

A short recap of the previous lecture I

- ✓ I] General Overview and applications of RMT
- ✓ II] Ensembles of RMT (Wigner vs rot. inv. ens.)
- III] Coulomb gas approach  $\leftarrow$
- IV] Local statistics (bulk & edge)

Let  $M = (M_{jk})$  be a  $N \times N$   $\left\{ \begin{array}{l} \text{real sym. } (\beta=1) \\ \text{matrix} \\ \text{complex Hermitian } (\beta=2) \end{array} \right.$

$$P(\lambda_1, \dots, \lambda_N) = B_N e^{-\beta \frac{N}{2} \sum_{i=1}^N \lambda_i^2} \prod_{i < j} |\lambda_i - \lambda_j|^\beta$$

joint proba. density function of the eigenvalues

$$\Rightarrow P(\lambda_1, \dots, \lambda_N) = B_N e^{-\beta E(\{\lambda_i\})}$$

$$E(\{\lambda_i\}) = \frac{N}{2} \sum_{i=1}^N \lambda_i^2 - \frac{1}{2} \sum_{i \neq j} \ln |\lambda_i - \lambda_j|$$

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$\Rightarrow$  Dyson's log-gas

$\Rightarrow$  competition between confinement and the repulsive interactions

• Typical scale of  $\lambda_i$ 's  $\equiv \lambda_{\text{typ}}$

$$\begin{aligned} \text{Potential energy: } \frac{N}{2} \sum_{i=1}^N \lambda_i^2 &\sim \frac{N}{2} \times \lambda_{\text{typ}}^2 \times N \\ &\sim O(N^2 \lambda_{\text{typ}}^2) \end{aligned}$$

$$\text{suppose that } \frac{1}{2} \sum_{i \neq j} |\lambda_i - \lambda_j| \sim O(N \times N^2)$$

$$\begin{aligned} \text{Interaction energy: } \frac{1}{2} \sum_{i \neq j} \ln |\lambda_i - \lambda_j| &= O\left(\frac{N(N-1)}{2}\right) \\ &= O(N^2) \end{aligned}$$

$\Rightarrow$  balancing the two terms:

$$\begin{aligned} N^2 \lambda_{\text{typ}}^2 \sim N^2 &\Rightarrow \lambda_{\text{typ}} = O(1) \\ &= O(N^0) \end{aligned}$$

• Empirical eigenvalue distribution:

$$P_N(\lambda) = \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i)$$

such that  $\int_{-\infty}^{\infty} d\lambda P_N(\lambda) = 1, \forall N.$

For GOE, GUE

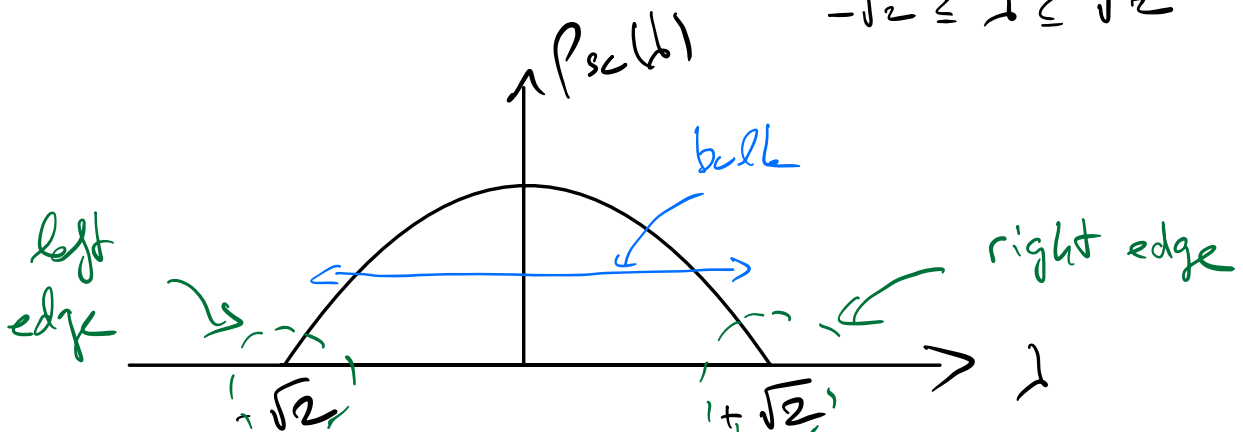
$$\lim_{N \rightarrow \infty} P_N(\lambda) = \langle P_N(\lambda) \rangle$$

$\langle \dots \rangle$  v. respect to  $P(\lambda_1, \dots, \lambda_N)$   
 $\hookrightarrow$  self-averaging for  $\lambda$  in the bulk.

Wigner semi-circle:

$$P_N(\lambda) \xrightarrow{N \rightarrow \infty} P_{sc}(\lambda) = \frac{1}{\pi} \sqrt{2 - \lambda^2}$$

$$-\sqrt{2} \leq \lambda \leq \sqrt{2}$$



. Universality of the Wigner semi-circle:

\* Wigner matrices: for independent entries w.  $\langle \Pi_{jk} \rangle = 0$ ,  $\langle \Pi_{jk}^2 \rangle < \infty$

then  $P_N(\lambda) \xrightarrow{N \rightarrow \infty} P_{sc}(\lambda)$ , + scale transformation

\* For rot. invariant ensemble

$$P(\lambda_1, \dots, \lambda_N) = \tilde{B}_N \prod_{i < j} |\lambda_i - \lambda_j|^\beta e^{-N \sum_{i=1}^N V(\lambda_i)}$$

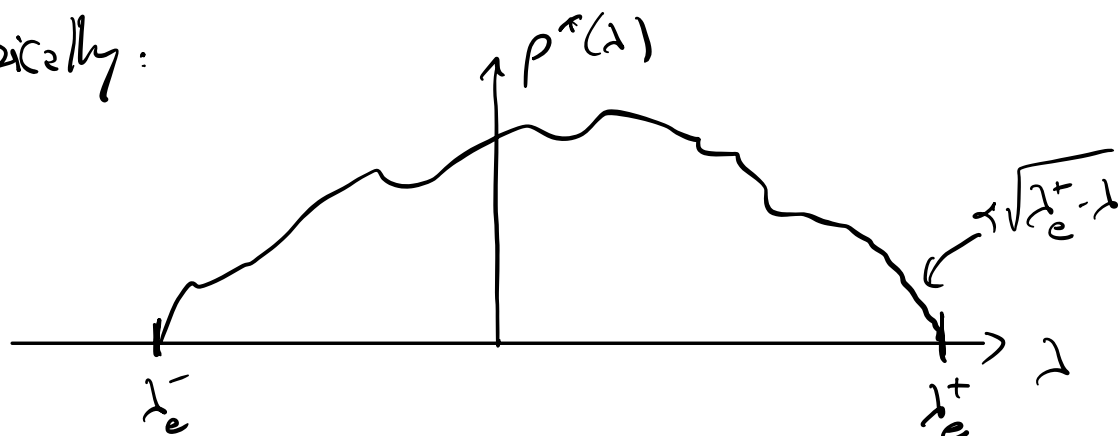
$$P(M) \propto e^{-N \text{Tr} V(M)}$$

$$\text{IF } \frac{V(\lambda)}{\ln \lambda} \xrightarrow{\lambda \rightarrow \infty} + \infty$$

then  $P_N(\lambda) \xrightarrow{N \rightarrow \infty} P^*(\lambda)$  which has a finite support.

But in  $g^{\text{sc}}$   $P^*(\lambda) \neq P_{sc}(\lambda)$

Typically:

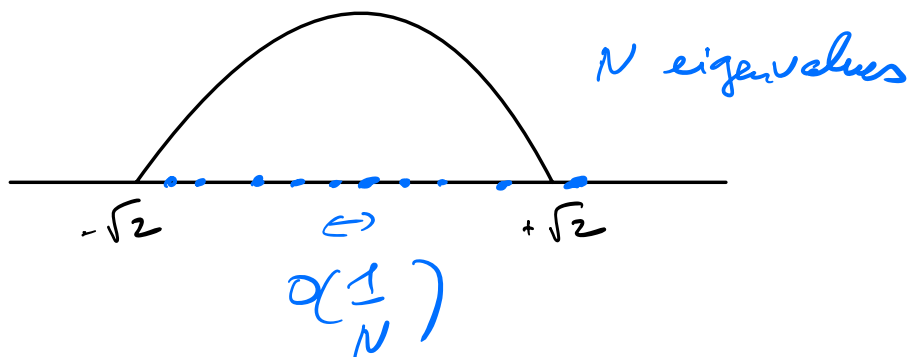


#### 4.1 Local statistics

Bulk vs edge

a) Bulk

\* Spacing between eigenvalues



$$\lambda_1 < \lambda_2 < \dots < \lambda_N$$

$$s = \lambda_{i+1} - \lambda_i = O\left(\frac{1}{N}\right)$$

Distribution of  $s$  ?

$P_N(s)$  = probab. density function of ' $s$ '

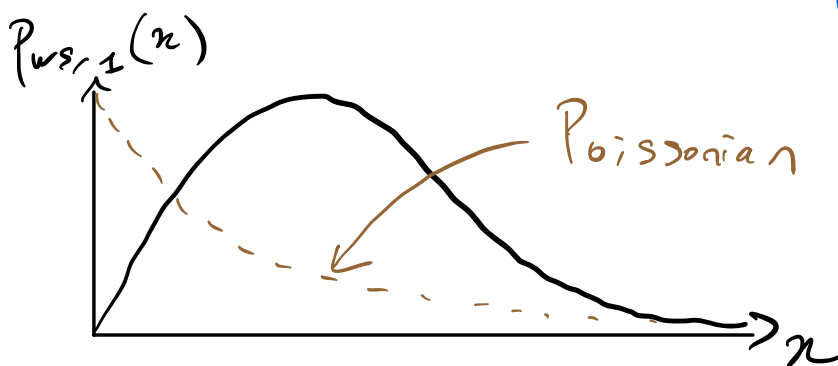
$$P_N(s) \xrightarrow[N \rightarrow \infty]{} \frac{1}{\langle s_N \rangle} P_\beta \left( \frac{s}{\langle s_N \rangle} \right)$$

$P_\beta(x)$  is well approximated by

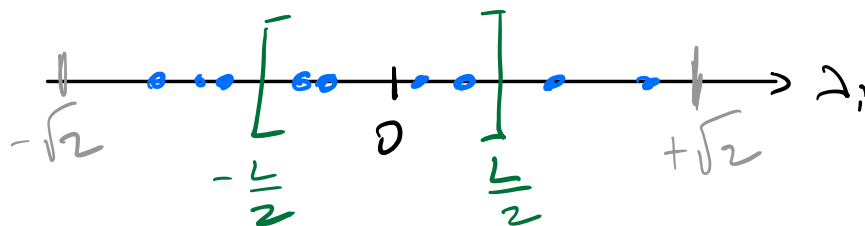
the *Wigner surmise*

$$P_\beta(x) \approx P_{WS,\beta}(x) = a_\beta x^\beta e^{-b_\beta x^2}$$

↑ level repulsion



\* Nber Variance:



$N_L$  = nber of eigenvalues in  $[-\frac{L}{2}, \frac{L}{2}]$

$$\langle N_L \rangle \simeq \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \rho_{sc}(x) = O(L)$$

$$\langle \text{Var } N_L \rangle \simeq \frac{2}{\beta \pi^2} \ln(NL) \ll \langle N_L \rangle$$

$\frac{L}{2} \ll \sqrt{L}$

$\Rightarrow$  rigidity of the spectrum

(more rigid than a Poisson point process)

Rk:  $\mathbb{P}(N_L = 0) \equiv$  gap proba.

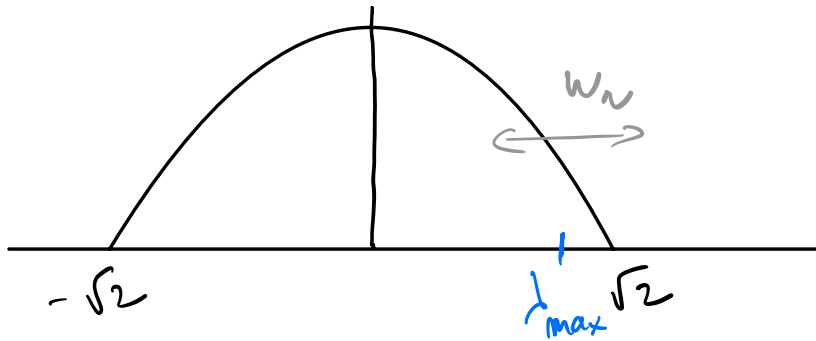
$\rightarrow$  Wigner surmise



## b) Edge

Width of the edge:  $w_N$

Define  $\lambda_{\max} = \max_{1 \leq i \leq N} \lambda_i$



When  $N \rightarrow \infty$ :  $\lambda_{\max} \rightarrow \sqrt{2}$

Argument from extreme value statistics:

Nber of eigenvalues in  $[\sqrt{2} - w_N, \sqrt{2}] \sim o(1)$

$$N_x \int_{\sqrt{2} - w_N}^{\sqrt{2}} \rho_{sc}(x) dx = o(1)$$

$\sim \sqrt{\sqrt{2} - x}$

$$\Rightarrow N_x w_N^{3/2} = o(1)$$



$$\Rightarrow W_N = O(N^{-2/3})$$

$\Rightarrow$  Tracy-Widom distributions ('94, '96)  
(TW)

$$\lambda_{\max} = \sqrt{2} + \frac{1}{\sqrt{2}} N^{-2/3} \chi_{\beta}$$

TW random variable

For  $\beta = 1, 2$ ,  $F_{\beta}(x) = P(\chi_{\beta} \leq x)$

Painlevé II:  $q''(s) = 2q^3(s) + sq(s)$

$s \in \mathbb{R}$

$$q(s) \underset{s \rightarrow \infty}{\sim} Ai(s)$$

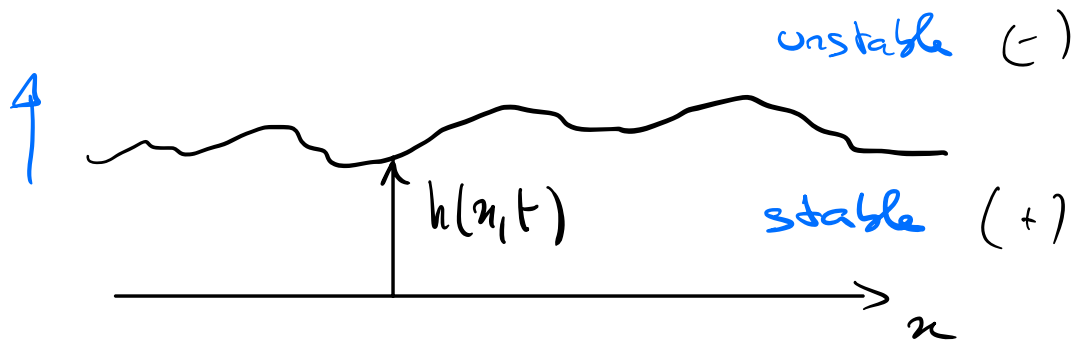
Airy-Function

$$\beta=2: F_2(x) = \exp\left[-\int_x^{\infty} ds (s-x) q^2(x)\right]$$

$$F_2'(x) \sim \begin{cases} \exp(-\frac{\beta}{24} |x|^3) & x \rightarrow -\infty \\ \exp(-\frac{2\beta}{3} x^{3/2}) & x \rightarrow +\infty \end{cases}$$

→ Connection w. Kardar-Parisi-Zhang equation  
(KPZ)

1+1 dim. fluctuating interface



$$\partial_t h(x,t) = \nu \Delta h(x,t) + \frac{\lambda}{2} (\partial_x h)^2 + \sqrt{D} \xi$$

white noise

• Flat initial condition:  $h(x,0) = 0, \forall x$

$$h(x,t) = c_f t + (\Gamma_f t)^{\frac{1}{3}} \chi_1$$

Tracy - Widom  
distribution  
 $\beta = 1/GOE$

• Curved initial profile



$$h(x,t) = c_2 t + \left( \frac{c_2 t}{\beta} \right)^{\frac{1}{3}} \chi_2$$

TW for

$$\beta = 2$$

Rh: Ising model + mag field

