An introduction to (Many-Body) Localization - extra 3

V. Ros - SFT school, GGI 2022

Ex. 4 . Convergence of perturbation theory and Forward Approximation. In this lecture we discussed Anderson's estimate of the critical disorder. This is an exercise to integrate some details of the argument. We have in mind the Anderson Hamiltonian in $\mathbb{Z}^d$, given by:

$$H = \sum_{a \in \mathbb{Z}^d} \epsilon_a |a\rangle\langle a| + \sum_{\langle a,b \rangle} V_{ab} \left(|a\rangle\langle b| + |b\rangle\langle a|\right) = H_0 + V, \tag{1}$$

where the hopping is between nearest-neighboring sites and the ϵ_a are independent, uniformly distributed in $\left[-\frac{W}{2}, \frac{W}{2}\right]$.

(i) **Distribution of paths/loops weights.**

- Show that the rescaled variable $x_n = \prod_{i=1}^n 1/|\epsilon_i|$ with ϵ_i independent and uniformly distributed in $[-1,1]$ has distribution:
$$P_{x_n}(x) = \frac{1}{x^2} \frac{(\log x)^{n-1}}{(n-1)!}.$$
Hint. Compute first the distribution of $\sigma_n = \log x_n$, by determining its Laplace transform $\Phi_{\sigma_n}(u) = \mathbb{E}_{\sigma_n}[e^{-u\sigma_n}]$ and subsequently inverting it.
- For n large, determine the typical value ω^{typ} of $|\omega_{\mathcal{L}}| = \left(\frac{2V}{W}\right)^n x_n$ and show that it scales exponentially with n, as $\omega^{\text{typ}} = \lambda_{\text{typ}}^n$ for some λ_{typ}. Show that for $\lambda > \lambda_{\text{typ}}$, the probability $\mathbb{P}\left(|\omega_{\mathcal{L}}| > \lambda^n\right)$ takes the large deviation form:
$$\mathbb{P}\left(|\omega_{\mathcal{L}}| > \lambda^n\right) \sim e^{-n\Lambda(\lambda) + o(n)},$$
and determine the large deviation rate $\Lambda(\lambda)$.
Hint. The large deviation probability can be written in terms of the probability of σ_n: determine $\Lambda(\lambda)$ via the saddle-point approximation.

(ii) **Estimate the localization length in FA.** We saw in the lecture/notes how to get an approximate expression for the eigenstate amplitudes $\psi_\alpha(b)$ in Forward Approximation, assuming $\psi_\alpha(a) \sim 1$:

$$\psi_\alpha(b) = \sum_{\mathcal{P}' \in \text{SDP}(a,b)} \prod_{s \in \mathcal{P}'} \frac{V}{\epsilon_a - \epsilon_s}, \tag{2}$$

where the sum is over the Shortest Directed Paths connecting the sites a, b. Use this expression to estimate the localization length for $W/V > (W/V)_c$, by computing the ξ such that

$$\mathbb{P}\left(\max_{b: d(a,b)=n} |\psi_\alpha(b)|^2 < e^{-\frac{n}{\xi}}\right) \stackrel{n \to \infty}{\longrightarrow} 1. \tag{3}$$

How does this diverge when $W \to W_c$? What does this criterion give in $1d$?
Notice. The $1d$ result illustrates that indeed the Forward Approximation gives an upper bound for the critical disorder strength W_c.

(iii) **Accounting for self-energy corrections.** We saw in the lecture/notes that the self energy corrections introduce an anti-correlation between consecutive locators in the expansion over self-avoiding paths/loops. One way to account approximately for this effect was proposed already by Anderson in his '58 paper, and consists in replacing $(E - \epsilon_i - S_i) \to E - \epsilon'_i$ where the distribution of ϵ'_i has a cutoff, i.e. $|\epsilon'_i| > 1/W$. Can you justify this? Show how the estimate for the critical disorder (in the case of a uniform distribution) gets modified with respect to the one obtained in the FA: did you expect the critical disorder to become larger or smaller?

I. Decay rates of local excitations: a functional order parameter.

- **Analytic properties of the resolvent and poles in second Reimann sheet.** See handwritten notes at the end of this file for more details on the computation of the decay rate, and on the poles in the second Riemann sheet. A nice place where to read about this is in Chapter 5 (on nuclear physics) of the book by R. Peierls "More surprises in theoretical physics" [Princeton University Press, 1991].

- **Spectral localization** In the mathematical literature the criterion for localization given in terms of pure point spectrum is referred to as "spectral localization", and the rigorous version of the argument relating the spectral properties of the Hamiltonian to the occurrence of bound states is given by the so called RAGE theorem. A standard approach to investigate localization also numerically (that extends also to the many-body case) consists in characterizing the properties of the energy levels, in particular the absence of level repulsion and Poisson statistics. For early applications of these approaches to interacting Hamiltonians see [Oganesyan and Huse, Physical review b 75(15), 155111]. Notice that Poisson statistics of the eigenvalues is used as a criterion to prove localization in systems where disorder is not given by local potentials, but rather by the structure of the underlying lattice: this is the case of Erdös–Rényi graphs, see [Alt, Ducatez, and Knowles, arXiv:2106.12519] and [Tarzia, arXiv:2112.11560] for very recent works.

II. Anderson's reasoning: a smart perturbation theory and its convergence.

- **Self-avoiding paths/loops.** It is quite intuitive that the number N_n of self-avoiding paths/loops of length n in a regular lattice in dimension d scales exponentially with n. For walks, $N_n \sim A\kappa^n n^{\gamma-1}$ where κ is the so called connective constant, see for instance [Bauerschmidt et al, Lectures on self-avoiding walks, Clay Mathematics Institute Proceedings 15 (2012): 395].

- **Role of dimensionality, beyond the FA.** In Anderson's argument, dimensionality enters as a parameter through the connective constant κ. A phenomenological description of the role of dimension in the single particle case is given within the scaling theory of localization developed in [Abrahams, Anderson, Licciardello, and Ramakrishnan, Physical Review Letters, 42(1979):673]. This predicts the lower critical dimension for the Anderson transition to be $d = 2$: in $d = 1$ and $d = 2$ (for system with orthogonal symmetry) all eigenstates are localized. The mechanisms for localization in these low dimensions is rather different than that emerging from the FA treatment: it relies more heavily on backscattering in $1d$ [Thouless, Journal of Physics C: Solid State Physics 5 (1972): 77], or on the interference between a path and its time reversed in $2d$ [Gorkov, Larkin and Khmelnitskii, JETP Lett. 30 (1979): 228]. Localization in $2d$ is not rigorously prove, at variance with the $1d$ case [Furstenberg and Kesten, Ann. Math. Stat. 31, 457-469 (1960)]. For accurate numerical estimates of the critical value of disorder in $d = 3 - 6$ see [Slevin and Ohtsuki, New. J. Phys. 16 (2014): 015012] and [Tarquini, Biroli and Tarzia, Phys. Rev. B 95 (2017): 094204]. For an example of how to study the Forward Approximation numerically using transfer matrices see [Pietracaprina et al, Phys. Rev. B 93 (2016):054201].

III. And beyond: from Bethe lattice to Many-Body, passing through directed polymers

- **Beyond Anderson's argument: the Bethe lattice.** The Anderson model on the Bethe lattice is discussed in [Abou-Chacra, Anderson and Thouless, Journal of Physics C: Solid State Physics 6.10 (1973): 1734]. An exact criterion for localization on the Bethe lattice has been proven by Aizenman and Warzel in [Journal of the European Mathematical Society, 15 (2013):1167]. Based on this, the large-connectivity asymptotics of the critical disorder W_c in the middle of the energy band has been worked out rigorously in [Bapst, Journal of Mathematical Physics 55.9 (2014): 092101].

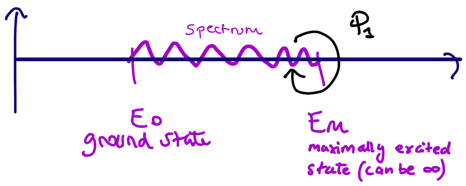
- **FA and connections with the Directed Polymer problem.** The FA naturally leads to the problem of estimating the free energy of a directed polymer in a random potential. A recipe to compute the quenched free energy of such polymer is given in the seminal paper by Deridda and Spohn [Journal of Statistical Physics 51.5-6 (1988): 817]. It can be easily shown that the Abou-Chacra, Anderson and Thouless equations are equivalent to the equations obtained asking that the directed polymer free energy becomes exactly equal to zero at the localization/delocalization transition. The low temperature phase $\beta > \beta_c$ is referred to as the "frozen" or "glassy" phase of the directed polymer: in this regime, the partition function is dominated by a sub-exponential number of paths contributing to the total sum. The localized phase is entirely contained in this frozen regime. The glassy features of the localized phase on the Bethe lattice have been investigated in a large variety of works, see for example [Monthus and Garel, Journal of Physics A: Mathematical and Theoretical 42.7 (2008): 075002] or the more recent [Lemarié, Physical review letters 122.3 (2019): 030401].

- **The many-body criterion.** The fate of localization in presence of interactions was already discussed by Anderson himself in [Fleishman and Anderson, Physical Review B, 21(1980), 2366]. The full extension of Anderson's argument to the case of interacting particles came however quite later, with the work [Basko, Aleiner, Altshuler, Annals of Physics 321 (2006):1126], see also [Gornyi, Mirlin, Polyakov, Phys. Rev. Lett. 95 (2005):206603]. These works build on the important precursor [Altshuler, Gefen, Kamenev, Levitov, Physical review letters, 78((1997), 2803]. A discussion of the implications of the self-energy criterion in the many-body case can be found in [Basko, Aleiner, Altshuler, Problems of Condensed Matter Physics (2006): 50]. Numerical diagnostics of MBL (based on the eigenstates structure) have been discussed for example in [De Luca and Scardicchio, Europhys. Lett., 101 (2013): 37003], [Pal and Huse. Phys. Rev. B 82 (2010): 174411]. A combination of these diagnostics has been used in [Luitz, Laflorencie and Alet, Physical Review B 91.8 (2015): 081103] to estimate numerically the phase diagram of the random-field Heisenberg chain.

SUPPLEMENT: analytic properties of $G_a(z)$, $S_a(z)$

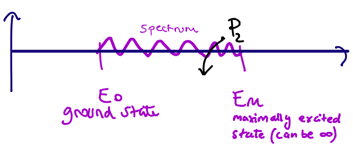
- When $|\Lambda| \to \infty$ and the spectrum becomes continuous, the poles on the real axis (eigenvalues) coalesce to form a branch cut. The function $G_a(z) = \int \frac{dE' \, \nu_a(E')}{z - E'}$ has a discontinuity (of its imaginary part) across the cut: one finds

$$\mathrm{Im}\{G_a(E \pm i\eta)\} = \mathrm{Im}\left\{\int \frac{dE' \, \nu_a(E')}{E \pm i\eta - E'}\right\} = \mp \int dE' \, \nu_a(E') \frac{\eta}{(E-E')^2 + \eta^2} \xrightarrow{\eta \downarrow 0} \mp i\pi \nu_a(E)$$

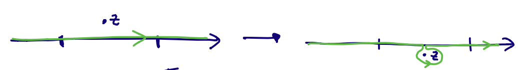


The function $G_a(z)$ for $\mathrm{Im}\, z < 0$ is the result of the analytic continuation of $G_a(z)$ for $\mathrm{Im}\, z > 0$ along the path P_1 that circles around the branch point.

- Because of the cut, the function is multivalued (like $\log z$, $\sqrt{z}$ etc...). The analytic continuation ACROSS the cut takes a Different value. To get it, we take Re z belonging to the spectrum. The integral $G_a(z) = \int_{E_0}^{E_m} \frac{dE' \, \nu_a(E')}{z - E'}$ has a pole at $E' = z$, and for $\mathrm{Im}(z) > 0$ the integration



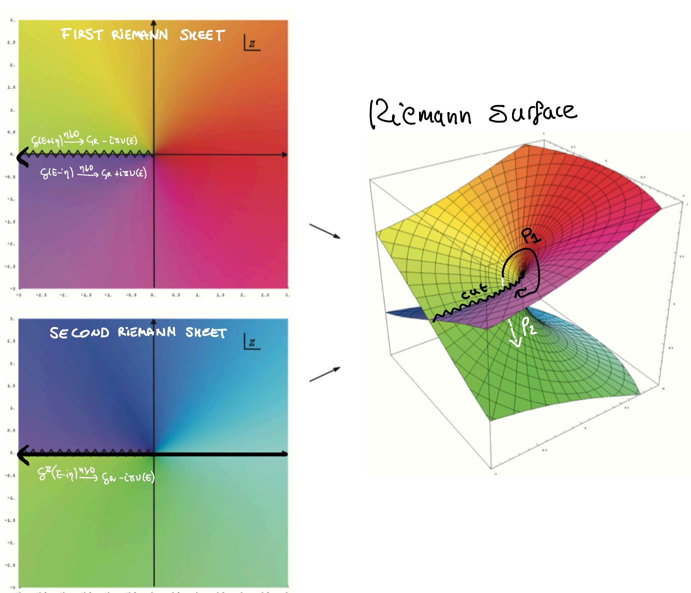
contour lies below the pole. To continue the function to $\mathrm{Im}\, z < 0$ we have to modify the contour so that the pole remains above:



This equals : $G_a^{II}(z)\Big|_{\mathrm{Im}\, z < 0} = \int_{\mathbb{R}} \frac{dE' \, \nu_a(E')}{z - E'} + 2i\pi \lim_{E' \to z} (E' - z) \frac{\nu_a(E')}{z - E'} = \int_{\mathbb{R}} dE' \frac{\nu_a(E')}{z - E'} - 2\pi i \nu_a(z)$

This function is continuous across the cut, as it should.

- As usually happens for multivalued functions, the analytic continuation depends on the path. This is because different paths brings you to different sheets on the Riemann surface, see below. The function $G_a^{II}(z)$ above, in particular, is defined on the 2nd sheet.



- The same things hold for the $S_a(z)$. On the 2nd sheet one finds

$$S_a^{II}(z) = \int_{\mathbb{R}} \frac{dE'}{2\pi} \frac{\Gamma_a(E')}{z - E'} - i\Gamma_a(z) = S_a(z) - i\Gamma_a(z)$$

Therefore,

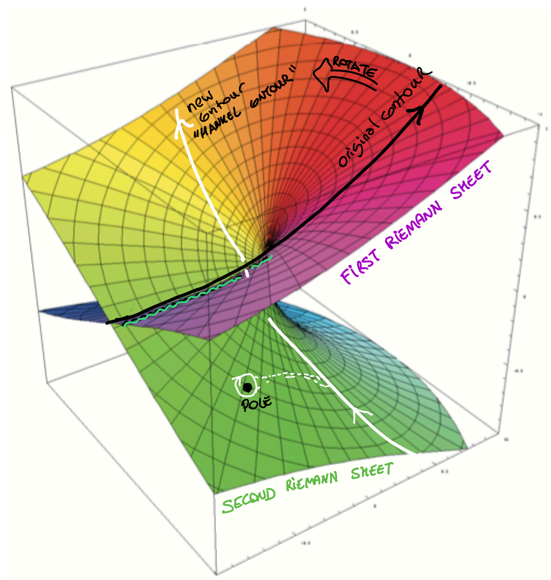
$$G_a^{II}(z) = \frac{1}{z - \varepsilon_a - S_a^{II}(z)} = \frac{1}{z - \varepsilon_a - S_a(z) + i\Gamma_a(z)}$$

has poles z^* in the lower-half plane, satisfying:

$$z^* - \varepsilon_a - S_a(z^*) + i\Gamma_a(z^*) = 0.$$

Assume there is one of such poles.

- The integral defining $A_a(t)$ can be computed extending the integrand to the complex plane & rotating the contour as shown below. Then we get the contribution of the pole, that gives a finite decay rate, + the contribution of the white contour (called the "Hankel contour").



[In this plot the spectrum is in the negative semiaxis. The corresponding branch-cut is the green line]

- The contribution of the Hankel contour has a power-law behavior at large times, going like $1/t^{\delta+1}$

  where $\Gamma_a(E) \sim (E-E_0)^{\delta}$ $\quad E \sim E_0$ (edge of Density)

  [try to show it!]

  This is the contribution that dominates at very large times, while the exponential term dominates at intermediate ones.

Refs

For details on this you may look at P. Facchi, "Quantum time evolution: free and controlled dynamics", doctoral thesis (2000).