Lectures on Statistical Field Theory
Random matrices and statistical physics – Solution of Homework 1

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The Wigner sumise

The two eigenvalues of M are the solutions of its characteristic equation

$$\lambda^2 - (M_{11} + M_{22})\lambda + (M_{11}M_{22} - M_{12}^2) = 0 \ , \tag{1}$$

which read

$$\frac{M_{11} + M_{22}}{2} \pm \frac{1}{2}\sqrt{(M_{11} + M_{22})^2 - 4(M_{11}M_{22} - M_{12}^2)} \ , \tag{2}$$

hence

$$\Delta = \sqrt{(M_{11} - M_{22})^2 + 4M_{12}^2} \ . \tag{3}$$

Denoting $X = M_{11} - M_{22}$ and $Y = 2M_{12}$, we realize that X and Y are two independent Gaussian random variables, both of variance 2, and that $\Delta = \sqrt{X^2 + Y^2}$ can be seen as the distance from the origin of a point drawn in the plane with this distribution. Hence the density of Δ is

$$\widehat{P}(\Delta) = \int_{\mathbb{R}^2} dxdy \frac{1}{4\pi} e^{-\frac{x^2+y^2}{4}} \delta(\Delta - \sqrt{x^2+y^2}) = \frac{1}{2}\int_0^\infty dr\, r\, e^{-\frac{r^2}{4}} \delta(\Delta - r) = \frac{1}{2}\Delta e^{-\frac{\Delta^2}{4}} \ , \tag{4}$$

after a change of variable towards polar coordinates. The average value of Δ is thus

$$\mathbb{E}[\Delta] = \int_0^\infty \mathrm{d}\Delta \frac{1}{2}\Delta e^{-\frac{\Delta^2}{4}} \Delta = \sqrt{\pi} \ . \tag{5}$$

Changing variables from Δ to $s = \Delta/\mathbb{E}[\Delta]$ yields the probability density

$$P(s) = \widehat{P}(\Delta = s\sqrt{\pi})\sqrt{\pi} = \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2} \ .$$

The Wigner semi-circular law via Coulomb gas

By taking $-a_1 = a_2 = a$, $g(t) = t$ and $C_0 = 1$ in Eq. (3) of the Homework sheet, we know the solution of Eq. (1) is given by

$$\rho(\lambda) = \frac{1}{\pi\sqrt{a^2 - \lambda^2}}[1 - \int_{-a}^{a} \frac{dt}{\pi} \frac{t\sqrt{a^2 - t^2}}{\lambda - t}] = \frac{1}{\pi\sqrt{a^2 - \lambda^2}}[1 - Hf(\lambda)],$$

where H is the Hilbert transform and $f(t) = t\sqrt{a^2 - t^2}\mathbb{1}_{[-a,a]}$.

Now by Plemelj's formula we know for any $x \in \mathbb{R}$

$$s(x)_\pm = \pi i(\pm f(x) + iHf(x)),$$

where

$$s(z) = \int_{-a}^{a} \frac{t\sqrt{a^2 - t^2}}{t - z} dt$$

is the Stieltjes transform of f for $z \in \mathbb{C}\backslash[-a, a]$. The stieltjes transform $s(z)$ can be computed as follows :

fix $z \in \mathbb{C}\backslash[-a,a]$, let C be any path in the complex plane $\mathbb{C}$ containing the interval $[-a,a]$ and not containing z, with counterclockwise orientation. Then we have

$$\int_{-a}^{a} \frac{t\sqrt{a^2-t^2}}{t-z}\, dt = -2\oint_C \frac{w\sqrt{a-w^2}}{w-z}\, dw,$$

where $\sqrt{a^2-w^2}$ is chosen such that $\lim_{w\in\mathbb{C}_+\to x} \sqrt{a^2-w^2} = \sqrt{a^2-x^2}$ for $x \in [-a,a]$.

Now deform C to C' which also contains z and by residue theorem we have

$$\begin{aligned}\oint_C \frac{w\sqrt{a^2-w^2}}{w-z}\, dw &= \oint_{C'} \frac{w\sqrt{a^2-w^2}}{w-z}\, dw + 2\pi i Res(\frac{w\sqrt{a^2-w^2}}{w-z}, z)\\ &= 2\pi i Res(\frac{w\sqrt{a^2-w^2}}{w-z}, z) + 2\pi i Res(\frac{w\sqrt{4-w^2}}{w-z}, \infty)\end{aligned}$$

Now it is clear that $Res(\frac{w\sqrt{a^2-w^2}}{w-z}, z) = z\sqrt{a^2-z^2}$.

By the laurent series expansion of $\frac{w\sqrt{a^2-w^2}}{w-z}$ at ∞ :

$$\frac{w\sqrt{a^2-w^2}}{w-z} = i[w + z + (z^2 - \frac{a^2}{2})\frac{1}{w} + ...),$$

we know $Res(\frac{w\sqrt{4-w^2}}{w-z}, \infty) = i(z^2 - \frac{a^2}{2})$.

Hence

$$s(z) = -\pi i(z\sqrt{a^2-z^2} + i(z^2 - \frac{a^2}{2}))$$

and for $x \in [-a,a]$

$$Hf(x) = -\frac{a^2}{2} + x^2.$$

Thus

$$\rho(\lambda) = \frac{1}{\pi\sqrt{a^2-\lambda^2}}[1 + \frac{a^2}{2} - \lambda^2]\mathbb{1}_{[-a,a]}(\lambda).$$

Since ρ is a probability measure, we must have $(1 + \frac{a^2}{2} - x^2) \geq 0$ for all $x \in [-a,a]$. This implies that $1 - \frac{a^2}{2} \geq 0$. Moreover, if we assume that $\rho(\lambda)$ is continuous in $\mathbb{R}$, then we must have $\rho(a) = 0$ which implies that $1 - \frac{a^2}{2} \geq 0$. Hence we conclude that $a = \sqrt{2}$ and $\rho(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}\mathbb{1}_{[-\sqrt{2},\sqrt{2}]}(\lambda)$ is the Wigner semi-circle.