Lecures on Statistical Field Theory
Random matrices and statistical physics – Homework 2

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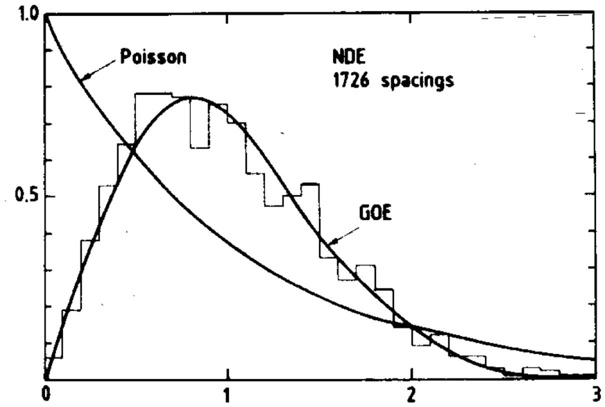

FIGURE 1 – Nearest neighbor spacing distribution for the "Nuclear Data Ensemble" comprising 1726 spacings (histogram) versus $s = S/D$ with D the mean level spacing and S the actual spacing. For comparison, the Wigner surmise labelled GOE is shown and compared to a Poisson distribution.

The Wigner sumise

Consider a two by two symmetric real random matrix M such that the matrix elements M_{11}, M_{12} and M_{22} are independent Gaussian random variables with zero mean and variances :

$$\mathbb{E}[M_{11}^2] = 1 \ , \qquad \mathbb{E}[M_{22}^2] = 1 \ , \qquad \mathbb{E}[M_{12}^2] = \frac{1}{2} \ ;$$

by symmetry $M_{21} = M_{12}$. We denote λ_1 and λ_2 the eigenvalues of M, and $\Delta = |\lambda_1 - \lambda_2|$ their spacing.

Find the probability density of Δ, its average value $\mathbb{E}[\Delta]$, and deduce that the normalized spacing $s = \Delta/\mathbb{E}[\Delta]$ has the probability density

$$P(s) = \frac{\pi}{2} s \, e^{-\frac{\pi}{4} s^2} \ ,$$

known as the Wigner surmise.

It is what Wigner proposed as an approximation for the probability density function of the normalised mean-level spacing of very complex nuclei, see Figure 1.

The Wigner semi-circular law via Coulomb gas

For the β-Gaussian ensemble of $N \times N$ random matrices, we have shown in the lectures, using the Coulomb gas approach, that the limiting equilibrium measure (in the limit $N \to \infty$) ρ is solution of the following Cauchy singular integral equation

$$\lambda = \fint_{-a}^{a} \frac{\rho(\lambda')}{\lambda - \lambda'} \, d\lambda' \ , \tag{1}$$

together with the normalization condition $\int_{-a}^{a} \rho(\lambda)\, d\lambda = 1$, while the real a is to be determined. In Eq. (1), $⨍$ denotes the principal value of the integral. This equation (1) belongs to the general class of Cauchy singular integral equations of the form

$$g(\lambda) = ⨍_{a_1}^{a_2} \frac{\rho(\lambda')}{\lambda - \lambda'}\, d\lambda' \ . \tag{2}$$

Fortunately, such singular integral equations can be explicitly inverted (assuming that ρ has a single compact support) using a formula due to Tricomi [see e.g. F. G. Tricomi, *Integral equations*, Dover publications (1985)] that reads

$$\rho(\lambda) = \frac{1}{\pi\sqrt{(a_2 - \lambda)(\lambda - a_1)}} \left[C_0 - ⨍_{a_1}^{a_2} \frac{dt}{\pi} \frac{\sqrt{(a_2 - t)(t - a_1)}}{\lambda - t}\, g(t) \right] \ , \tag{3}$$

where $C_0 = \int_{a_1}^{a_2} \rho(\lambda) d\lambda$ is a constant. In our case, the source function $g(\lambda) = \lambda$ and the constant $C_0 = 1$ due to the normalization $\int_{a_1}^{a_2} \rho_w(\lambda) d\lambda = 1$.

Apply this result (3) to show that the solution to Eq. (1) is given by the Wigner semi-circular law

$$\rho(\lambda) = \frac{1}{\pi}\sqrt{2 - \lambda^2} \ \ , \ \ \lambda \in [-\sqrt{2}, +\sqrt{2}] \ . \tag{4}$$