Lectures on Statistical Field Theory
Random matrices and statistical physics – Solution of Homework 3

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Cauchy-Binet formula

Let $\Lambda \subset \mathbb{R}$ and let ϕ_i, ψ_j, $1 \leq i, j \leq N$ as well as w be integrable functions on Λ such that $\phi_i \psi_j w$ is also integrable on Λ for any i, j. Below we prove the Cauchy-Binet identity :

$$\int_{\Lambda^N} \det_{1\leq i,j\leq N}[\phi_i(x_j)] \det_{1\leq i,j\leq N}[\psi_i(x_j)] \prod_{i=1}^N w(x_i)dx_i = N! \det_{1\leq i,j\leq N} \left(\int_\Lambda \phi_i(x)\psi_j(x)\, w(x)\, dx \right) . \tag{1}$$

By the Leibniz formula and Fubini's theorem,

$$\begin{aligned} D_1 := \det_{i,j=1}^N \left(\int_\Lambda \phi_i(y)\phi_j(y)w(y)dy \right) &= \sum_{\pi \in S_N} \mathrm{sgn}(\pi) \prod_{i=1}^N \int_\Lambda \phi_i(y)\psi_{\pi(i)}(y)w(y)dy \\ &= \sum_{\pi\in S_N} \mathrm{sgn}(\pi) \int_{\Lambda^N} \prod_{i=1}^N \phi_i(x_i)\psi_{\pi(i)}(x_i) \prod_{i=1}^N w(x_i)dx_i \end{aligned}$$

and similarly

$$D_2 := \int_{\Lambda^N} \det_{i,j=1}^N[\phi_i(x_j)] \det_{i,j=1}^N[\psi_i(x_j)]\mathrm{d}\lambda^N(x) = \sum_{\sigma,\rho\in S_N} \mathrm{sgn}(\sigma)\mathrm{sgn}(\rho) \int_{\Lambda^N} \prod_{i=1}^N \phi_i(x_{\sigma(i)})\psi_i(x_{\rho(i)})\mathrm{d}\lambda^N(x)$$

(S_N denotes the symmetric group of order N, and sgn the signature, or parity of a permutation). We show that $N!D_1 = D_2$. Note that, since the N-fold product measure of $w(x_1)\dots w(x_n)dx_1 \dots dx_N$ is invariant with respect to any permutation of the components x_i, we can write

$$\begin{aligned} N!D_1 &= \sum_{\tau,\pi\in S_N} \mathrm{sgn}(\pi) \int_{\Lambda^N} \prod_{i=1}^N \phi_i(x_{\tau(i)})\psi_{\pi(i)}(x_{\tau(i)}) \prod_{i=1}^N w(x_i)dx_i \\ &= \sum_{\tau,\pi\in S_N} \mathrm{sgn}(\pi) \int_{\Lambda^N} \prod_{i=1}^N \phi_i(x_{\tau(i)})\psi_i(x_{\tau\circ\pi^{-1}(i)}) \prod_{i=1}^N w(x_i)dx_i \ , \end{aligned}$$

where in the rightmost equality above we have simply changed the order of appearance of the elements in the product inside the integral. Let us pose a change of variable $\kappa = \tau \circ \pi^{-1}$, so that $\pi = \kappa^{-1} \circ \tau$. Given that $\mathrm{sgn}(\kappa^{-1} \circ \tau) = \mathrm{sgn}(\kappa^{-1})\mathrm{sgn}(\tau) = \mathrm{sgn}(\kappa)\mathrm{sgn}(\tau)$, the result follows.

Derivation of the Airy kernel

Consider the eigenvalues of random matrices belonging to the Gaussian Unitary Ensemble (GUE). Their joint PDF is given by

$$P_{\mathrm{joint}}(\lambda_1, \lambda_2, \dots, \lambda_N) = \frac{1}{Z_N} \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^2 e^{-N\sum_{i=1}^N \lambda_i^2} \ , \tag{2}$$

where Z_N is a normalization constant. They form a determinantal process with a Christoffel-Darboux kernel $K_N(x,y)$ given by

$$K_N(x,y) = \sqrt{\frac{N}{\pi}} \frac{1}{2^N (N-1)!} e^{-N\frac{(x^2+y^2)}{2}} \frac{H_N(x\sqrt{N})H_{N-1}(y\sqrt{N}) - H_N(y\sqrt{N})H_{N-1}(x\sqrt{N})}{\sqrt{N}(x-y)} \tag{3}$$

where $H_k(x)$ denotes the Hermite polynomial of degree k. They satisfy the following orthogonality condition (note the choice of the physicists' convention)

$$\int_{-\infty}^{\infty} H_k(x)H_{k'}(x)e^{-x^2} = \delta_{k,k'}h_k \ , \ h_k = 2^k k! \sqrt{\pi} \ . \tag{4}$$

Using the following asymptotic expansion [1]

$$\exp(-x^2/2)H_{N+m}(x) = (2N)^{m/2}\pi^{1/4}2^{N/2+1/4}(N!)^{1/2}N^{-1/12}\left(\mathrm{Ai}(t) - \frac{m}{N^{1/3}}\mathrm{Ai}'(t) + \mathcal{O}(N^{-2/3})\right) \tag{5}$$

where $x = (2N)^{1/2} + 2^{-1/2}N^{-1/6}t$, with $\mathrm{Ai}(x)$ denoting the Airy function, show that

$$\lim_{N\to\infty} \frac{1}{\sqrt{2}N^{2/3}} K_N\left(\sqrt{2} + \frac{u}{\sqrt{2}N^{2/3}}, \sqrt{2} + \frac{v}{\sqrt{2}N^{2/3}}\right) = K_{\mathrm{Ai}}(u,v) \tag{6}$$

where $K_{\mathrm{Ai}}(u,v)$ is the Airy kernel, given by

$$K_{\mathrm{Ai}}(u,v) = \frac{\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}'(u)\mathrm{Ai}(v)}{u-v} \ . \tag{7}$$

We assume the following asymptotic expansion holds :

$$\exp(-x^2/2)H_{N+m}(x\sqrt{N}) = \pi^{1/4}2^{(N+m)/2+1/4}((N+m)!)^{1/2}N^{-1/12}(\mathrm{Ai}(t) - \frac{m}{N^{1/3}}\mathrm{Ai}'(t) + O(N^{-2/3}))$$

for $x = (2N)^{1/2} + 2^{-1/2}N^{-1/6}t$ and $m = O(1)$. Using the above for $m = 0, -1$, we get

$$\exp(-x^2/2)H_N(x\sqrt{N}) = \pi^{1/4}2^{N/2+1/4}(N!)^{1/2}N^{-1/12}(\mathrm{Ai}(u) + O(N^{-2/3}))$$

and

$$\exp(-x^2/2)H_{N-1}(x\sqrt{N}) = \pi^{1/4}2^{(N-1)/2+1/4}((N-1)!)^{1/2}N^{-1/12}(\mathrm{Ai}(u) + \frac{\mathrm{Ai}'(u)}{N^{1/3}} + O(N^{-2/3}))$$

Using the same expansions with y in place of x and substituting into the Christoffel-Darboux kernel gives

$$\begin{aligned}
K_N(x,y) &= \sqrt{\frac{N}{\pi}} \frac{1}{2^N(N-1)!} e^{-N\frac{x^2+y^2}{2}} \frac{H_N(x\sqrt{N})H_{N-1}(y\sqrt{N}) - H_N(y\sqrt{N})H_{N-1}(x\sqrt{N})}{\sqrt{N}(x-y)} \\
&= \sqrt{\frac{N}{\pi}} \frac{1}{2^N(N-1)!} \pi^{1/2}2^N(N-1)!\sqrt{N}N^{-1/6} \frac{N^{-1/3}(\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}'(u)\mathrm{Ai}(v)) + O(N^{-2/3})}{2^{-1/2}N^{-1/6}(u-v)} \\
&= \sqrt{2}N^{2/3} \frac{(\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}'(u)\mathrm{Ai}(v)) + O(N^{-2/3})}{(u-v)}
\end{aligned}$$

with $x = \sqrt{2} + \frac{u}{\sqrt{2}N^{2/3}}$ and $y = \sqrt{2} + \frac{v}{\sqrt{2}N^{2/3}}$. Dividing by $\sqrt{2}N^{2/3}$ and taking the limit $N \to \infty$ gives the Airy kernel $K_{\mathrm{Ai}}(u,v)$.

Free fermions in presence of a hard wall potential and random matrices

Consider N non-interacting fermions, confined on the *positive* real axis $\mathbb{R}^+$ by a *quantum potential* of the form

$$V(x) = \frac{1}{2}b^2x^2 + \frac{\alpha(\alpha-1)}{2x^2} \ , \ b > 0 \ \& \ \alpha > 1 \ . \tag{8}$$

1. This is same as $\exp(-x^2/2)H_{N+m}(x\sqrt{N}) = \pi^{1/4}2^{(N+m)/2+1/4}((N+m)!)^{1/2}N^{-1/12}(\mathrm{Ai}(t) - \frac{m}{N^{1/3}}\mathrm{Ai}'(t) + O(N^{-2/3}))$ as one can check using the asymptotics of factorials.

3.1) Show that the solution of the Schrödinger equation for this type of "hard wall" potential (8) with the boundary conditions $\varphi_k(0) = 0$ (due to the hard wall at $x = 0$) and $\lim_{x\to\infty} \varphi_k(x) = 0$ are of the form

$$\varphi_k(x) = c_k e^{-\frac{bx^2}{2}} x^\alpha \mathcal{L}_k^{(\alpha-1/2)}(b\,x^2)\ , \text{ with } E_k = b\,(2k+\alpha+1/2)\ , \tag{9}$$

with k a non-negative integer and where $\mathcal{L}_k^{(\alpha-1/2)}$ is a generalized Laguerre polynomial of degree k and index $\alpha - 1/2$ and c_k some constant. We recall that $\mathcal{L}_k^\gamma$ can be written as

$$\mathcal{L}_k^\gamma(x) = \sum_{i=0}^{k} \binom{k+\gamma}{k-i} \frac{(-x)^i}{i!}\ . \tag{10}$$

Solution : It is easy to verify that $\varphi_k(0) = 0$ due to the x^α term, and $\lim_{x\to\infty} \varphi_k(x) = 0$ due to the $e^{-bx^2/2}$ term. We verify that $\varphi_k(x)$ satisfies the Schrodinger equation

$$-\frac{1}{2}\frac{d^2}{dx^2}\varphi_k(x) + V(x)\varphi_k(x) = E_k\varphi_k(x).$$

For simplicity, we denote $p(x) := \mathcal{L}_k^{(\alpha-1/2)}(x)$, and we can calculate that

$$\frac{d}{dx}\varphi_k(x) = c_k\left(-bx^{\alpha+1} + \alpha x^{\alpha-1}\right)e^{-bx^2/2}p(bx^2) + c_k 2bx^{\alpha+1}e^{-bx^2/2}p'(bx^2), \tag{11}$$

and

$$\begin{aligned}\frac{d^2}{dx^2}\varphi_k(x) =& c_k\left[-(\alpha+1)bx^\alpha + \alpha(\alpha-1)x^{\alpha-2} + b^2x^{\alpha+2} - \alpha bx^\alpha\right]e^{-bx^2/2}p(bx^2) + c_k\left(-2b^2x^{\alpha+2} + 2b\alpha x^\alpha\right)e^{-bx^2/2}p'(bx^2) \\ &+ c_k\left[2b(\alpha+1)x^\alpha - 2b^2x^{\alpha+2}\right]e^{-bx^2/2}p'(bx^2) + c_k 4b^2x^{\alpha+2}e^{-bx^2/2}p''(bx^2).\end{aligned}$$

We can simplify the above expression and get

$$\begin{aligned}\frac{d^2}{dx^2}\varphi_k(x) =& c_k\left[-(2\alpha+1)bx^\alpha + \alpha(\alpha-1)x^{\alpha-2} + b^2x^{\alpha+2}\right]e^{-bx^2/2}p(bx^2) \\ &+ 4bc_kx^\alpha\left[\left(\alpha + \frac{1}{2} - bx^2\right)p'(bx^2) + bx^2p''(bx^2)\right]e^{-bx^2/2}.\end{aligned} \tag{12}$$

The Laguerre polynomial of degree k and index $\alpha - 1/2$ satisfies the equation

$$xp''(x) + \left(\alpha + \frac{1}{2} - x\right)p'(x) + kp(x) = 0,$$

so we have

$$bx^2p''(bx^2) + \left(\alpha + \frac{1}{2} - bx^2\right)p'(bx^2) + kp(bx^2) = 0.$$

Plug it into (12), we get that

$$\begin{aligned}\frac{d^2}{dx^2}\varphi_k(x) =& c_k\left[-(2\alpha+1)bx^\alpha + \alpha(\alpha-1)x^{\alpha-2} + b^2x^{\alpha+2}\right]e^{-bx^2/2}p(bx^2) - 4bkc_kx^\alpha e^{-bx^2/2}p(bx^2) \\ =& \left[-(2\alpha+1+4k)b + \alpha(\alpha-1)x^{-2} + b^2x^2\right]c_kx^\alpha e^{-bx^2/2}p(bx^2).\end{aligned}$$

Thus we have

$$\begin{aligned}-\frac{1}{2}\frac{d^2}{dx^2}\varphi_k(x) + V(x)\varphi_k(x) &= \left[\left(2k+\alpha+\frac{1}{2}\right)b - \frac{1}{2}\alpha(\alpha-1)x^{-2} - \frac{1}{2}b^2x^2\right]\varphi_k(x) + \left(\frac{1}{2}b^2x^2 + \frac{\alpha(\alpha-1)}{2x^2}\right)\varphi_k(x) \\ &= \left(2k+\alpha+\frac{1}{2}\right)b\varphi_k(x) = E_k\varphi_k(x).\end{aligned}$$

Thus $\varphi_k(x)$ satisfies the Schrodinger equation. Now by the uniqueness of the solution to the ODE, we conclude that $\phi_k(x)$ is of the form in (13).

3.2) Deduce that the joint PDF of the N non-interacting fermions (in the ground state) reads

$$P_{\text{joint}}(x_1, x_2, \cdots, x_N) = \frac{1}{Z_N}\prod_{i=1}^{N} x_i^{2\alpha} \prod_{1\le i<j\le N} |x_i^2 - x_j^2|^2 e^{-b\sum_{i=1}^N x_i^2}\ . \tag{13}$$

Solution : We plug the functions ϕ_k in equation (6) to get

$$P_{joint}(x_1,...,x_N) = \frac{1}{N!}(\det_{1\leq i,j\leq N}[\phi_i(x_j)])^2 = \frac{1}{N!}\left(\det_{1\leq i,j\leq N}\left[c_i e^{\frac{-bx_j^2}{2}} x_j^\alpha \mathcal{L}_i^{(\alpha-1/2)}(bx_j^2)\right]\right)^2$$
$$= \frac{1}{Z_N} e^{-b\sum_{j=1}^N x_j^2} \prod_{j=1}^N x_J^{2\alpha} \prod_{i<j}(x_j^2 - x_i^2)^2$$

where we pulled out all the common factors in the rows and columns of the matrix, the Vandermonde determinant in the polynomial form and we denoted the product of all the constant factors by $1/Z_N$.

3.3) Besides the GUE studied in a lecture, a well known ensemble is the so-called Laguerre-Wishart ensemble for which the eigenvalues are distributed according to

$$P_{\rm joint}(\lambda_1,\cdots,\lambda_N) = \frac{1}{\tilde{Z}_N}\prod_{i<j}(\lambda_i-\lambda_j)^2\left(\prod_{i=1}^N \lambda_i^a\right)e^{-\sum_{i=1}^N \lambda_i}\ , \quad \text{for} \quad \lambda_i \geq 0 \quad \forall i=1,\cdots,N\ . \tag{14}$$

Conclude form (13) that the variables $y_i = x_i^2$ are distributed like the eigenvalues of random matrices belonging to the Laguerre-Wishart ensemble. What is the corresponding parameter a?

Solution : Comparing the last expression with $y_i = x_i^2$ with the Laguerre-Wishart ensemble, we see that they coincide (taking into account the Jacobian of the transformation $x_i \to y_i$) when $a = \alpha - 1/2$.