Lectures on Statistical Field Theory
Random matrices and statistical physics – Homework 3

Grégory Schehr

Cauchy-Binet formula

Let $\Lambda \subset \mathbb{R}$ and let ϕ_i, ψ_j, $1 \le i, j \le N$ as well as w be integrable functions on Λ such that $\phi_i \psi_j w$ is also integrable on Λ for any i, j. Prove the Cauchy-Binet identity :

$$\int_{\Lambda^N} \det_{1\le i,j \le N}[\phi_i(x_j)] \det_{1\le i,j \le N}[\psi_i(x_j)] \prod_{i=1}^N w(x_i) dx_i = N! \det_{1\le i,j\le N} \left(\int_\Lambda \phi_i(x) \psi_j(x) \, w(x) \, dx \right) . \quad (1)$$

<u>Hint :</u> Use the Leibniz expansion of the determinant of a square matrix.

Derivation of the Airy kernel

Consider the eigenvalues of random matrices belonging to the Gaussian Unitary Ensemble (GUE). Their joint PDF is given by

$$P_{\rm joint}(\lambda_1, \lambda_2, \dots, \lambda_N) = \frac{1}{Z_N} \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j)^2 e^{-N \sum_{i=1}^N \lambda_i^2} , \quad (2)$$

where Z_N is a normalization constant. They form a determinantal process with a Christoffel-Darboux kernel $K_N(x,y)$ given by

$$K_N(x,y) = \sqrt{\frac{N}{\pi}} \frac{1}{2^N (N-1)!} e^{-N \frac{(x^2+y^2)}{2}} \frac{H_N(x\sqrt{N}) H_{N-1}(y\sqrt{N}) - H_N(y\sqrt{N}) H_{N-1}(x\sqrt{N})}{\sqrt{N}(x-y)} \quad (3)$$

where $H_k(x)$ denotes the Hermite polynomial of degree k. They satisfy the following orthogonality condition (note the choice of the physicists' convention)

$$\int_{-\infty}^{\infty} H_k(x) H_{k'}(x) e^{-x^2} = \delta_{k,k'} h_k \ , \ h_k = 2^k k! \sqrt{\pi} \ . \quad (4)$$

Using the following asymptotic expansion (known as Plancherel-Rotach formula)

$$\exp(-x^2/2) H_{N+m}(x) = (2N)^{m/2} \pi^{1/4} 2^{N/2+1/4} (N!)^{1/2} N^{-1/12} \left({\rm Ai}(t) - \frac{m}{N^{1/3}} {\rm Ai}'(t) \right) + \mathcal{O}(N^{-2/3})) \quad (5)$$

where $x = (2N)^{1/2} + 2^{-1/2} N^{-1/6} t$, with ${\rm Ai}(x)$ denoting the Airy function, show that

$$\lim_{N\to\infty} \frac{1}{\sqrt{2} N^{2/3}} K_N \left(\sqrt{2} + \frac{u}{\sqrt{2} N^{2/3}}, \sqrt{2} + \frac{v}{\sqrt{2} N^{2/3}} \right) = K_{\rm Ai}(u,v) \quad (6)$$

where $K_{\rm Ai}(u,v)$ is the Airy kernel, given by

$$K_{\rm Ai}(u,v) = \frac{{\rm Ai}(u){\rm Ai}'(v) - {\rm Ai}'(u){\rm Ai}(v)}{u-v} \ . \quad (7)$$

Free fermions in presence of a hard wall potential and random matrices

Consider N non-interacting fermions, confined on the *positive* real axis $\mathbb{R}^+$ by a *quantum potential* of the form

$$V(x) = \frac{1}{2} b^2 x^2 + \frac{\alpha(\alpha - 1)}{2x^2} \;,\; b > 0 \;\; \& \;\; \alpha > 1 \;. \tag{8}$$

3.1) Show that the solution of the Schrödinger equation for this type of "hard wall" potential (8) with the boundary conditions $\varphi_k(0) = 0$ (due to the hard wall at $x = 0$) and $\lim_{x\to\infty} \varphi_k(x) = 0$ are of the form

$$\varphi_k(x) = c_k e^{-\frac{bx^2}{2}} x^\alpha \mathcal{L}_k^{(\alpha-1/2)}(b\,x^2) \;,\; \text{with } E_k = b\,(2k + \alpha + 1/2) \;, \tag{9}$$

with k a non-negative integer and where $\mathcal{L}_k^{(\alpha-1/2)}$ is a generalized Laguerre polynomial of degree k and index $\alpha - 1/2$ and c_k some constant. We recall that $\mathcal{L}_k^\gamma$ can be written as

$$\mathcal{L}_k^\gamma(x) = \sum_{i=0}^{k} \binom{k+\gamma}{k-i} \frac{(-x)^i}{i!} \;. \tag{10}$$

3.2) Deduce that the joint PDF of the N non-interacting fermions (in the ground state) reads

$$P_{\text{joint}}(x_1, x_2, \cdots, x_N) = \frac{1}{Z_N} \prod_{i=1}^{N} x_i^{2\alpha} \prod_{1\le i<j\le N} |x_i^2 - x_j^2|^2 e^{-b\sum_{i=1}^N x_i^2} \;. \tag{11}$$

3.3) Besides the GUE studied in a lecture, a well known ensemble is the so-called Laguerre-Wishart ensemble for which the eigenvalues are distributed according to

$$P_{\text{joint}}(\lambda_1, \cdots, \lambda_N) = \frac{1}{\tilde{Z}_N} \prod_{i<j} (\lambda_i - \lambda_j)^2 \left(\prod_{i=1}^{N} \lambda_i^a \right) e^{-\sum_{i=1}^N \lambda_i} \;, \quad \text{for} \quad \lambda_i \ge 0 \quad \forall i = 1, \cdots, N \;. \tag{12}$$

Conclude form (11) that the variables $y_i = x_i^2$ are distributed like the eigenvalues of random matrices belonging to the Laguerre-Wishart ensemble. What is the corresponding parameter a?