Lecures on Statistical Field Theory
Random matrices and statistical physics – Homework 1

Grégory Schehr

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The Wigner semi-circular law via replicas

We consider an $N \times N$ random matrix $\mathbf{J}$ drawn from the Gaussian Orthogonal Ensemble (GOE), i.e. $\mathbf{J}$ is a real symmetric matrix such that, for $k \leq l$, the entries J_{kl} are independent Gaussian entries of zero mean and variances $\mathbb{E}(J_{kk}^2) = 2/N$ and $\mathbb{E}(J_{kl}^2) = 1/N$ for $k < l$. We denote by $\lambda_1, \lambda_2, \cdots, \lambda_N$ the N real eigenvalues of $\mathbf{J}$ and the goal is to compute the average density of eigenvalues

$$\rho(x) = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\delta(x-\lambda_i)\right] , \tag{1}$$

in the limit $N \to \infty$. In the following, ϵ denotes a strictly positive real number $\epsilon > 0$. You will find, at the end, some results on Gaussian integrals, which may be useful in the following computations.

1 Preamble

1.) Using the identity, valid for any real x in the sense of distribution,

$$\lim_{\epsilon\to 0^+}\frac{1}{x-i\epsilon} = \mathrm{PV}\frac{1}{x} + i\,\pi\delta(x) , \tag{2}$$

where PV denotes the Cauchy principal value, show that

$$\rho(x) = -\frac{1}{N\pi}\lim_{\epsilon\to 0^+}\mathrm{Im}\left(\mathbb{E}\left[\mathrm{Tr}(\mathbf{J}-x_\epsilon\mathbb{I}_N)^{-1}\right]\right) , \; x_\epsilon = x - i\epsilon , \tag{3}$$

where $\mathbb{I}_N$ denotes the identity matrix of size N.

2.) Consider the N-fold Gaussian integral

$$\begin{aligned} Z_J(x_\epsilon) &= \int_{-\infty}^{\infty}\mathrm{d}\phi_1\cdots\int_{-\infty}^{\infty}\mathrm{d}\phi_N\exp\left[-\frac{i}{2}x_\epsilon\sum_{k=1}^{N}\phi_k^2 + \frac{i}{2}\sum_{k,l}J_{kl}\,\phi_k\phi_l\right] & (4)\\ &= \int_{\mathbb{R}^N}\mathrm{d}\vec{\phi}\,\exp\left[-\frac{1}{2}\vec{\phi}^T(ix_\epsilon\mathbb{I}_N - i\,\mathbf{J})\vec{\phi}\right] . & (5)\end{aligned}$$

Justify that $Z_J(x_\epsilon)$ is well defined and show, using results for Gaussian integrals, that

$$\mathrm{Tr}(\mathbf{J}-x_\epsilon\mathbb{I}_N)^{-1} = 2\,\frac{\partial}{\partial x}\ln Z_J(x_\epsilon) . \tag{6}$$

2 Annealed computation

3.) To compute $\rho(x)$, we will first make the approximation $\mathbb{E}\left[\ln Z_J(x_\epsilon)\right] \approx \ln\left(\mathbb{E}[Z_J(x_\epsilon)]\right)$. When do you expect this approximation to be valid?

4.) Show that the averaged "partition function" $\mathbb{E}[Z_J(x_\epsilon)]$ can be written as

$$\mathbb{E}[Z_J(x_\epsilon)] = \int_{-\infty}^{\infty} \mathrm{d}\phi_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}\phi_N \exp\left[-\frac{i}{2}\, x_\epsilon \sum_{k=1}^{N} \phi_k^2 - \frac{1}{4N}\left(\sum_{k=1}^{N} \phi_k^2\right)^2\right] . \tag{7}$$

5.) By using the standard Gaussian identity (18) for $m = 1$ to disentangle the ϕ_k variables show that $\mathbb{E}[Z_J(x_\epsilon)]$ can be written as

$$\mathbb{E}[Z_J(x_\epsilon)] = \sqrt{\frac{N}{\pi}} \int_{-\infty}^{\infty} \mathrm{d}q\, \mathrm{e}^{-N\,\varphi_x(q)} \;,\; \varphi_x(q) = q^2 - \frac{1}{2}\ln\left(\frac{2\pi}{i(x_\epsilon - 2q)}\right) . \tag{8}$$

6.) Explain why, to compute the density $\rho(x)$ in the large N limit, we only need to retain terms in (8) which are of order $\mathcal{O}(\mathrm{e}^N)$.

7.) At this order of accuracy, the integral over q in (8) can be evaluated, for large N, by the Laplace (or saddle-point) method. To find this saddle point, we need to solve $\partial_q \varphi_x(q) = 0$. Show that this equation has actually two roots $q = q_+$ or $q = q_-$ given by

$$q_\pm = \frac{1}{4}\left(x_\epsilon \pm \sqrt{x_\epsilon^2 - 4}\right) . \tag{9}$$

8.) It can be shown that the correct saddle point is given by $q = q^+$. Deduce then that in the limit $N \to \infty$ one has

$$\lim_{N\to\infty} \rho(x) = \frac{2}{\pi} \lim_{\epsilon\to 0^+} \mathrm{Im} \frac{\partial}{\partial x}\left[\varphi_x(q_+)\right] . \tag{10}$$

9.) Obtain finally that, when $N \to \infty$, $\rho(x) = \sqrt{4 - x^2}/(2\pi)$ if $|x| \leq 2$ while $\rho(x) = 0$ if $x > 2$.

3 Quenched computation (More complicated)

To compute the average $\mathbb{E}\left[\ln Z_J(x_\epsilon)\right]$, going beyond the approximation $\mathbb{E}\left[\ln Z_J(x_\epsilon)\right] \approx \ln\left(\mathbb{E}[Z_J(x_\epsilon)]\right)$, we will use the replica trick which we find convenient here to write as

$$\mathbb{E}[\ln Z_J(x_\epsilon)] = \lim_{n\to 0} \frac{1}{n} \ln\left(\mathbb{E}[Z_J(x_\epsilon)^n]\right) . \tag{11}$$

10.) Use this replica trick (11) to write $\rho(x)$ in terms of $\mathbb{E}\left[Z_J(x_\epsilon)^n\right]$.

11.) By performing the averages over the random variables J_{kl}, show that

$$\mathbb{E}[Z_J(x_\epsilon)^n] = \int_{-\infty}^{\infty} \prod_{k=1}^{N}\prod_{a=1}^{n} \mathrm{d}\phi_k^a \exp\left[-\frac{i}{2}\, x_\epsilon \sum_{k=1}^{N}\sum_{a=1}^{n} (\phi_k^a)^2 - \frac{1}{4N}\sum_{a,b=1}^{n}\left(\sum_{k=1}^{N} \phi_k^a \phi_k^b\right)^2\right] . \tag{12}$$

12.) By using $n(n+1)/2$ times the Gaussian identity (18) show that

$$\mathbb{E}[Z_J(x_\epsilon)^n] = A_{N,n} \int_{-\infty}^{\infty} \prod_{k=1}^{N} \prod_{a=1}^{n} \mathrm{d}\phi_k^a \prod_{1=a\le b\le n} \mathrm{d}Q_{ab} \times \exp\left[-N \sum_{a,b=1}^{n} (Q_{ab})^2 - \frac{i}{2} x_\epsilon \sum_{k=1}^{N}\sum_{a=1}^{n} (\phi_k^a)^2 + i \sum_{a,b=1}^{n} Q_{ab} \sum_{k=1}^{N} \phi_k^a \phi_k^b\right] \tag{13}$$

where $A_{N,n} = 2^{n(n-1)/4} \left(\frac{N}{\pi}\right)^{n(n+1)/4}$, with the convention, in (13), that $Q_{ab} = Q_{ba}$ when $a > b$.

13.) Observe that the n-dimensional vectors $\{\phi_k^1, \phi_k^2, \cdots, \phi_k^n\}$, for different values of $k = 1, \cdots, N$, are now independent on this expression (13), and show that

$$\mathbb{E}[Z_J(x_\epsilon)^n] = A_{N,n} \prod_{1\le a\le b\le n} \int_{-\infty}^{\infty} \mathrm{d}Q_{ab} \exp\left[-N\Phi_x(\mathbf{Q})\right] \tag{14}$$

where $\mathbf{Q}$ is the (symmetric) overlap matrix, of size $n \times n$, with matrix elements Q_{ab} for $a \le b$ (and thus $Q_{ab} = Q_{ba}$ if $a > b$), while $\Phi_x(\mathbf{Q})$ is given by

$$\Phi_x(\mathbf{Q}) = \mathrm{Tr}[\mathbf{Q}^2] - \frac{1}{2}\mathrm{Tr}\left[\ln\left(\frac{2\pi}{i}\left(x_\epsilon\, \mathbb{I}_n - 2\mathbf{Q}\right)^{-1}\right)\right] , \tag{15}$$

where $\mathbb{I}_n$ denotes the identity matrix of size n.

14.) In the large N limit, this multiple integral (14) can be evaluated by the Laplace (or saddle-point) method. Show that the saddle point equations read

$$2\,Q_{ab} - (x_\epsilon \mathbb{I}_n - 2\mathbf{Q})^{-1}_{ab} = 0 \,, \quad \forall\, a,b = 1, \cdots, n \,. \tag{16}$$

<u>**Hint:**</u> we recall that, for an invertible matrix $\mathbf{Y}$ depending on a parameter z, one has $\partial_z(\det\mathbf{Y}) = (\det\mathbf{Y})\,\mathrm{Tr}(\mathbf{Y}^{-1}\partial_z\mathbf{Y})$.

15.) Show that these equations (16) admit the (replica symmetric) solution $Q_{ab} = q\,\delta_{ab}$ with $q = q_\pm$ as obtained before (9).

16.) Using the ansatz $Q_{ab} = q_+\delta_{ab}$, assuming that this is the correct saddle-point solution to (16), compute the density $\rho(\lambda)$ in the large N limit and show that it gives back the Wigner semi-circle.

Some Gaussian integrals

We will find useful the following Gaussian integrals which hold for an invertible symmetric squared matrix $\mathbf{A}$ of size $m \times m$ (not necessarily real, provided the corresponding integrals are well defined) and a complex vector $\vec{b} \in \mathbb{C}^m$:

$$\int_{\mathbb{R}^m} \mathrm{d}\vec{x}\; \mathrm{e}^{-\frac{1}{2}\vec{x}^T\mathbf{A}\vec{x}} = \frac{(2\pi)^{m/2}}{\sqrt{\det\mathbf{A}}} \,, \tag{17}$$

$$\int_{\mathbb{R}^m} \mathrm{d}\vec{x}\; \mathrm{e}^{-\frac{1}{2}\vec{x}^T\mathbf{A}\vec{x}+\vec{b}^T\vec{x}} = \frac{(2\pi)^{m/2}}{\sqrt{\det\mathbf{A}}} \mathrm{e}^{\frac{1}{2}\vec{b}^T\mathbf{A}^{-1}\vec{b}} \tag{18}$$

$$\int_{\mathbb{R}^m} \mathrm{d}\vec{x}\; x_k x_l\; \mathrm{e}^{-\frac{1}{2}\vec{x}^T\mathbf{A}\vec{x}} = \frac{(2\pi)^{m/2}}{\sqrt{\det\mathbf{A}}} (\mathbf{A})^{-1}_{kl} \,. \tag{19}$$