.

Question 4: How can we understand everything in a unified framework?

The notion of quasi-local conserved quantities and the effective model for the MBL phase have been introduced in [Huse, Nandkishore and Oganesyan, Phys. Rev. B 90 (2014):174202] and [Serbyn, Papic and Abanin, Physical review letters 111 (2013):127201]. The explicit, perturbative construction of these conserved quantities for Hamiltonians of interacting fermions is given in [Ros, Mueller and Scardicchio, Nuclear Physics B 891 (2015): 420]. For the mathematical proof mentioned in the lecture see [Imbrie, Jour. Stat. Phys. 163 (1026):998]. A broader account of the works on integrability in MBL can be found in the review [Imbrie et al, *Local integrals of motion in many-body localized systems*, Annalen der Physik 529 (7), 1600278].

Question 5: Why is it challenging, and why is it worth the effort?

- **MBL as quantum device.** The experiment on quantum correlations measurements in a system of coupled superconducting qbits is given in [Chiaro et al, arxiv:1910.06024].
- **Order in highly-excited states.** The spin-glass ordered phase in the interacting version of the disordered transverse field Ising model was first discussed in [Huse et al, Phys. Rev. B 88, 014206 (2013)], see also [Laflorencie et al, arXiv:2201.00556] for more recent numerical results. A review on localization-protected quantum order is [Parameswaran and Vasseur, *Many-body localization, symmetry and topology*, Reports on Progress in Physics 81.8 (2018): 082501]. Out-of-equilibrium phases enabled by MBL have been found also for periodically driven systems, see for instance [Khemani et al, Phys. Rev. Lett. 116 (2016): 250401].
- **The dynamical phase transition.** The MBL-delocalization transition (and, more generally, the stability of the MBL phase) are subjects of very active theoretical research; the a picture for the delocalization transition being driven by avalanches was introduced in [T. Thiery, F. Huveneers, M. Müller and W. De Roeck, Physical review letters, 121 (2018)]. Notice that in disordered systems the delocalized phase of disordered systems has some peculiar properties in itself, see for example [Luitz and Bar Lev, Annalen der Physik 529 (7), 201600350]. For another recent review, see *Dynamics and Transport at the Threshold of Many-Body Localization* by Gopalakrishnan and Parameswaran [Physics Reports 862, 1 (2020)].