

LECTURES ON STATISTICAL FIELD THEORY  
RANDOM MATRICES AND STATISTICAL PHYSICS – Solution of Homework 1

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## The Wigner semi-circular law via replicas

We recall here the definition of the average density of eigenvalues,

$$\rho(x) = \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N \delta(x - \lambda_i) \right], \quad (1)$$

and the Gaussian identities :

$$\int_{\mathbb{R}^m} d\vec{x} e^{-\frac{1}{2}\vec{x}^T \mathbf{A} \vec{x}} = \frac{(2\pi)^{m/2}}{\sqrt{\det \mathbf{A}}}, \quad (2)$$

$$\int_{\mathbb{R}^m} d\vec{x} e^{-\frac{1}{2}\vec{x}^T \mathbf{A} \vec{x} + \vec{b}^T \vec{x}} = \frac{(2\pi)^{m/2}}{\sqrt{\det \mathbf{A}}} e^{\frac{1}{2}\vec{b}^T \mathbf{A}^{-1} \vec{b}}, \quad (3)$$

$$\int_{\mathbb{R}^m} d\vec{x} x_k x_l e^{-\frac{1}{2}\vec{x}^T \mathbf{A} \vec{x}} = \frac{(2\pi)^{m/2}}{\sqrt{\det \mathbf{A}}} (\mathbf{A}^{-1})_{kl}. \quad (4)$$

### 1 Preamble

1.) Using the distributional identity recalled in the text we obtain

$$\frac{1}{N} \sum_{i=1}^N \delta(x - \lambda_i) = \frac{1}{\pi N} \lim_{\epsilon \rightarrow 0^+} \text{Im} \left( \sum_{i=1}^N \frac{1}{x - i\epsilon - \lambda_i} \right) = -\frac{1}{\pi N} \lim_{\epsilon \rightarrow 0^+} \text{Im} \left( \sum_{i=1}^N \frac{1}{\lambda_i - x_\epsilon} \right) \quad (5)$$

where  $x_\epsilon = x - i\epsilon$ . Therefore, using that

$$\sum_{i=1}^N \frac{1}{\lambda_i - x_\epsilon} = \text{Tr}(\mathbf{J} - x_\epsilon \mathbb{I}_N)^{-1}, \quad (6)$$

where  $\mathbb{I}_N$  denotes the identity matrix of size  $N$ , together with the definition of  $\rho(x)$  in (1) one obtains

$$\boxed{\rho(x) = -\frac{1}{N\pi} \lim_{\epsilon \rightarrow 0^+} \text{Im} \left( \mathbb{E} [\text{Tr}(\mathbf{J} - x_\epsilon \mathbb{I}_N)^{-1}] \right), \quad x_\epsilon = x - i\epsilon.} \quad (7)$$

2.) The  $N$ -dimensional Gaussian integral

$$Z_J(x_\epsilon) = \int_{-\infty}^{\infty} d\phi_1 \cdots \int_{-\infty}^{\infty} d\phi_N \exp \left[ -\frac{i}{2} x_\epsilon \sum_{k=1}^N \phi_k^2 + \frac{i}{2} \sum_{k,l} J_{kl} \phi_k \phi_l \right] \quad (8)$$

$$= \int_{\mathbb{R}^N} d\vec{\phi} \exp \left[ -\frac{1}{2} \vec{\phi}^T (i x_\epsilon \mathbb{I}_N - i \mathbf{J}) \vec{\phi} \right] \quad (9)$$

is convergent because the absolute value of the integrand is  $\exp[-\frac{\epsilon}{2} \sum_k \phi_k^2]$ , which is integrable for every  $\epsilon > 0$ . Taking the derivative of  $\ln Z_J(x_\epsilon)$  with respect to  $x$  one obtains

$$\frac{d \ln Z_J(x_\epsilon)}{dx} = \frac{\int_{\mathbb{R}^N} d\vec{\phi} \left( -\frac{i}{2} \sum_{k=1}^N \phi_k^2 \right) \exp \left[ -\frac{1}{2} \vec{\phi}^T (ix_\epsilon \mathbb{I}_N - i\mathbf{J}) \vec{\phi} \right]}{\int_{\mathbb{R}^N} d\vec{\phi} \exp \left[ -\frac{1}{2} \vec{\phi}^T (ix_\epsilon \mathbb{I}_N - i\mathbf{J}) \vec{\phi} \right]} \quad (10)$$

$$= -\frac{i}{2} \sum_{k=1}^N ((ix_\epsilon \mathbb{I}_N - i\mathbf{J})^{-1})_{kk} = \frac{1}{2} \sum_{k=1}^N ((\mathbf{J} - x_\epsilon \mathbb{I}_N)^{-1})_{kk} \quad (11)$$

where, in the second line, we have used the Gaussian identities (2) and (4). This finally leads to

$$\boxed{\text{Tr}(\mathbf{J} - x_\epsilon \mathbb{I})^{-1} = 2 \frac{d}{dx} \ln Z_J(x_\epsilon) .} \quad (12)$$

## 2 Annealed computation

3.) The replacement  $\mathbb{E}[\ln Z_J(x_\epsilon)] \approx \ln(\mathbb{E}[Z_J(x_\epsilon)])$  is the annealed approximation. In the more general context of disordered systems, this approximation is valid in the large  $N$  limit if  $Z_J(x_\epsilon)$  does not fluctuate too much, i.e. if it is self-averaging :  $Z_J(x_\epsilon) \rightarrow \mathbb{E}[Z_J(x_\epsilon)]$  as  $N \rightarrow \infty$ . This is the case when the disorder is weak, or the temperature high.

4.) The average ‘‘partition function’’  $\mathbb{E}[Z_J(x_\epsilon)]$  is given by

$$\mathbb{E}[Z_J(x_\epsilon)] = \int_{\mathbb{R}^N} d\vec{\phi} e^{-\frac{i}{2} x_\epsilon \sum_{k=1}^N \phi_k^2} \prod_{k=1}^N \mathbb{E} \left[ e^{\frac{i}{2} J_{kk} \phi_k^2} \right] \prod_{k < l} \mathbb{E} \left[ e^{i J_{kl} \phi_k \phi_l} \right] . \quad (13)$$

Note that in the last factors, the argument of the exponential is indeed  $i J_{kl} \phi_k \phi_l$  and not  $(i/2) J_{kl} \phi_k \phi_l$  since the product over  $k$  and  $l$  is restricted to  $k < l$ , while there is no restriction on these indices in Eq. (8). Using the Gaussian integration formulae (2,3) for  $m = 1$ , one finds

$$\mathbb{E} \left[ e^{\frac{i}{2} J_{kk} \phi_k^2} \right] = e^{-\frac{\phi_k^4}{8} \mathbb{E}[J_{kk}^2]} = e^{-\frac{\phi_k^4}{4N}} , \quad k = 1, \dots, N , \quad (14)$$

and similarly

$$\mathbb{E} \left[ e^{i J_{kl} \phi_k \phi_l} \right] = e^{-\frac{\phi_k^2 \phi_l^2}{2} \mathbb{E}[J_{kl}^2]} = e^{-\frac{\phi_k^2 \phi_l^2}{2N}} , \quad k < l . \quad (15)$$

Injecting these results (14) and (15) in Eq. (13) one finds

$$\mathbb{E}[Z_J(x_\epsilon)] = \int_{\mathbb{R}^N} d\vec{\phi} e^{-\frac{i}{2} x_\epsilon \sum_{k=1}^N \phi_k^2} e^{-\frac{1}{4N} \sum_{k=1}^N \phi_k^4 - \frac{1}{2N} \sum_{k < l} \phi_k^2 \phi_l^2} , \quad (16)$$

which finally yields

$$\boxed{\mathbb{E}[Z_J(x_\epsilon)] = \int_{-\infty}^{\infty} d\phi_1 \cdots \int_{-\infty}^{\infty} d\phi_N \exp \left[ -\frac{i}{2} x_\epsilon \sum_{k=1}^N \phi_k^2 - \frac{1}{4N} \left( \sum_{k=1}^N \phi_k^2 \right)^2 \right] .} \quad (17)$$

5.) We use the identity (3) with  $m = 1$  to write

$$e^{-\frac{1}{4N} (\sum_{k=1}^N \phi_k^2)^2} = \sqrt{\frac{N}{\pi}} \int_{-\infty}^{\infty} dq e^{-Nq^2 + iq \sum_{k=1}^N \phi_k^2} . \quad (18)$$

Hence  $\mathbb{E}[Z_J(x_\epsilon)]$  can be written as

$$\begin{aligned} \mathbb{E}[Z_J(x_\epsilon)] &= \sqrt{\frac{N}{\pi}} \int_{-\infty}^{\infty} dq e^{-Nq^2} \int_{\mathbb{R}^N} d\vec{\phi} e^{i(q - \frac{x_\epsilon}{2}) \sum_{k=1}^N \phi_k^2} \\ &= \sqrt{\frac{N}{\pi}} \int_{-\infty}^{\infty} dq e^{-Nq^2} \left[ \int_{-\infty}^{\infty} d\phi e^{i(q - \frac{x_\epsilon}{2}) \phi^2} \right]^N . \end{aligned} \quad (19)$$

The integral over  $\phi$  can be evaluated using the relation (2) which, specified for  $m = 1$ , yields here

$$\int_{-\infty}^{\infty} d\phi e^{i(q - \frac{x_\epsilon}{2})\phi^2} = \left[ \frac{i\pi}{(q - \frac{x_\epsilon}{2})} \right]^{1/2}. \quad (20)$$

Therefore, by substituting (20) in (19) one finds that  $\mathbb{E}[Z_J(x_\epsilon)]$  can be written as

$$\mathbb{E}[Z_J(x_\epsilon)] = \sqrt{\frac{N}{\pi}} \int_{-\infty}^{\infty} dq e^{-N\varphi(q, x_\epsilon)}, \quad \varphi(q, x_\epsilon) = q^2 - \frac{1}{2} \ln \left( \frac{2\pi}{i(x_\epsilon - 2q)} \right). \quad (21)$$

6.) By substituting the identity found in (12) in the expression for the density  $\rho(x)$  in Eq. (7) one finds

$$\rho(x) = -\frac{2}{N\pi} \lim_{\epsilon \rightarrow 0^+} \text{Im} \frac{d}{dx} \mathbb{E}[\ln(Z_J(x_\epsilon))] \approx -\frac{2}{N\pi} \lim_{\epsilon \rightarrow 0^+} \text{Im} \frac{d}{dx} \ln(\mathbb{E}[Z_J(x_\epsilon)]), \quad (22)$$

where the last relation holds in the annealed approximation considered here. We see that we need to evaluate  $N^{-1} \ln(\mathbb{E}[Z_J(x_\epsilon)])$  in the limit of large  $N$  and therefore, anticipating that  $\rho(x)$  is of order  $\mathcal{O}(1)$ , we only need to retain the exponentially large terms in (21).

7.) The saddle point equation  $\partial_q \varphi(q, x_\epsilon) = 0$  yields the quadratic equation

$$2q + \frac{1}{2q - x_\epsilon} = 0, \quad 4q^2 - 2x_\epsilon q + 1 = 0, \quad (23)$$

which admits the two solutions

$$q_{\pm} = \frac{1}{4} \left( x_\epsilon \pm \sqrt{x_\epsilon^2 - 4} \right). \quad (24)$$

8.) In the limit of large  $N$  the Laplace method applied to (21) yields

$$\lim_{N \rightarrow \infty} \frac{\ln \mathbb{E}[Z_J(x_\epsilon)]}{N} = -\varphi(q_+, x_\epsilon). \quad (25)$$

Therefore, using this result (25) in Eq. (22) one finds

$$\lim_{N \rightarrow \infty} \rho(x) = \frac{2}{\pi} \lim_{\epsilon \rightarrow 0^+} \text{Im} \frac{d}{dx} \varphi(q_+, x_\epsilon). \quad (26)$$

9.) To compute  $\frac{d}{dx} \varphi(q_+, x_\epsilon)$  we notice that  $\varphi(q_+, x_\epsilon)$  depends on  $x$  through the dependence of  $q_+$  on  $x$  (24) as well as through the  $x$ -dependence of the function  $\varphi(q, x_\epsilon)$  itself (21). Therefore differentiating using the chain rule one finds

$$\frac{d}{dx} \varphi(q_+, x_\epsilon) = \frac{\partial q_+}{\partial x} \frac{\partial \varphi(q, x_\epsilon)}{\partial q} \Big|_{q=q_+} + \frac{\partial \varphi(q, x_\epsilon)}{\partial x} \Big|_{q=q_+}. \quad (27)$$

The first term vanishes since  $q_+$  is a saddle point, solution of  $\partial_q \varphi = 0$ , and the second term yields

$$\frac{d}{dx} \varphi(q_+, x_\epsilon) = \frac{\partial \varphi(q, x_\epsilon)}{\partial x} \Big|_{q=q_+} = \frac{1}{2(x_\epsilon - 2q_+)} = q_+ = \frac{1}{4} \left( x_\epsilon + \sqrt{x_\epsilon^2 - 4} \right), \quad (28)$$

where in the last two equalities we have used (23) and (24). Finally, injecting this result (28) in Eq. (26) one finds

$$\lim_{N \rightarrow \infty} \rho(x) = \frac{1}{2\pi} \lim_{\epsilon \rightarrow 0^+} \text{Im} \left[ x_\epsilon + \sqrt{x_\epsilon^2 - 4} \right], \quad (29)$$

which leads to the Wigner semi-circle law

$$\rho(x) \xrightarrow{N \rightarrow \infty} \begin{cases} \frac{1}{2\pi} \sqrt{4 - x^2}, & |x| < 2 \\ 0, & |x| > 2. \end{cases} \quad (30)$$

### 3 Quenched computation

10.) By using the replica trick

$$\mathbb{E}[\ln Z_J(x_\epsilon)] = \lim_{n \rightarrow 0} \frac{1}{n} \ln (\mathbb{E}[Z_J(x_\epsilon)^n]) . \quad (31)$$

together with the first equality in (22) one obtains

$$\rho(x) = -\frac{2}{N\pi} \lim_{\epsilon \rightarrow 0^+} \lim_{n \rightarrow 0} \operatorname{Im} \frac{d}{dx} \frac{1}{n} \ln (\mathbb{E}[Z_J(x_\epsilon)^n]) . \quad (32)$$

11.) By replicating the system  $n$  times to write the  $n$ -th power  $Z_J(x_\epsilon)^n$  and performing the averages over the random variables  $J_{kl}$ , one has

$$\mathbb{E}[Z_J(x_\epsilon)^n] = \int_{-\infty}^{\infty} \prod_{k=1}^N \prod_{a=1}^n d\phi_k^a e^{-\frac{ix_\epsilon}{2} \sum_{k=1}^N \sum_{a=1}^n (\phi_k^a)^2} \mathbb{E}[e^{\frac{i}{2} \sum_{k,l} J_{kl} \sum_{a=1}^n \phi_k^a \phi_l^a}] \quad (33)$$

$$= \int_{-\infty}^{\infty} \prod_{k=1}^N \prod_{a=1}^n d\phi_k^a e^{-\frac{ix_\epsilon}{2} \sum_{k=1}^N \sum_{a=1}^n (\phi_k^a)^2} \prod_{k=1}^N \mathbb{E}[e^{\frac{i}{2} J_{kk} \sum_{a=1}^n (\phi_k^a)^2}] \prod_{k < l} \mathbb{E}[e^{i J_{kl} \sum_{a=1}^n \phi_k^a \phi_l^a}]$$

$$= \int_{-\infty}^{\infty} \prod_{k=1}^N \prod_{a=1}^n d\phi_k^a e^{-\frac{ix_\epsilon}{2} \sum_{k=1}^N \sum_{a=1}^n (\phi_k^a)^2} e^{-\frac{1}{4N} \sum_{k=1}^N \sum_{a,b=1}^n (\phi_k^a)^2 (\phi_k^b)^2} \quad (34)$$

$$\times e^{-\frac{1}{2N} \sum_{k < l} \sum_{a,b=1}^n \phi_k^a \phi_l^a \phi_k^b \phi_l^b} \quad (35)$$

which can be written (by inverting the sum over  $k$  and the double sum over  $a, b$  in the argument of the last two exponentials) as

$$\mathbb{E}[Z_J(x_\epsilon)^n] = \int_{-\infty}^{\infty} \prod_{k=1}^N \prod_{a=1}^n d\phi_k^a \exp \left[ -\frac{i}{2} x_\epsilon \sum_{k=1}^N \sum_{a=1}^n (\phi_k^a)^2 - \frac{1}{4N} \sum_{a,b=1}^n \left( \sum_{k=1}^N \phi_k^a \phi_k^b \right)^2 \right] . \quad (36)$$

12.) We write the exponentials containing quartic terms in Eq. (36) as

$$\exp \left[ -\frac{1}{4N} \sum_{a,b=1}^n \left( \sum_{k=1}^N \phi_k^a \phi_k^b \right)^2 \right] = \exp \left[ -\frac{1}{4N} \sum_{a=1}^n \left( \sum_{k=1}^N (\phi_k^a)^2 \right)^2 - \frac{1}{2N} \sum_{1 \leq a < b \leq n} \left( \sum_{k=1}^N \phi_k^a \phi_k^b \right)^2 \right] \quad (37)$$

and we use  $n(n+1)/2$  times the Gaussian identity (3) to rewrite it as

$$\begin{aligned} \exp \left[ -\frac{1}{4N} \sum_{a,b=1}^n \left( \sum_{k=1}^N \phi_k^a \phi_k^b \right)^2 \right] &= \prod_{a=1}^n \left( \sqrt{\frac{N}{\pi}} \int_{-\infty}^{\infty} dQ_{aa} e^{-NQ_{aa}^2 + iQ_{aa} \sum_{k=1}^N (\phi_k^a)^2} \right) \\ &\times \prod_{1 \leq a < b \leq n} \left( \sqrt{\frac{2N}{\pi}} \int_{-\infty}^{\infty} dQ_{ab} e^{-2NQ_{ab}^2 + 2iQ_{ab} \sum_{k=1}^N \phi_k^a \phi_k^b} \right) \end{aligned} \quad (38)$$

which can be re-written as

$$\begin{aligned} \exp \left[ -\frac{1}{4N} \sum_{a,b=1}^n \left( \sum_{k=1}^N \phi_k^a \phi_k^b \right)^2 \right] &= 2^{n(n-1)/4} \left( \frac{N}{\pi} \right)^{n(n+1)/4} \\ &\times \int_{-\infty}^{\infty} \prod_{1 \leq a < b \leq n} dQ_{ab} \exp \left[ -N \sum_{a,b=1}^n (Q_{ab})^2 + i \sum_{a,b=1}^n Q_{ab} \sum_{k=1}^N \phi_k^a \phi_k^b \right] , \end{aligned} \quad (39)$$

with the convention, in the argument of the exponential, that  $Q_{ab} = Q_{ba}$  if  $a > b$ . Therefore, substituting this identity (39) in (36) one obtains

$$\begin{aligned} \mathbb{E}[Z_J(x_\epsilon)^n] &= A_{N,n} \int_{-\infty}^{\infty} \prod_{k=1}^N \prod_{a=1}^n d\phi_k^a \int_{-\infty}^{\infty} \prod_{1 \leq a \leq b \leq n} dQ_{ab} \\ &\times \exp \left[ -N \sum_{a,b=1}^n (Q_{ab})^2 - \frac{i}{2} x_\epsilon \sum_{k=1}^N \sum_{a=1}^n (\phi_k^a)^2 + i \sum_{a,b=1}^n Q_{ab} \sum_{k=1}^N \phi_k^a \phi_k^b \right] \end{aligned} \quad (40)$$

where  $A_{N,n} = 2^{n(n-1)/4} \left(\frac{N}{\pi}\right)^{n(n+1)/4}$ .

13.) We observe that the  $n$ -dimensional vectors  $\{\phi_k^1, \phi_k^2, \dots, \phi_k^n\}$ , for different values of  $k = 1, \dots, N$ , are now independent in (40). Therefore one has

$$\mathbb{E}[Z_J(x_\epsilon)^n] = A_{N,n} \int_{-\infty}^{\infty} \prod_{1 \leq a \leq b \leq n} dQ_{ab} e^{-N \sum_{a,b=1}^n (Q_{ab})^2} \left( \int_{-\infty}^{\infty} \prod_{a=1}^n d\phi^a e^{-\frac{1}{2} \sum_{a,b=1}^n \phi^a (ix_\epsilon \delta_{ab} - 2iQ_{ab}) \phi^b} \right)^N. \quad (41)$$

The multiple integral over the  $\phi^a$ 's can be explicitly evaluated using (2) with  $m = n$ , which yields

$$\mathbb{E}[Z_J(x_\epsilon)^n] = A_{N,n} \prod_{1 \leq a \leq b=1}^n \int_{-\infty}^{\infty} dQ_{ab} \exp[-N\Phi(\mathbf{Q}, x_\epsilon)] \quad (42)$$

where  $\mathbf{Q}$  is the (symmetric) overlap matrix, of size  $n \times n$ , with matrix elements  $Q_{ab}$  for  $a \leq b$  (and  $Q_{ab} = Q_{ba}$  for  $a > b$ ), and

$$\Phi(\mathbf{Q}, x_\epsilon) = \text{Tr}[\mathbf{Q}^2] - \frac{1}{2} \ln \left[ \det \left( \frac{2\pi}{i} (x_\epsilon \mathbb{I}_n - 2\mathbf{Q})^{-1} \right) \right], \quad (43)$$

where  $\mathbb{I}_n$  denotes the identity matrix matrix of size  $n$ .

14.) Expliciting the independent parameters in  $\mathbf{Q}$  we rewrite  $\Phi$  as

$$\Phi(\mathbf{Q}, x_\epsilon) = \sum_{a=1}^n Q_{aa}^2 + 2 \sum_{a < b} Q_{ab}^2 - \ln \left( \int_{-\infty}^{\infty} \prod_{a=1}^n d\phi^a e^{-\frac{1}{2} x_\epsilon \sum_a (\phi^a)^2 + i \sum_a Q_{aa} (\phi^a)^2 + 2i \sum_{a < b} Q_{ab} \phi^a \phi^b} \right). \quad (44)$$

Taking the derivative with respect to one diagonal element  $Q_{cc}$  or one off-diagonal element  $Q_{cd}$  with  $c < d$  yields

$$\frac{\partial \Phi}{\partial Q_{cc}} = 2Q_{cc} - i \frac{\int_{-\infty}^{\infty} \prod_{a=1}^n d\phi^a (\phi^c)^2 e^{-\frac{1}{2} \sum_{a,b=1}^n \phi^a (ix_\epsilon \delta_{ab} - 2iQ_{ab}) \phi^b}}{\int_{-\infty}^{\infty} \prod_{a=1}^n d\phi^a e^{-\frac{1}{2} \sum_{a,b=1}^n \phi^a (ix_\epsilon \delta_{ab} - 2iQ_{ab}) \phi^b}} = 2Q_{cc} - i((ix_\epsilon \mathbb{I}_n - 2i\mathbf{Q})^{-1})_{cc}, \quad (45)$$

$$\frac{\partial \Phi}{\partial Q_{cd}} = 4Q_{cd} - 2i \frac{\int_{-\infty}^{\infty} \prod_{a=1}^n d\phi^a \phi^c \phi^d e^{-\frac{1}{2} \sum_{a,b=1}^n \phi^a (ix_\epsilon \delta_{ab} - 2iQ_{ab}) \phi^b}}{\int_{-\infty}^{\infty} \prod_{a=1}^n d\phi^a e^{-\frac{1}{2} \sum_{a,b=1}^n \phi^a (ix_\epsilon \delta_{ab} - 2iQ_{ab}) \phi^b}} = 4Q_{cd} - 2i((ix_\epsilon \mathbb{I}_n - 2i\mathbf{Q})^{-1})_{cd}, \quad (46)$$

where we used the Gaussian integrals (2,4). Simplifying the factors  $i$  gives the saddle-point equation of the text,

$$2Q_{ab} - ((x_\epsilon \mathbb{I}_n - 2\mathbf{Q})^{-1})_{ab} = 0, \quad \forall a, b = 1, \dots, n. \quad (47)$$

Alternatively one can use the identity  $\partial_z(\det \mathbf{Y}) = (\det \mathbf{Y}) \text{Tr}(\mathbf{Y}^{-1} \partial_z \mathbf{Y})$ , hence  $\partial_z \ln(\det \mathbf{Y}) = \text{Tr}(\mathbf{Y}^{-1} \partial_z \mathbf{Y})$  applied to  $\mathbf{Y} = x_\epsilon \mathbb{I}_n - 2\mathbf{Q}$  and  $z = Q_{ab}$ ; one finds

$$\frac{\partial \ln \det(x_\epsilon \mathbb{I}_n - 2\mathbf{Q})}{\partial Q_{ab}} = \sum_{c,d=1}^n ((x_\epsilon \mathbb{I}_n - 2\mathbf{Q})^{-1})_{cd} \frac{\partial (x_\epsilon \delta_{cd} - 2Q_{cd})}{\partial Q_{ab}} = -2((x_\epsilon \mathbb{I}_n - 2\mathbf{Q})^{-1})_{ab}, \quad (48)$$

which yields the same saddle-point equations (47) when applied to the expression (43) of  $\Phi$ .

15.) The replica symmetric ansatz  $Q_{ab} = q \delta_{ab}$  corresponds to  $\mathbf{Q} = q \mathbb{I}_n$ ; the saddle point equations (47) are then equivalent to

$$2q - \frac{1}{x_\epsilon - 2q} = 0, \quad (49)$$

which is precisely the equation found in the annealed approximation in (23). It admits the two solutions  $q = q_{\pm}$  as given in (24).

16.) Assuming that  $Q_{ab} = q_+ \delta_{ab}$  is the correct saddle point solution, one finds, from Eq. (42) that

$$\lim_{N \rightarrow \infty} \frac{\ln(\mathbb{E}[Z(x_\epsilon)^n])}{N} = -\Phi(\mathbf{Q} = q_+ \mathbb{I}_n, x_\epsilon) = -n\varphi(q_+, x_\epsilon), \quad (50)$$

where we have used the explicit expression of  $\Phi(\mathbf{Q}, x_\epsilon)$  in (43) together with the expression of the function  $\varphi(q, x_\epsilon)$  found before in the annealed approximation and given in (21). Substituting this result (50) in Eq. (32) one obtains (assuming that the limit  $N \rightarrow \infty$  and  $n \rightarrow 0$  can be exchanged)

$$\lim_{N \rightarrow \infty} \rho(x) = \frac{2}{\pi} \lim_{\epsilon \rightarrow 0^+} \text{Im} \frac{d}{dx} \varphi(q_+, x_\epsilon), \quad (51)$$

which coincides with the result obtained above in the annealed approximation (26) and therefore this quenched computation also yields the Wigner semi-circle (30) for the average density  $\rho(x)$  in the limit of large  $N$ .