Exercise set 1

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1. Derive the equation of motion for the flux inside an rf SQUID. How does this compare to a Brownian particle?

2. Consider a system interacting with a reservoir, which is described by a Lagrangian

$$L = L_S + L_I + L_R$$

where

$$\begin{aligned} L_S &= \frac{1}{2} M \dot{q}^2 - V_0(q) \\ L_I &= -\sum_k C_k \, q \, q_k \\ L_R &= \sum_k \left(\frac{1}{2} m_k \dot{q}_k^2 - \frac{1}{2} m_k \omega_k^2 q_k^2 \right) \end{aligned}$$

are respectively, the Lagrangian of the system (S), of interaction (I), and of the reservoir (R), which is formed by an ensemble of harmonic oscillators with coordinates q_k, mass m_k, and coupling constants C_k.

   (a) Derive the equations of motion for q and q_k.

   (b) Take the Laplace transform for the q_k equation and show that

$$\tilde{q}_k(s) = \frac{\dot{q}_k(0)}{s^2 + \omega_k^2} + \frac{s q_k(0)}{s^2 + \omega_k^2} + \frac{C_k \tilde{q}(s)}{m_k (s^2 + \omega_k^2)}.$$

Hint: Look, for instance, at the book of Arfken, Mathematical methods for physics, and find that $\mathcal{L}\left[f^{(n)}(t)\right] = s^n \mathcal{L}\left[f(t)\right] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$.

3. Now, take the inverse transform of $\tilde{q}_k(s)$ and substitute it into the equation of motion for q. Using the identity

$$\frac{1}{s^2 + \omega_k^2} = \frac{1}{\omega_k^2} \left[1 - \frac{s^2}{s^2 + \omega_k^2} \right],$$

derive Eq. (42) of the script.

4. Is there any difference between coupling the system of interest to a reservoir via coordinate-coordinate or via velocity-velocity?

5. Find some physical examples where the potential is renormalized by the coupling to the reservoir.

6. Show that the spectral function $J(\omega)$ defined in Eq. (45) of the script is nothing but the imaginary part of the Fourier transform of the (retarded) dynamical susceptibility of the oscillators bath in the classical limit,

$$J(\omega) = \mathrm{Im} \mathcal{F} \left\{ -i\theta(t - t') \left\langle \left[\sum_k C_k q_k(t), \sum_{k'} C_{k'} q_{k'}(t') \right] \right\rangle \right\}.$$

7. Assuming an Ohmic spectral function and using Eq. (46) of the script, show that the last term in the LHS of Eq. (42) becomes $\eta\dot{q}(t)$.

8. Use the equipartition theorem for an Ohmic bath of harmonic oscillators to show that $\langle f(t)\rangle = 0$ and $\langle f(t)f(t')\rangle = 2\eta k_B T\delta(t-t')$.

9. Collect all the above results to show that the Caldeira-Leggett model reproduces the Langevin equation in the classical limit.

10. The Caldeira-Leggett model describes the environment as a bath of harmonic oscillators. How good is this approximation, and when is it valid?