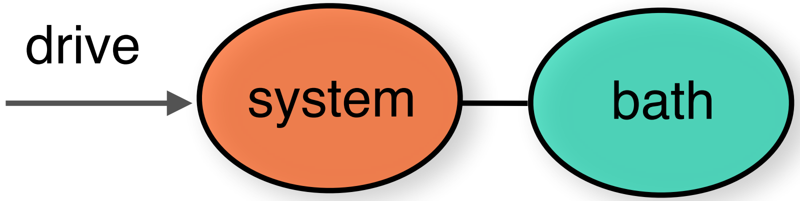
Open Quantum Optical Systems - Theory Perspectives

Quantum optical systems as open quantum systems

- **open system**

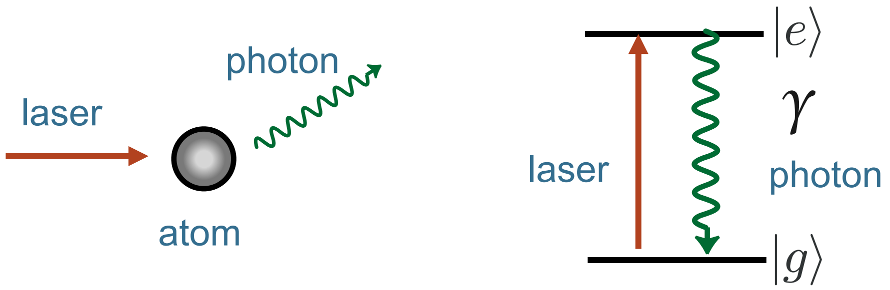
  system of interest coupled to an environment (bath, reservoir)



Examples

- **Driven, radiatively damped two-level atoms**

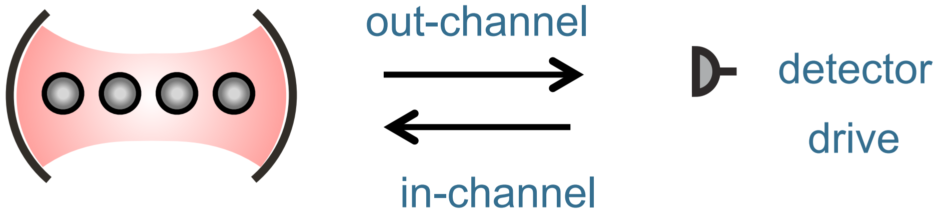
  atom driven by laser emitting photons in the 3D vacuum modes of the electromagnetic radiation field



- **Cavity QED**

  atoms + cavity mode (system), coupled to waveguide / optical fiber (electromagnetic bath)

  environment as *input & output channels* to drive and monitor evolution of quantum system



Various Scenarios of Open System Dynamics

- **We do not monitor the quantum system**

  dynamics of *unobserved* system / master equation / *a priori* dynamics

- **We (continuously) observe the systems**

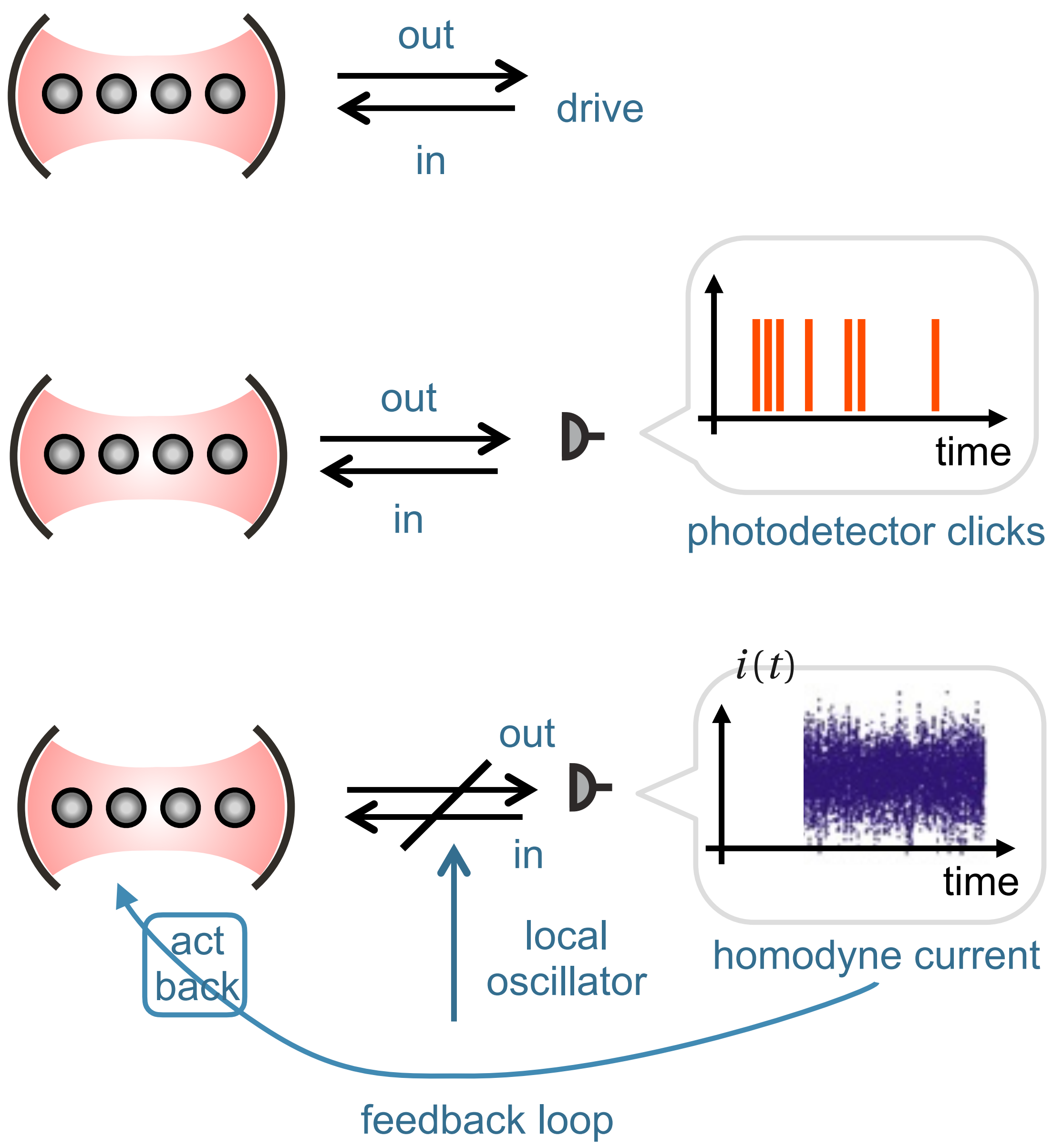
  photodetector / photon counting: photon statistics

  homo- / heterodyne current: current correlations

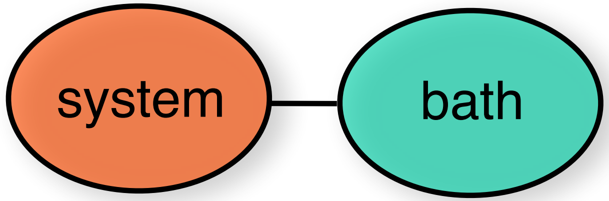
  dynamics of system *conditional* to observed measurement trajectory *in a single run* / stochastic Schrödinger Equation / *a posteriori* dynamics

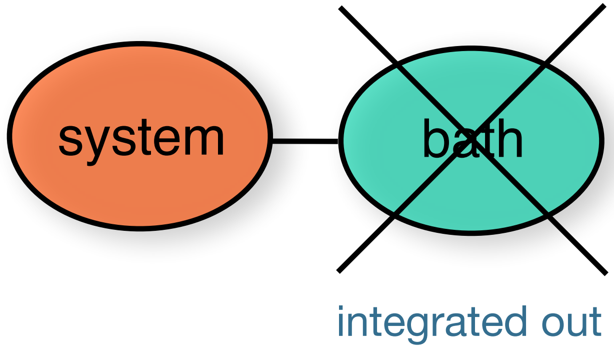
- **… and quantum feedback**

  act back on quantum system conditional to measurement read out in a *single run* of the experiment



Theory Overview - Quantum Markov Processes





- Dynamics of system + bath

  Quantum Stochastic Schrödinger Equation

  Quantum Langevin Equations
  input-output formalism

- Reduced system dynamics

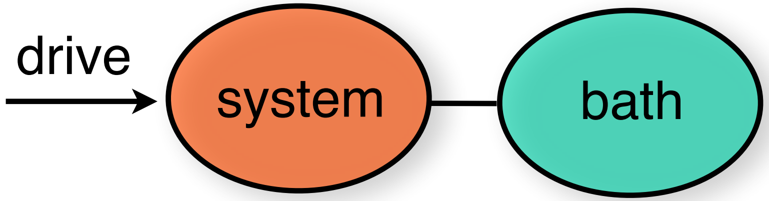
  Master Equations

  Stochastic Schrödinger Equation
  quantum trajectories (jump / diffusive)

- Topics

  - Ito & Stratonovich (Quantum) Calculus
  - Solution of QSSE with Matrix Product States

Part II



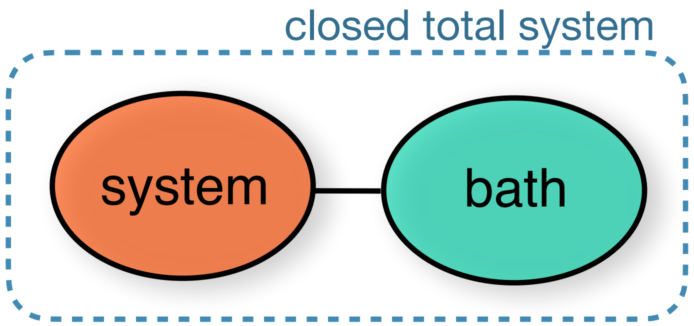
Theory of Quantum Noise in Quantum Optics

Quantum Stochastic Schrödinger Equation (QSSE)

- **Quantum Operations, Kraus operators**
  - formal quantum information theory
- **QSSE, quantum trajectories, master equations etc.**
  - quantum Markov processes, quantum stochastic formalism
  - formulated in context of quantum optical problems

System coupled to a Bath - a Quantum Information Perspective

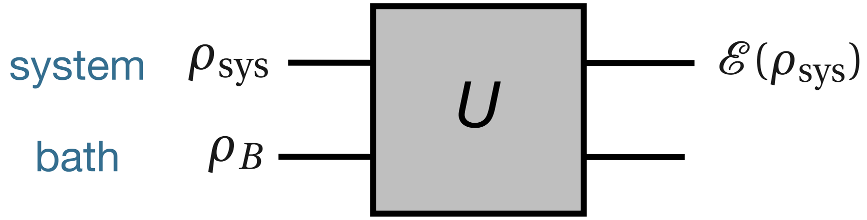
Quantum Operations



Open quantum system: system of interest coupled to a bath. They form a closed quantum system where time evolution U in Hilbert space $\mathscr{H}_{\text{sys}} \otimes \mathscr{H}_B$.

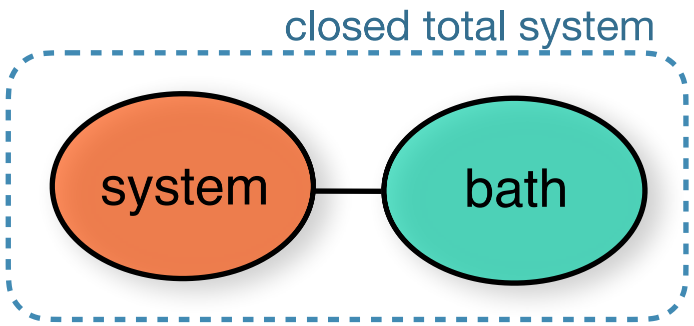
Evolution of system coupled to a bath:

$$\rho_{\text{sys}} \longrightarrow \mathscr{E}(\rho_{\text{sys}}) = \text{tr}_B \left[U \left(\rho_{\text{sys}} \otimes \rho_B \right) U^{\dagger} \right]$$



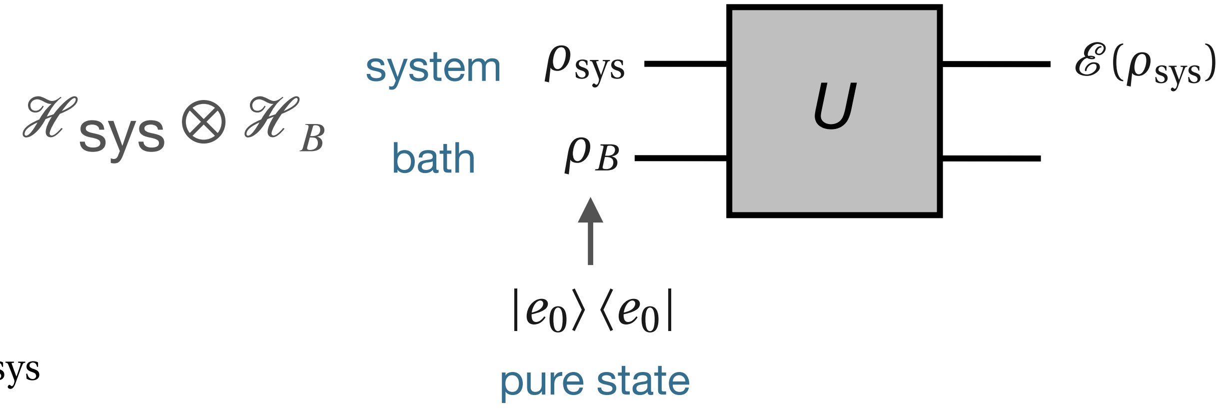
Lit.: M. A. Nielsen and I. I. Chuang, Quantum Computation and Quantum Information, Cambridge (2000): Chapter 8

Operator sum representation



Evolution of the composite system

$$\mathcal{E}(\rho_{\text{sys}}) = \sum_k \langle e_k | U \left[\rho_{\text{sys}} \otimes | e_0 \rangle \langle e_0 | \right] U^\dagger | e_k \rangle$$
$$= \sum_k E_k \rho_{\text{sys}} E_k^\dagger$$



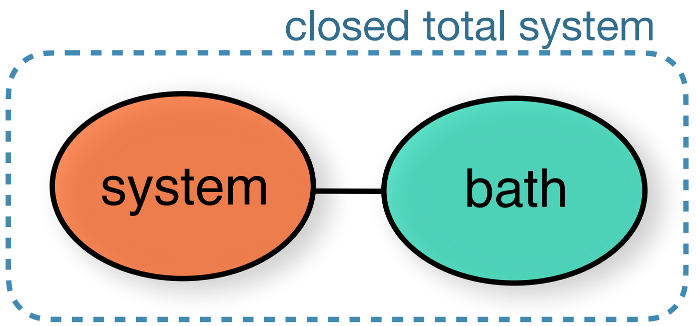
with operational elements $E_k \equiv \langle e_k | U | e_0 \rangle$ acting in $\mathscr{H}_{\text{sys}}$

Completeness relation:

$$\sum_k E_k^\dagger E_k = \hat{1}_{\text{sys}}$$

Proof: $1 = \text{tr}_{\text{sys}} \mathcal{E}(\rho_{\text{sys}}) = \text{tr}_{\text{sys}} \left(\sum_k E_k \rho_{\text{sys}} E_k^\dagger \right) = \text{tr}_{\text{sys}} \left(\sum_k E_k^\dagger E_k \rho_{\text{sys}} \right)$

Operator sum representation

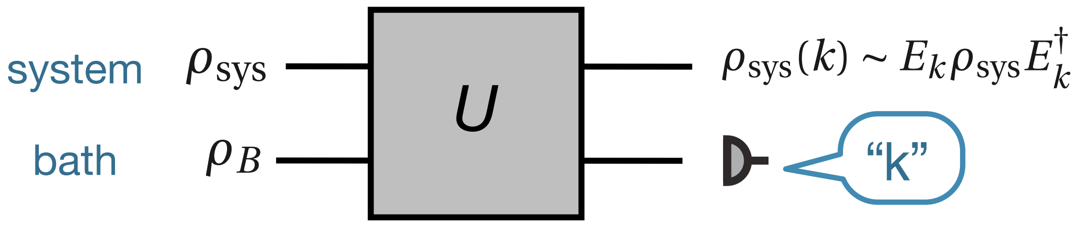


Performing a measurement in the basis $\{|e_k\rangle\}$, the state of the system after reading "k" is

$$\rho_{\text{sys}}(k) \sim \text{tr}_B\left[|e_k\rangle\langle e_k| U\left(\rho_{\text{sys}} \otimes |e_0\rangle\langle e_0|\right) U^\dagger |e_k\rangle\langle e_k|\right]$$

$$\sim \langle e_k| U\left(\rho_{\text{sys}} \otimes |e_0\rangle\langle e_0|\right) U^\dagger |e_k\rangle$$

$$= E_k \rho_{\text{sys}} E_k^\dagger / \text{tr}_{\text{sys}}(\ldots)$$



Probability for the outcome k

$$p(k) = \text{tr}_{\text{sys}+B}\left[|e_k\rangle\langle e_k| U\left(\rho_{\text{sys}} \otimes |e_0\rangle\langle e_0|\right) U^\dagger |e_k\rangle\langle e_k|\right]$$

$$= \text{tr}_{\text{sys}}\left(E_k \rho_{\text{sys}} E_k^\dagger\right)$$

Not reading the measurements

$$\mathcal{E}(\rho) = \sum_k p(k) \rho_{\text{sys}}(k) = \sum_k E_k \rho_{\text{sys}} E_k^\dagger$$

Open Quantum Optical Systems

Example: Radiatively Damped and Driven Two-Level Atom

Hamiltonian ($\hbar = 1$)

$$H_{\text{tot}} = H_{\text{sys}} + H_B + H_{\text{int}}$$

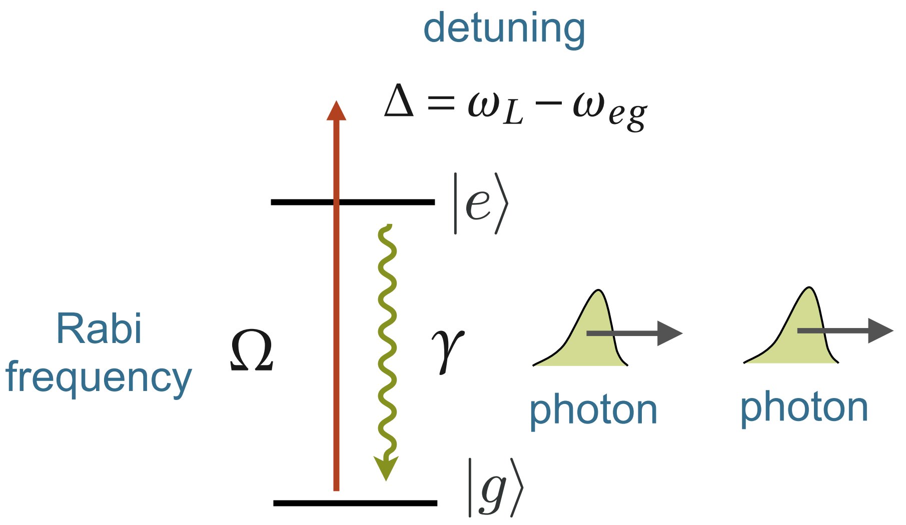
- **System:** with Paulioperators $\sigma_- = |g\rangle\langle e|$ etc.

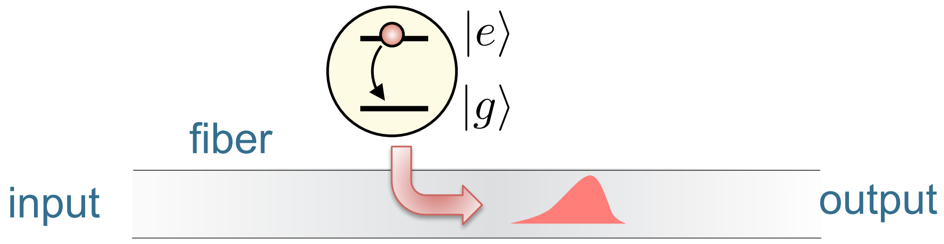
$$H_{\text{sys}} = \omega_{eg}|e\rangle\langle e| - \frac{1}{2}\Omega e^{-i\omega_L t}\sigma_+ - \frac{1}{2}\Omega^* e^{+i\omega_L t}\sigma_-$$

- **Bath:** vacuum modes of 1D radiation field

$$H_B = \int_{\omega_L-\vartheta}^{\omega_L+\vartheta} d\omega\, \omega\, b^\dagger(\omega)\, b(\omega)$$

with $[b(\omega), b^\dagger(\omega')] = \delta(\omega - \omega')$ and cutoff ϑ ($\ll \omega_{eg} \approx \omega_L$)





photonic nanostructure as engineered em environment

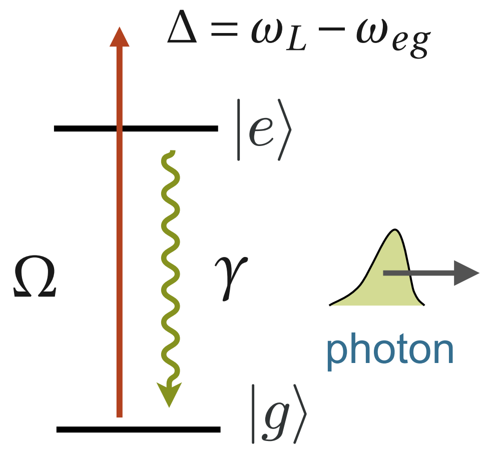
- **Interaction Hamiltonian system-bath in RWA**

$$H_{\rm int} \quad = \quad -\mu_{eg}\sigma_- E^{(-)}(0) - \mu_{eg}\sigma_+ E^{(+)}(0)$$

with 1D electric field operator $E^{(+)}(x) = i\int_{\omega_L-\vartheta}^{\omega_L+\vartheta} d\omega \sqrt{\frac{\omega}{4\pi\mathscr{A}\epsilon_0}}\, b(\omega) e^{i\omega x/c}$.

We rewrite

$$H_{\rm int} = i\int_{\omega_L-\vartheta}^{\omega_L+\vartheta} d\omega \kappa(\omega)\left[b^\dagger(\omega)\sigma_- - \sigma_+ b(\omega)\right]$$



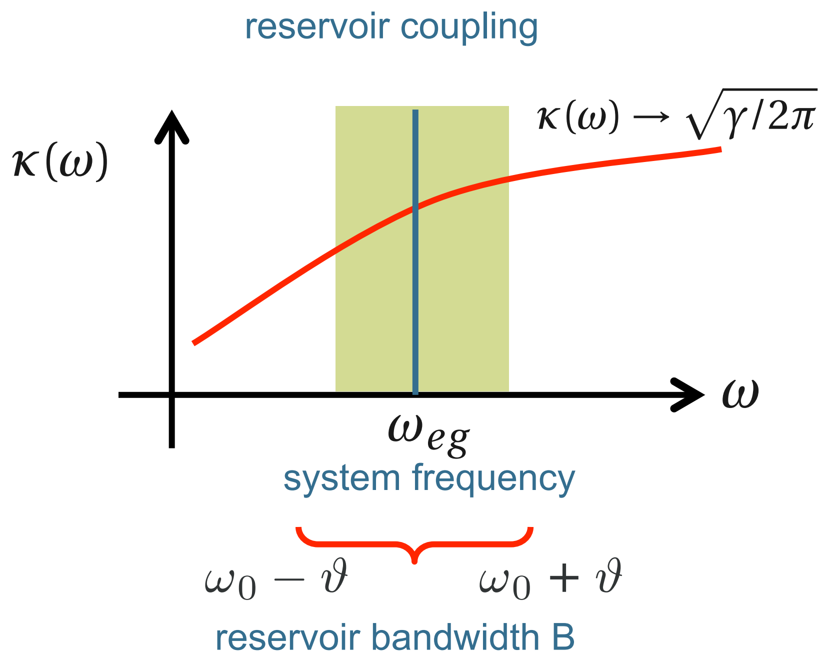
- **First Markov approximation:** replace in $[\omega_{eg} - \vartheta, \omega + \vartheta]$

$$\kappa(\omega) \rightarrow \sqrt{\frac{\gamma}{2\pi}}$$

Below γ will be identified with the spontaneous emission rate.

Validity and hierarchy of time scales:

$$\Omega, \Delta, \gamma \ll \vartheta \ll \omega_{eg} \approx \omega_L$$

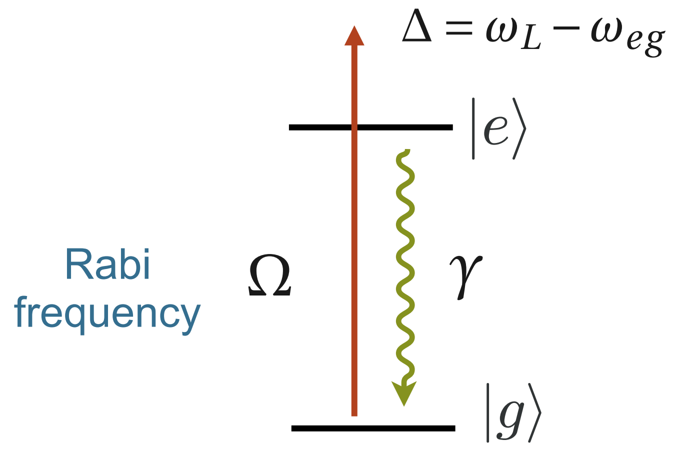


Schrödinger Equation

Schrödinger Equation in interaction picture:

$$i\hbar\frac{d}{dt}|\tilde{\Psi}(t)\rangle = \left[\tilde{H}_{\rm sys} + \tilde{H}_{\rm int}(t)\right]|\tilde{\Psi}(t)\rangle$$

with $U_I(t) = \exp\left(-i[H_B + H_{\rm sys}^{(0)}]t\right)$ with $H_{\rm sys}^{(0)} = \omega_L|e\rangle\langle e|$



- **System Hamiltonian** in rotating frame

$$\tilde{H}_{\rm sys} = -\Delta|e\rangle\langle e| - \frac{1}{2}\Omega\sigma_+ - \frac{1}{2}\Omega^*\sigma_-$$

- **Interaction Hamiltonian** in RWA

$$\tilde{H}_{\rm int}(t) = i\int_{\omega_{eg}-\vartheta}^{\omega_{eg}+\vartheta} d\omega\kappa(\omega)\left[b^\dagger(\omega)e^{-i(\omega-\omega_L)t}\sigma_- - \sigma_+ b(\omega)e^{-i(\omega-\omega_L)t}\right]$$

$$\equiv i\sqrt{\gamma}\left[b^\dagger(t)\sigma_- - b(t)\sigma_+\right]$$

Remarks:

- optical frequencies disappeared

$$\Omega, \Delta, \gamma \ll \vartheta \ll \omega_{eg} \approx \omega_L \quad \text{RWA}$$

weak coupling; hierarchy of time scales

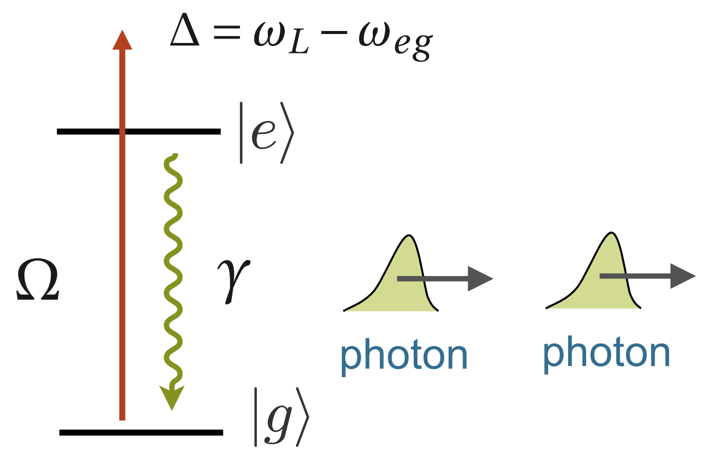
- below take white noise limit

$$\vartheta \to \infty$$

Quantum Stochastic Schrödinger Equation (QSSE)

$$\frac{d}{dt}|\tilde{\Psi}(t)\rangle = \left[-i\tilde{H}_{\rm sys} + \sqrt{\gamma}b^\dagger(t)\sigma_- - \sqrt{\gamma}\sigma_+ b(t)\right]|\tilde{\Psi}(t)\rangle$$

with initial condition $|\tilde{\Psi}(0)\rangle = |\psi_{\rm sys}\rangle \otimes |{\rm vac}\rangle$.



- **Operator 'white' noise**

$$b(t) = \frac{1}{\sqrt{2\pi}}\int_{\omega_L-\vartheta}^{\omega_L+\vartheta} b(\omega)e^{-i(\omega-\omega_L)t}d\omega$$

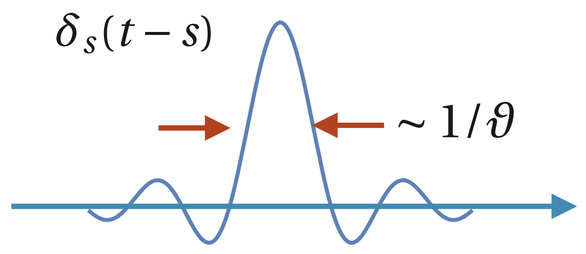
with commutator

$$\left[b(t), b^\dagger(s)\right] = \delta_s(t-s) \quad \text{with} \quad \delta_s(t-s) = \frac{1}{2\pi}\int_{-\vartheta}^{+\vartheta} d\omega e^{-i\omega(t-s)}$$

with δ_s a 'δ–function' with time scale $1/\vartheta$ → 0

white noise limit

slowly varying
∂-function

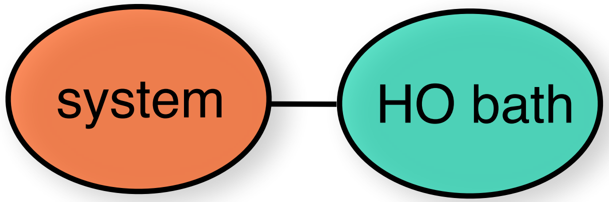


Note: QSSE can be given a rigorous mathematical definition within Stratonovich (or Ito Calculus.) Below will integrate this equation 'naively.'

Generic Quantum Optical Model - Summary

Hamiltonian of system + bath

$$H_{\text{tot}} = H_{\text{sys}} + H_B + H_{\text{int}}$$

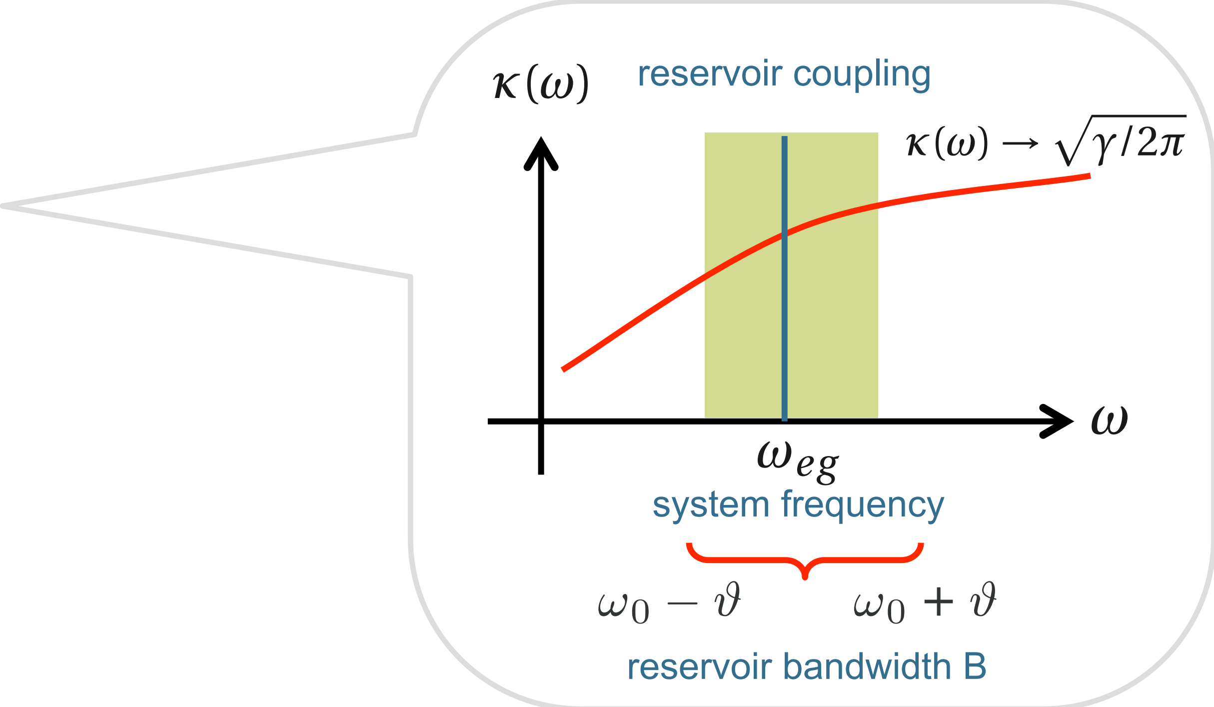


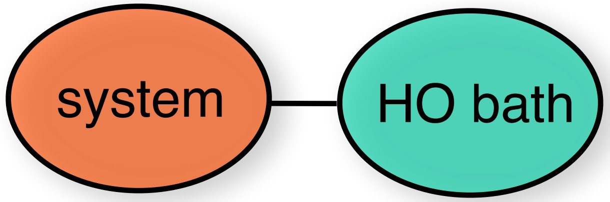
H_{sys} (unspecified)

$$H_B = \int_{\omega_0-\vartheta}^{\omega_0+\vartheta} d\omega\, \omega\, b^\dagger(\omega)\, b(\omega)$$ with bosonic $\left[b(\omega), b^\dagger(\omega')\right] = \delta(\omega - \omega')$

$$H_{\text{int}}(t) = i \int_{\omega_0-\vartheta}^{\omega_0+\vartheta} d\omega \kappa(\omega) \left[b^\dagger(\omega) c - c^\dagger b(\omega)\right]$$

in RWA with c a system operator





Quantum Stochastic Schrödinger Equation

$$\frac{d}{dt}|\Psi(t)\rangle = \left\{-iH_{\text{sys}} + \sqrt{\gamma}\, b^\dagger(t)c - \sqrt{\gamma}\, c^\dagger b(t)\right\}|\Psi(t)\rangle \quad \text{with } |\Psi(0)\rangle = |\psi_{\text{sys}}\rangle \otimes |\text{vac}\rangle$$

quantum noise

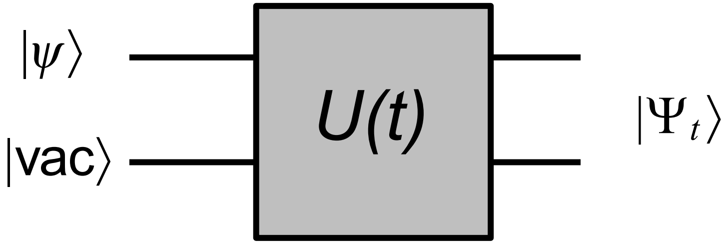
$$b(t) = \frac{1}{\sqrt{2\pi}} \int_{\omega_0-\vartheta}^{\omega_0+\vartheta} b(\omega)\, e^{-i(\omega-\omega_0)t}\, d\omega$$

$$\left[b(t), b^\dagger(s)\right] = \delta_s(t-s)$$

Rem.: Here we assumed a RWA, and all optical frequencies have been transformed away

$\Omega, \Delta, \gamma \ll \vartheta \ll \omega_{eg} \approx \omega_L$ TLS

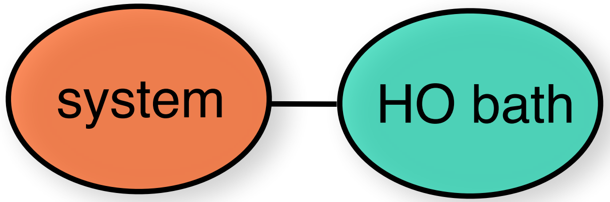
Below we will integrate the QSSE



$$|\Psi_0\rangle \to |\Psi_t\rangle = e^{-iH_{\text{tot}}t}|\Psi_0\rangle$$

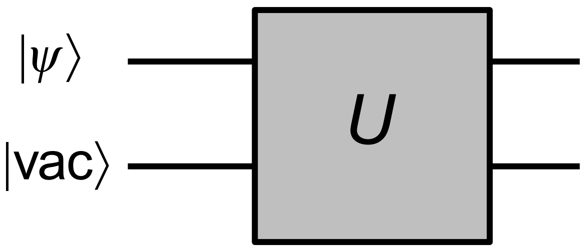
Schrödinger equation:
system + environment

Time Evolution of System + Bath



Questions:

- **We do not observe the bath / environment:** reduced density operator

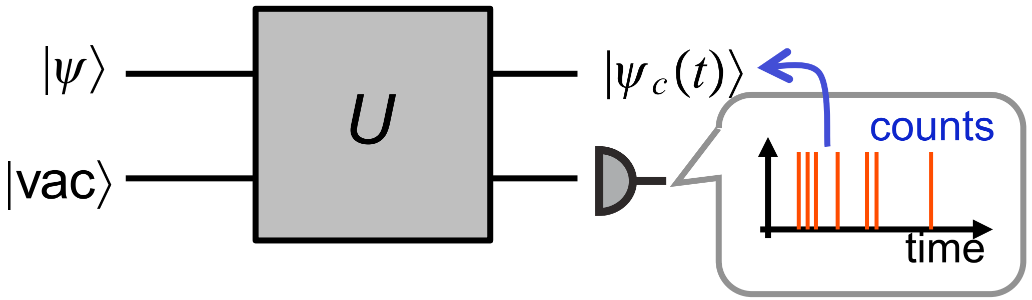


$$\rho_t = \mathrm{tr}_B |\Psi_t\rangle\langle\Psi_t|$$

master equation

- decoherence
- preparation of the system
  (e.g. optical pumping, laser cooling)

- **We monitor the bath / environment:** continuous measurement

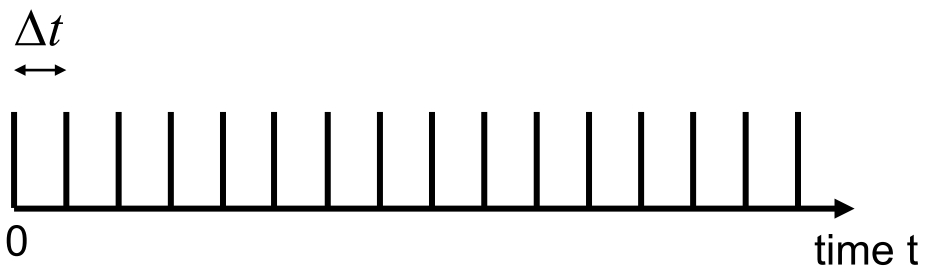


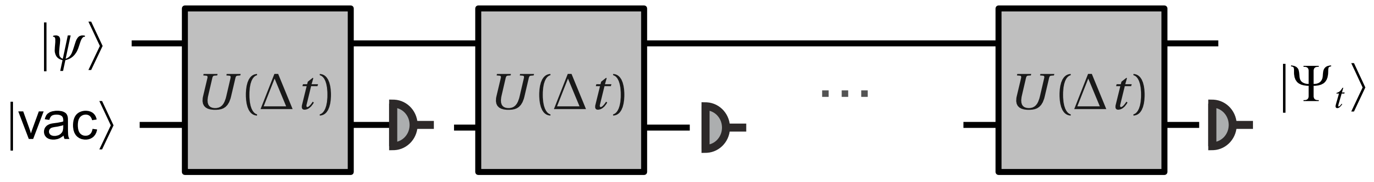
conditional wavefunction

- counting statistics
- effect of observation on system evolution
  (e.g. preparation of the (single) quantum system)

Integration of the Quantum Stochastic Schrödinger Equation

"Coarse Grained" Time Integration





Stroboscopic evolution of the state vector:

$$
\begin{aligned}
|\Psi(t = n\Delta t)\rangle &= U(t,0)|\Psi(0)\rangle \\
&= U(t, t-\Delta t)\ldots U(2\Delta t, \Delta t)U(\Delta t, 0)|\Psi(0)\rangle
\end{aligned}
$$

with $U(\Delta t, 0)$ etc. the time evolution operators.

Idea: we integrate the QSSE in small time steps Δt

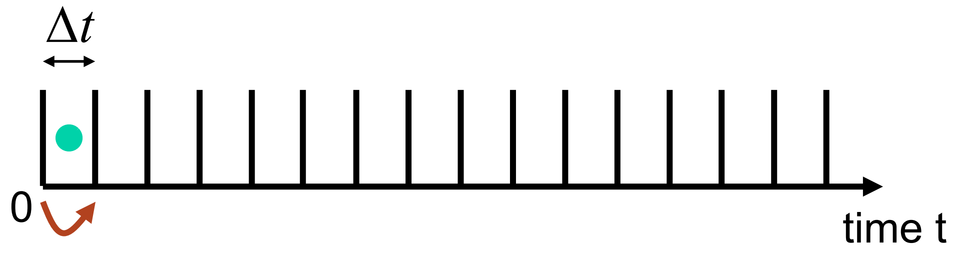
$$\tau_{\text{sys}} \gg \Delta t \gg 1/\vartheta \quad (\gg \tau_{\text{opt}})$$

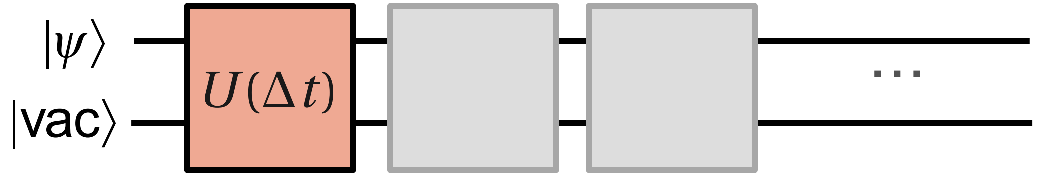
hierarchy of time scales

RWA

Example TLS: $\tau_{\text{sys}} \sim 1/\Omega, 1/\Delta, 1/\gamma$

The first time step





We wish to do an expansion of $U(\Delta t, 0)$ to keeping terms up to order Δt

$$U(\Delta t, 0)|\Psi(0)\rangle = \quad \ldots$$

Background Material: Perturbation theory

The SE equation for the time evolution operator,

$$\frac{d}{dt}U(t,t_0) = -iH_{\text{int}}(t)U(t,t_0) \quad (U(t_0,t_0) = \hat{1}),$$

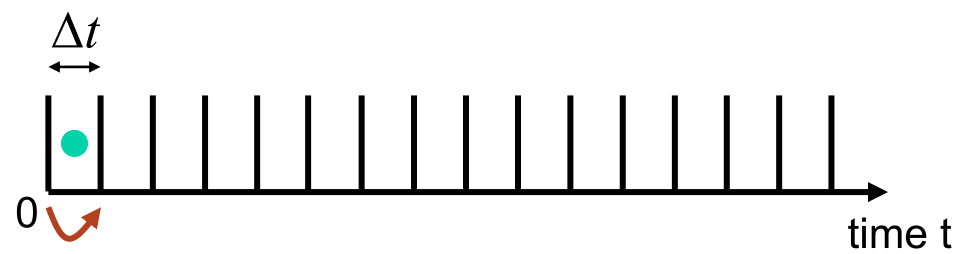
is equivalent to the integral equation

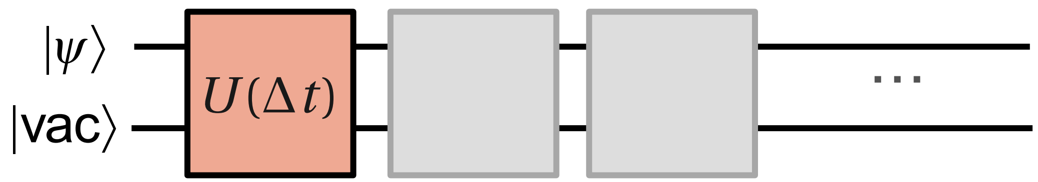
$$U(t,t_0) = \hat{1} - \int_{t_0}^{t} dt'\, H_{\text{int}}(t')U(t',t_0).$$

A perturbative solution is derived by iteration:

$$U(t,t_0) = \hat{1} - i\int_{t_0}^{t} dt_1\, H_{\text{int}}(t_1) + (-)^2 \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1\, H_{\text{int}}(t_2)\, H_{\text{int}}(t_1) + \ldots.$$

The first time step





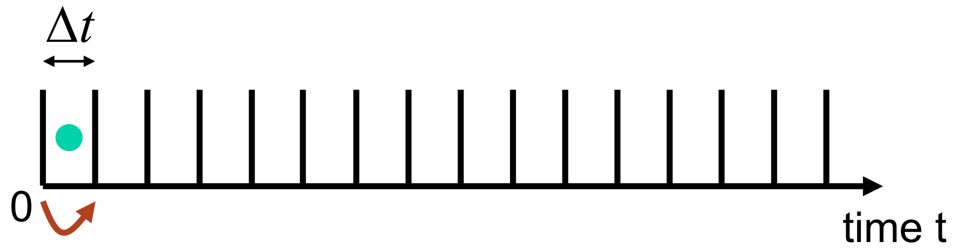
We wish to do an expansion of $U(\Delta t, 0)$ to keeping terms up to order Δt

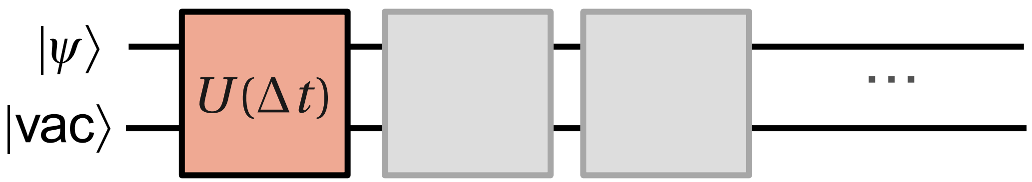
$$U(\Delta t, 0)|\Psi(0)\rangle = \left\{ \hat{1} - iH_{\text{sys}}\Delta t + \sqrt{\gamma}\, c \int_0^{\Delta t} b^\dagger(t)\, dt - \sqrt{\gamma}\, c^\dagger \int_0^{\Delta t} \cancel{b(t)}\, dt \right.$$

$$\left. + \quad \dots \right\} |\Psi(0)\rangle$$

with $|\Psi(0))\rangle = |\psi\rangle \otimes$ |vac⟩

The first time step





We wish to do an expansion of $U(\Delta t, 0)$ to keeping terms up to order Δt

$$U(\Delta t, 0)|\Psi(0)\rangle = \left\{ \hat{1} - iH_{\text{sys}}\Delta t + \sqrt{\gamma}\, c \int_0^{\Delta t} b^\dagger(t)\, dt - \sqrt{\gamma} c^\dagger \int_0^{\Delta t} b(t)\, dt \right.$$

$$\left. + (-i)^2 \gamma c^\dagger c \int_0^{\Delta t} dt \int_0^{t_2} dt'\, b(t) b^\dagger(t') + \ldots + \ldots \right\} |\Psi(0)\rangle \qquad \text{with } |\Psi(0))\rangle = |\psi\rangle \otimes |\text{vac}\rangle$$

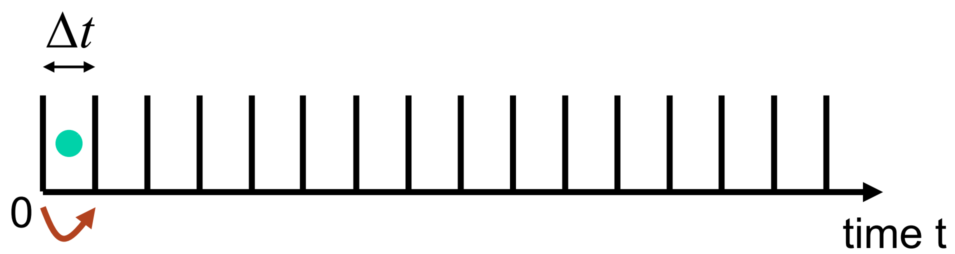
$$\int_0^{\Delta t} dt \int_0^{t} dt'\, b(t)\, b^\dagger(t')|\text{vac}\rangle = \int_0^{\Delta t} dt \int_0^{t} dt' \left[b(t), b^\dagger(t') \right] |\text{vac}\rangle$$

$$= \int_0^{\Delta t} dt \int_0^{t} dt'\, \delta_s(t - t')|\text{vac}\rangle$$

$$= \frac{1}{2}\Delta t|\text{vac}\rangle \qquad \text{for } \Delta t \gg 1/\vartheta.$$

Due to the singluar nature of $\left[b(t), b^\dagger(s) \right] = \delta_s(t - s)$ *the second* order perturbation theory term is actually *first* order in Δt

The first time step



Result: the wave function after the first time step order Δt

$$|\Psi(\Delta t)\rangle = U(\Delta t, 0)|\Psi(0)\rangle$$
$$= \left\{ \hat{1} - iH_{\text{eff}}\Delta t + \sqrt{\gamma}\, c\, \Delta B^{\dagger}(0) \right\} |\Psi(0)\rangle$$

with $|\Psi(0))\rangle = |\psi\rangle \otimes |\text{vac}\rangle$

with the definitions

- **effective (non-Hermitian) system Hamiltonian**

$$H_{\text{eff}} \equiv H_{\text{sys}} - \frac{i}{2}\gamma c^{\dagger} c$$

Wigner-Weisskopf Hamiltonian

- A system wave function evolving under H_{eff} will decayloose norm
- TLS Wigner-Weisskopf Hamiltonian $H_{\text{sys}} = \left(\omega_{eg} - \frac{i}{2}\gamma\right)|e\rangle\langle e| + \ldots$

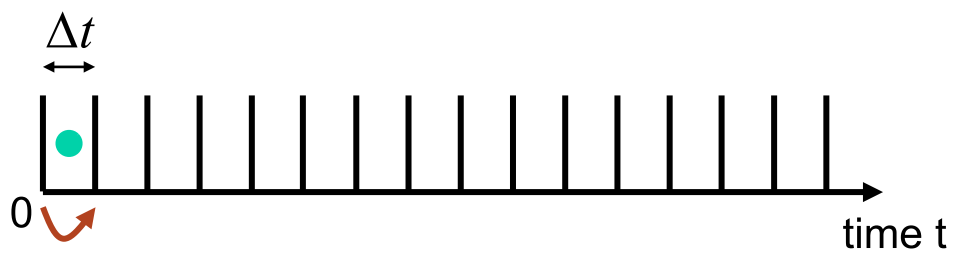
- **increment of noise operator**

$$\Delta B(t) \equiv \int_t^{t+\Delta t} b(s)\, ds$$

integral of what op. noise "points to the future"

- annihilation (creation) operator of photon in $(t, t+\Delta]$
- quantum version of Wiener increment

The first time step



Result: the wave function after the first time step order Δt

$$
\begin{aligned}
|\Psi(\Delta t)\rangle &= U(\Delta t, 0)|\Psi(0)\rangle \\
&= \left\{ \hat{1} - iH_{\text{eff}}\Delta t + \sqrt{\gamma}\, c\, \Delta B^{\dagger}(0) \right\} |\Psi(0)\rangle
\end{aligned}
$$

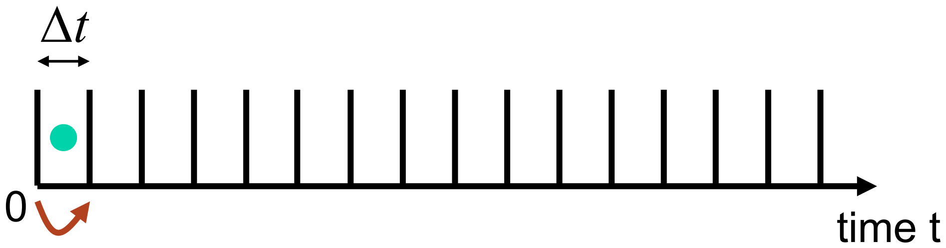
with $|\Psi(0))\rangle = |\psi\rangle \otimes |\text{vac}\rangle$

with the definitions

Note: The state $|\Psi(\Delta t)\rangle$ is normalized to first order in Δt

$$
\begin{aligned}
\|\Psi(\Delta t)\|^2 = 1 &+ \langle \text{vac}|\langle\psi| + \frac{i}{\hbar} H_{\text{eff}}^{\dagger}\Delta t - \frac{i}{\hbar} H_{\text{eff}}\Delta t \\
&+ \sqrt{\gamma}\, c^{\dagger}\Delta B(0) \sqrt{\gamma}\, c \Delta B^{\dagger}(0) |\psi\rangle|\text{vac}\rangle \\
= 1 &+ O(\Delta t^2)
\end{aligned}
$$

The first time step



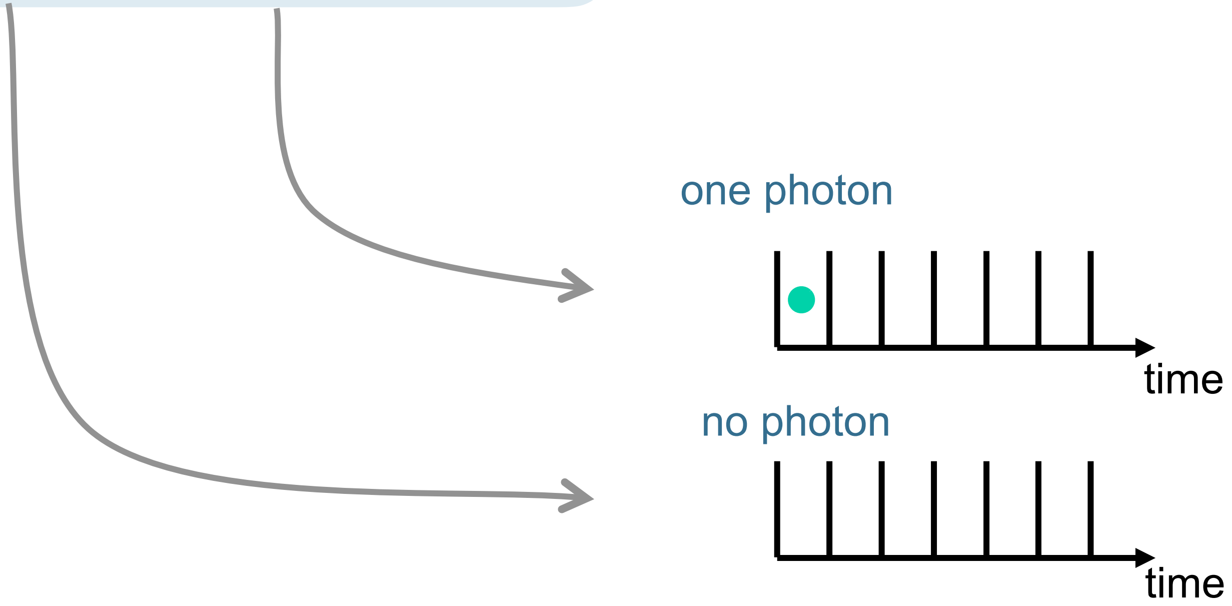
Result: the wave function after the first time step order Δt

$$|\Psi(\Delta t)\rangle = U(\Delta t, 0)|\Psi(0)\rangle$$
$$= \left\{\hat{1} - i H_{\text{eff}} \Delta t + \sqrt{\gamma}\, c\, \Delta B^{\dagger}(0)\right\} |\Psi(0)\rangle$$

with $|\Psi(0))\rangle = |\psi\rangle \otimes |\text{vac}\rangle$

superposition state of atom
an no & one photon state

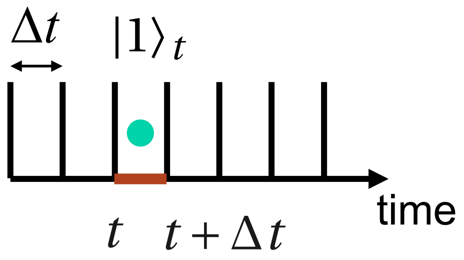
with the definitions



Physical Meaning and Properties of $\Delta B(t)$ and $\Delta B^\dagger(t)$

$$\Delta B(t) \equiv \int_t^{t+\Delta t} b(s)\, ds$$

increment "pointing to the future" (Ito)



Properties

- $\Delta B(t)|\mathrm{vac}\rangle = 0$
- commutation relations (independent increments)

$$\left[\Delta B(t), \Delta B^\dagger(t')\right] = \begin{cases} \Delta t & t = t' \quad \text{overlapping} \\ 0 & t \neq t' \quad \text{non-overlapping} \end{cases}$$

- one photon wave packet in time slot $(\Delta t, t+\Delta t]$

$$\frac{\Delta B^\dagger(t)}{\sqrt{\Delta t}}|\mathrm{vac}\rangle \equiv |1\rangle_t$$

with zero-photon state $\Delta B(t)|0\rangle_t = 0$.

- photon number operators

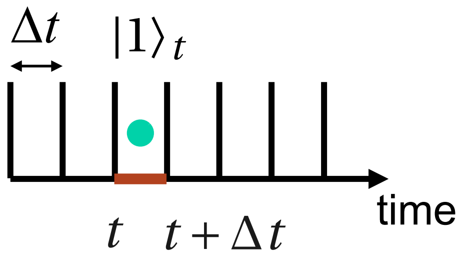
$$N(t) = \frac{\Delta B^\dagger(t)}{\sqrt{\Delta t}} \frac{\Delta B(t)}{\sqrt{\Delta t}}$$

with eigenstates $N(t)|0\rangle_t = 0,\ N(t)|1\rangle_t = 1|1\rangle_t$

Physical Meaning and Properties of $\Delta B(t)$ and $\Delta B^\dagger(t)$

$$\Delta B(t) \equiv \int_t^{t+\Delta t} b(s)\, ds$$

increment pointing "to the future" (Ito)



Properties (continued)

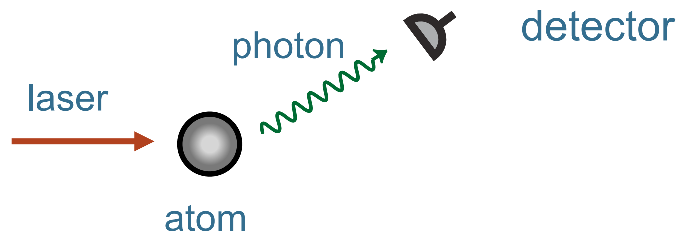
- $\{N(t)\}$ form a set of commuting operators,

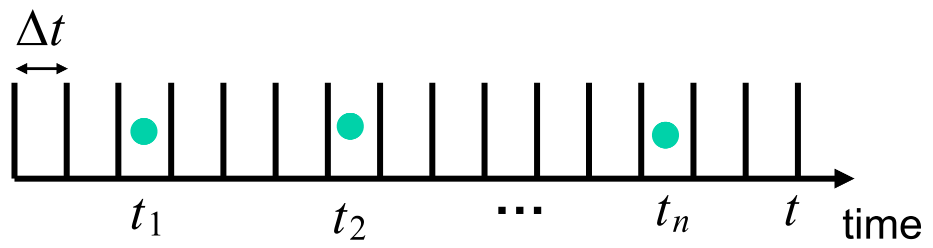
  $$[N(t), N(t')] = 0$$

  with eigenstates

  $$\frac{\Delta B^\dagger(t_n)}{\sqrt{\Delta t}} \ldots \frac{\Delta B^\dagger(t_1)}{\sqrt{\Delta t}} |\text{vac}\rangle \equiv |1_{t_n}, \ldots, 1_{t_2}, 1_{t_1}\rangle$$

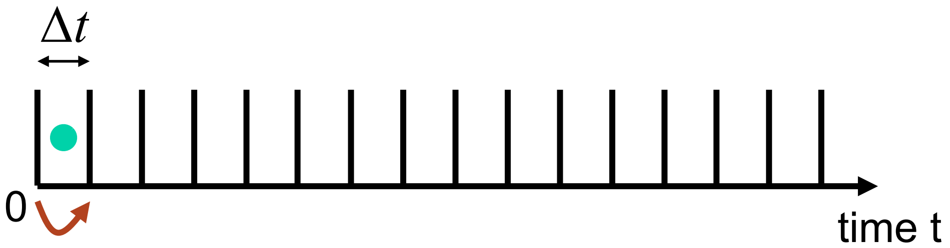
  corresponding to photons emitted at times $t_1 < t_2 < \ldots < t_n$.





Rem.: We will use this below to calculate the *photon statistics* of the emitted light

The first time step: quantum operations



Summary of first time step: to first order in Δt

$$|\Psi(\Delta t)\rangle = \left[\hat{1} - iH_{\text{eff}}\,\Delta t + \sqrt{\gamma}c\,\Delta B^{\dagger}(0)\right]|\Psi(0)\rangle$$

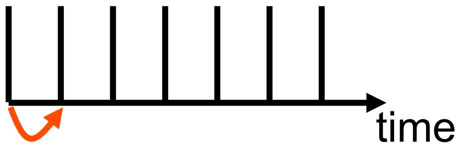
superposition of a vacuum and one-photon state.

$$= |\text{vac}\rangle \otimes \left(\hat{1} - iH_{\text{eff}}\,\Delta t\right)|\psi(0)\rangle + |1\rangle_{t=0} \otimes \sqrt{\gamma \Delta t}\, c|\psi(0)\rangle$$

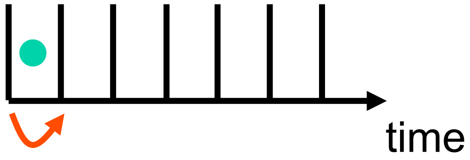
$$\equiv |\text{vac}\rangle \otimes E_0|\psi(0)\rangle + |1\rangle_{t=0} \otimes E_1|\psi(0)\rangle$$

We read off the operation elements

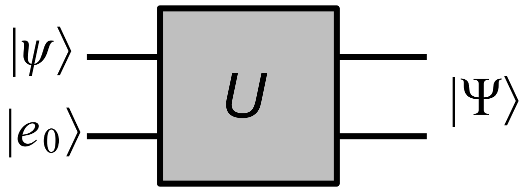
$E_0 = \hat{1} - iH_{\text{eff}}\,\Delta t$ (no photon in $(0, \Delta t]$)



$E_1 = \sqrt{\gamma \Delta t}\, c$ (one photon in $(0, \Delta t]$)



Reminder: Quantum operations



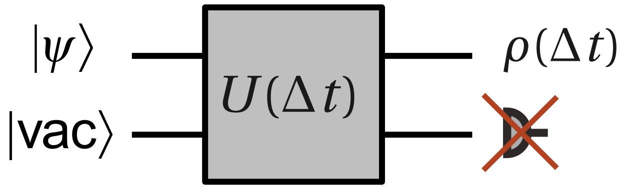
$$|\psi\rangle|e_0\rangle \longrightarrow |\Psi\rangle = U|\psi\rangle|e_0\rangle$$

$$= \sum_k |e_k\rangle\langle e_k|U|e_0\rangle|\psi\rangle \equiv \sum_k |e_k\rangle\, E_k|\psi\rangle$$ with $E_k = \langle e_k|U|e_0\rangle$ (operation elements)

The first time step

Discussion:

- **We *do not read* the detector:** reduced density operator



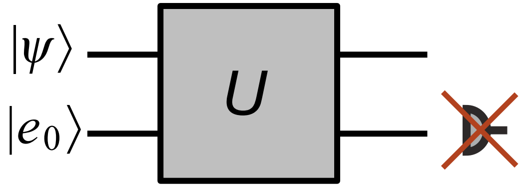
$$
\begin{aligned}
\rho(\Delta t) &= \mathrm{tr}_B |\Psi(\Delta t)\rangle\langle\Psi(\Delta t)| \\
&= E_0 \rho(0) E_0^\dagger + E_1 \rho(0) E_1^\dagger \\
&= \left(\hat{1} - i H_{\mathrm{eff}} \Delta t\right) \rho(0) \left(\hat{1} - i H_{\mathrm{eff}} \Delta t\right)^\dagger + \gamma c \rho(0) c^\dagger \Delta t
\end{aligned}
$$

master equation:

$$
\begin{aligned}
\rho(\Delta t) - \rho(0) &= -i \left(H_{\mathrm{eff}} \rho(0) - \rho(0) H_{\mathrm{eff}}^\dagger\right) \Delta t + \gamma c \rho(0) c^\dagger \Delta t \\
&\equiv -i \left[H_{\mathrm{sys}}, \rho(0)\right] \Delta t + \frac{1}{2} \gamma \left(2 c \rho(0) c^\dagger - c^\dagger c \rho(0) - \rho(0) c^\dagger c\right) \Delta t
\end{aligned}
$$

Lindblad form of master equation; optical Bloch equations

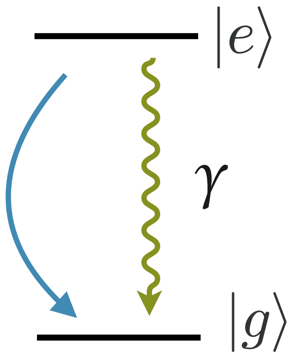
Reminder: Quantum operations



$$
\rho = |\psi\rangle\langle\psi| \longrightarrow \mathscr{E}(\rho) = \mathrm{Tr}_B \left[U \rho \otimes |e_0\rangle\langle e_0| U^\dagger\right] = \sum_k E_k \rho E_k^\dagger
$$

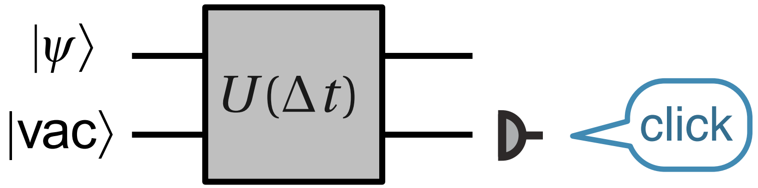
The first time step

$c \equiv \sigma_- = |g\rangle\langle e|$



Discussion:

- **We *read* the detector:**



- **Click:** resulting state after emission / detection of a photon

$$E_1|\psi(0)\rangle \equiv |\psi^{\text{click}}(\Delta t)\rangle = \sqrt{\gamma \Delta t}\, c|\psi(0)\rangle$$ (quantum jump) $\rightarrow |g\rangle$

with probability

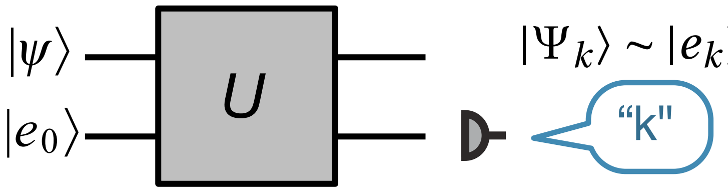
$$\mathscr{P}^{\text{click}} = \text{tr}_{\text{sys}}\left(E_1 \rho(0) E_1^\dagger\right) = \gamma \Delta t \, \|c\psi(0)\|^2$$

$$\mathscr{P}^{\text{click}} = \gamma \Delta t \, \|\langle e|\psi\rangle\|^2$$

Rem.: density matrix after click

$$\rho_1(\Delta t) = E_1 \rho(0) E_1^\dagger / \text{tr}_{\text{sys}}(\ldots)$$

Reminder: Quantum operations

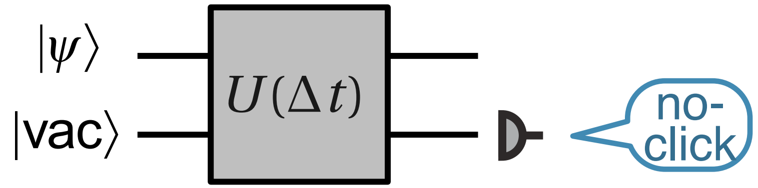


$|\Psi_k\rangle \sim |e_k\rangle E_k|\psi\rangle$ probability $p_k = \|E_k \psi\|^2$

The first time step

Discussion:

- **We *read* the detector:**



- **No click:** resulting state

$$E_0|\psi(0)\rangle \equiv |\psi^{\text{no click}}(\Delta t)\rangle = \left(\hat{1} - i H_{\text{eff}} \Delta t\right)|\psi(0)\rangle \approx e^{-iH_{\text{eff}}\Delta t/\hbar}|\psi(0)\rangle$$

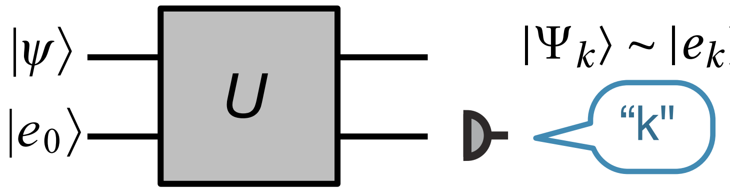
with probability

non-Hermitian evolution

$$\mathscr{P}^{\text{no click}} = \text{tr}_{\text{sys}}\left(E_0\rho(0)E_0^\dagger\right) = \left\| e^{-iH_{\text{eff}}\Delta t/\hbar}\psi(0)\right\|^2$$

Rem.: density matrix $\rho_0(\Delta t) = E_0\rho(0)E_0/\text{tr}_{\text{sys}}(\ldots)$

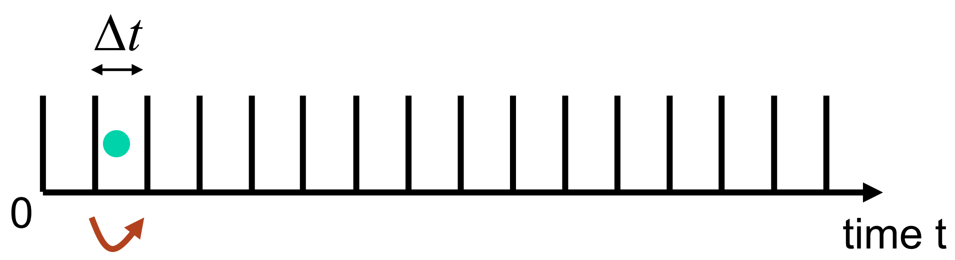
Reminder: Quantum operations



$|\Psi_k\rangle \sim |e_k\rangle E_k|\psi\rangle$

probability $p_k = \left\|E_k\psi\right\|^2$

Second integration step



Wave function after second step

$$|\Psi(2\Delta t)\rangle = U(\Delta t)\,|\Psi(\Delta t)\rangle$$

explicitly

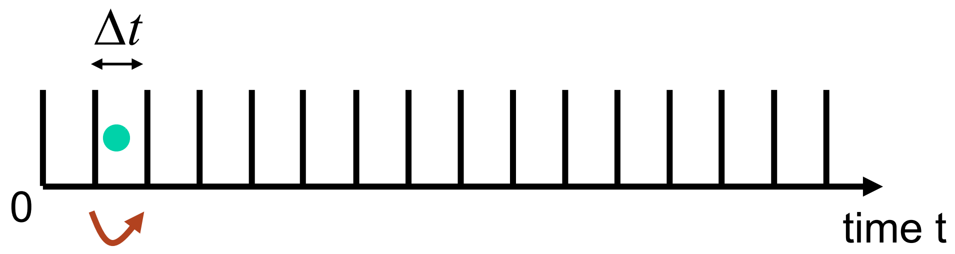
$$|\Psi(2\Delta t)\rangle = \Big\{ \hat{1} - iH_{\text{sys}}\Delta t + \sqrt{\gamma}\,c\int_{\Delta t}^{2\Delta t} b^\dagger(t)\,dt - \sqrt{\gamma}\,c^\dagger \int_{\Delta t}^{2\Delta t} b(t)\,dt$$
$$+ (-i)^2\gamma c^\dagger c \int_{\Delta t}^{2\Delta t} dt \int_{\Delta t}^{t_2} dt'\, b(t) b^\dagger(t') + \ldots \Big\}$$
$$\times \Big\{ \hat{1} - iH_{\text{eff}}\,\Delta t + \sqrt{\gamma}\,c \int_0^{\Delta t} b^\dagger(t)\,dt \Big\} |\Psi(0)\rangle$$

commute due to

$$\left[b(t), b^\dagger(s)\right] = \delta_s(t-s)$$

The increments $\Delta B^\dagger(\Delta t)$ and $\Delta B(\Delta t)$ etc. *commute* with $\Delta B^\dagger(0)$ because the time intervals do not overlap, i.e. we can commute $\Delta B(\Delta t)$ through the $\Delta B^\dagger(0)$ until we can apply $\Delta B(\Delta t)|\text{vac}\rangle = 0$.

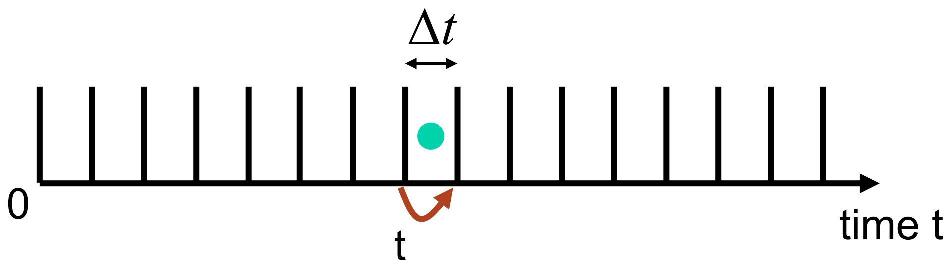
Second integration step

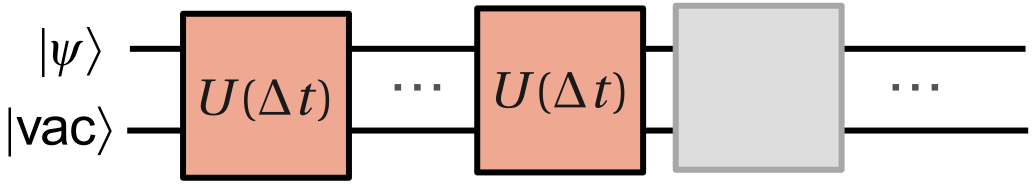


Wave function after the second step:

$$|\Psi(2\Delta t)\rangle = \left(\hat{1} - iH_{\text{eff}}\Delta t + \sqrt{\gamma}c\Delta B^{\dagger}(\Delta t)\right)\left(\hat{1} - iH_{\text{eff}}\Delta t + \sqrt{\gamma}c\Delta B^{\dagger}(0)\right)|\Psi(0)\rangle$$

Integration up to time t





Many time steps:

$$|\Psi(t+\Delta t)\rangle = \left(\hat{1} - iH_{\text{eff}}\Delta t + \sqrt{\gamma} c \Delta B^\dagger(t)\right)|\Psi(t)\rangle$$

or

$$|\Psi(t)\rangle = \prod_{i=0}^{n-1}\left(1 - iH_{\text{eff}}\Delta t + \sqrt{\gamma} c \Delta B^\dagger(t_i)\right)|\Psi(0)\rangle$$

Schrödinger Equation for $|\Psi(t)\rangle$ & Properties

- Schrödinger Equation with a discretized time steps Δt:

$$\begin{aligned}\Delta|\Psi(t)\rangle &:= |\Psi(t+\Delta t)\rangle - |\Psi(t)\rangle \\ &= \left(-iH_{\text{eff}}\Delta t + \sqrt{\gamma} c \Delta B^\dagger(t)\right)|\Psi(t)\rangle\end{aligned}$$

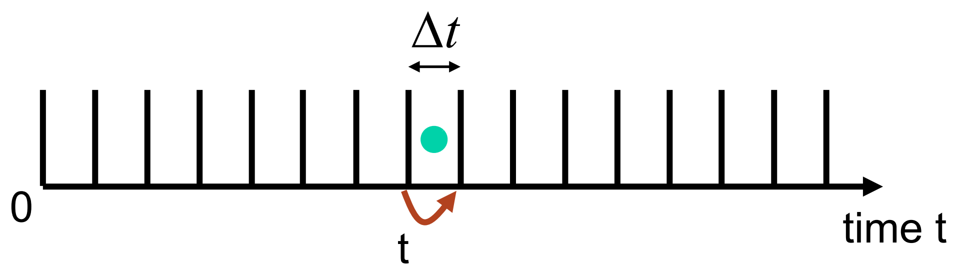
- $|\Psi(t)\rangle$ depends only on $\Delta B^\dagger(t_i)$ in the *past* $(0, t]$, and thus

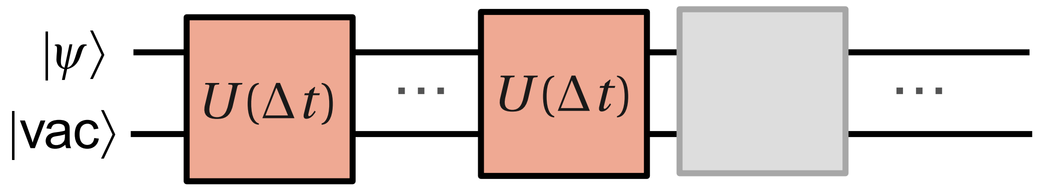
$$\Delta B(t)\, U(t)\, |\psi\rangle|\text{vac}\rangle = \Delta B(t)\, U(t)\, |\psi\rangle|\text{vac}\rangle = 0,$$

and

$$\Delta B(t)\, |\Psi(t)\rangle = 0$$

Integration up to time t





Many time steps:

$$|\Psi(t+\Delta t)\rangle = \left(\hat{1} - iH_{\text{eff}}\Delta t + \sqrt{\gamma}c\Delta B^\dagger(t)\right)|\Psi(t)\rangle$$

or

$$|\Psi(t)\rangle = \prod_{i=0}^{n-1}\left(1 - iH_{\text{eff}}\Delta t + \sqrt{\gamma}c\Delta B^\dagger(t_i)\right)|\Psi(0)\rangle$$

Schrödinger Equation for $|\Psi(t)\rangle$ & Properties

- Schrödinger Equation with a discretized time steps Δt:

$$\Delta|\Psi(t)\rangle := |\Psi(t+\Delta t)\rangle - |\Psi(t)\rangle$$
$$= \left(-iH_{\text{eff}}\Delta t + \sqrt{\gamma}c\Delta B^\dagger(t)\right)|\Psi(t)\rangle$$

Ito Quantum Stochastic Schrödinger Equation ($\Delta t \to dt$)

$$(I) \quad d|\Psi(t)\rangle = |\Psi(t+dt)\rangle - |\Psi(t)\rangle$$
$$= \left(-iH_{\text{eff}}\Delta t + \sqrt{\gamma}c\,dB^\dagger(t)\right)|\Psi(t)\rangle$$

with Ito table (vaccum input)

$$dB(t)dB^\dagger(t) = dt$$ other elements of Ito table are zero

$$\Delta B(t)\Delta B^\dagger(t)|\text{vac}\rangle = \left[\Delta B(t), \Delta B^\dagger(t)\right]|\text{vac}\rangle = \Delta t|\text{vac}\rangle$$

Interpretation of the state vector

Entangled state system + (emitted) photons … huge!

$$|\Psi(t)\rangle = e^{-iH_{\text{eff}}t}|\psi(0)\rangle \otimes |\text{vac}\rangle$$

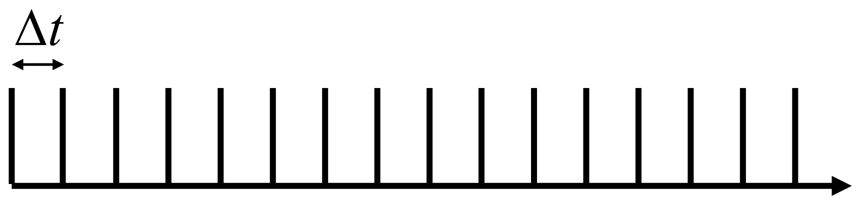
$$+\sum_{t_1} e^{-iH_{\text{eff}}(t-t_1)}\sqrt{\gamma}c\,e^{-iH_{\text{eff}}t_1}|\psi(0)\rangle \otimes \Delta B^\dagger(t_1)|\text{vac}\rangle$$

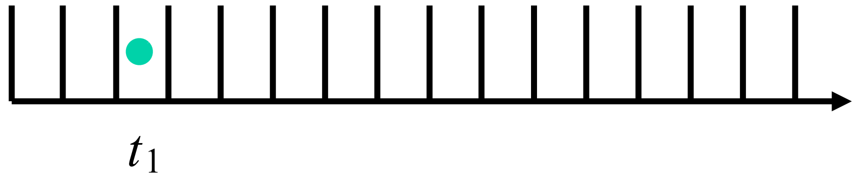
$$+\ldots$$

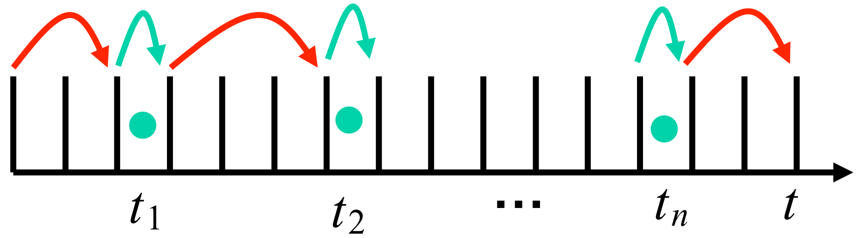
$$+\sum_{t_n>\ldots>t_1} e^{-iH_{\text{eff}}(t-t_n)}\sqrt{\gamma}c\,e^{-iH_{\text{eff}}(t_n-t_{n-1})}\ldots e^{-iH_{\text{eff}}(t_2-t_1)}\sqrt{\gamma}c\,e^{-iH_{\text{eff}}t_1}|\psi(0)\rangle \otimes \Delta B^\dagger(t_n)\ldots\Delta B^\dagger(t_1)|\text{vac}\rangle$$

atomic wavefunction conditional to sequence of photon emission $\equiv |\psi(t|t_n,\ldots,t_1)\rangle$ — photon state

$$+\ldots$$







Continuous observation with a photodetector will collapse the superposition

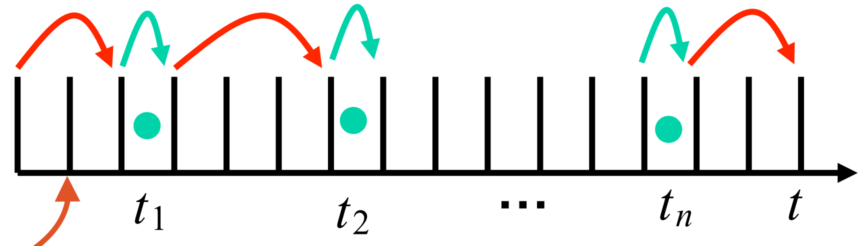
Discussion:

- Under **continuous observation** the **system wave function** evolves during time intervals of **no photon emission** as

$$|\psi(t_0)\rangle \to |\psi(t)\rangle = e^{-iH_{\text{eff}}(t-t_0)}|\psi(t_0)\rangle$$

while **emission of a photon** is associated with a **quantum jump**

$$|\psi(t)\rangle \to |\psi(t+dt)\rangle \sim c|\psi(t)\rangle$$ (quantum jump)



- **Probability of no photon emission in** $(0, t]$

$$p_0^t = \|\psi(t|)\|^2 \equiv \left\| e^{-iH_{\text{eff}}t/\hbar}\psi(0) \right\|^2$$

- **Probability density for** n**-photon emission** at times $t_1 < t_2 < \ldots < t_n$ in $(0, t]$

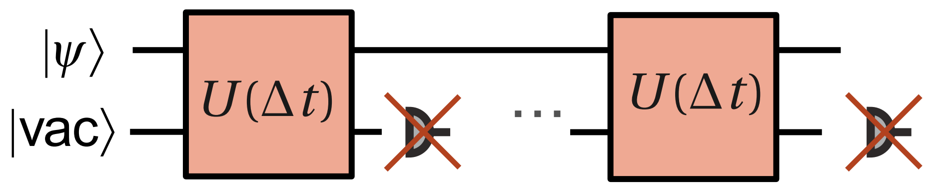
$$p_0^t(t_1, \ldots, t_n) = \|\psi(t|t_n, \ldots, t_1)\|^2 \equiv \left\| e^{-iH_{\text{eff}}(t-t_n)/\hbar}\sqrt{\gamma}ce\ldots\sqrt{\gamma}ce^{-iH_{\text{eff}}t_1/\hbar}\psi(0) \right\|^2$$

Instead of *solving* for ρ with the master equation, we can *simulate* stochastic wavefunctions ψ and *average* over quantum trajectories $\rho = \langle\langle\, |\psi\rangle\langle\psi|\, \rangle\rangle_{\text{st}}$

We have calculated the complete photon statistics

Master Equation

Reduced system density operator



$$\rho(t) \equiv \mathrm{tr}_B |\Psi(t)\rangle\langle\Psi(t)|$$

Master equation: taking the $\Delta t \to dt$ (coarse grained derivative)

$$\begin{aligned}\dot{\rho}(t) &= -iH_{\text{eff}}\rho(t) + \rho(t)\, iH_{\text{eff}}^{\dagger} + \gamma c \rho(t) c^{\dagger} \\ &= -i\left[H_{\text{sys}}, \rho(t)\right] + \frac{1}{2}\gamma\left(2c\rho(t)c^{\dagger} - c^{\dagger}c\rho(t) - \rho(t)c^{\dagger}c\right) \\ &\equiv \mathscr{L}\rho(t)\end{aligned}$$

Lindblad

jump operator

Rem.: check $\mathrm{tr}_{\text{sys}}\rho = 1$

Master Equation

Proof:

$$
\begin{aligned}
\Delta\rho(t) &:= \rho(t+\Delta t) - \rho(t) \\
&= \mathrm{Tr}_B\left\{\left(1 - iH_{\text{eff}}\Delta t + \sqrt{\gamma}c\Delta B^\dagger(t)\right)|\Psi(t)\rangle\langle\Psi(t)|\left(1 + iH_{\text{eff}}^\dagger\Delta t + \sqrt{\gamma}c^\dagger\Delta B(t)\right)\right\} \\
&\quad - \mathrm{Tr}_B\left\{|\Psi(t)\rangle\langle\Psi(t)|\right\} \\
&= \mathrm{Tr}_B\{-iH_{\text{eff}}\Delta t|\Psi(t)\rangle\langle\Psi(t)| + |\Psi(t)\rangle\langle\Psi(t)|iH_{\text{eff}}^\dagger\Delta t \\
&\quad + \sqrt{\gamma}c\Delta B^\dagger(t)|\Psi(t)\rangle\langle\Psi(t)| + |\Psi(t)\rangle\langle\Psi(t)|\sqrt{\gamma}c^\dagger\Delta B(t) \\
&\quad + \gamma c\Delta B^\dagger(t)|\Psi(t)\rangle\langle\Psi(t)|c^\dagger\Delta B(t)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\Delta\rho(t) &:= \rho(t+\Delta t) - \rho(t) \\
&= \mathrm{tr}_B\left\{\ldots\Delta B^\dagger\cancel{(t)|\Psi(t)\rangle}\langle\Psi(t)| + \ldots|\Psi\cancel{(t)\rangle\langle\Psi(t)|}\Delta B(t) + \ldots\Delta B^\dagger(t)|\Psi(t)\rangle\langle\Psi(t)|\Delta B(t)\ldots\right\}
\end{aligned}
$$

use $\Delta B(t)|\Psi(t)\rangle = 0$, $\langle\Psi(t)|\Delta B^\dagger = 0$ and $\left[\Delta B(t), \Delta B^\dagger(t)\right] = \Delta t$

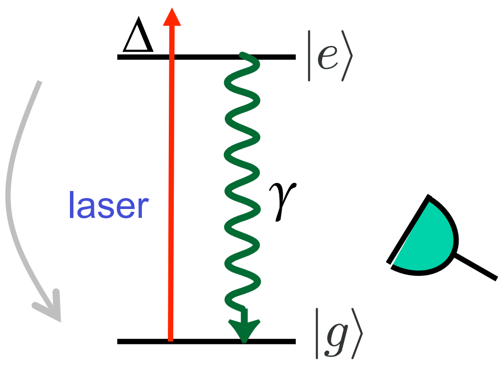
Some simple, and not so simple examples

1. Driven Two-Level System undergoing Spontaneous Emission

Quantum jumps and master equation

- **two-level system**

$c \rightarrow \sigma_- = |g\rangle\langle e|$



Hamiltonian

$$\tilde{H}_{\rm sys} = -\Delta |e\rangle\langle e| - \frac{1}{2}\Omega\sigma_+ - \frac{1}{2}\Omega^*\sigma_-$$

jump operator

$$c \rightarrow \sigma_- = |g\rangle\langle e|$$

- **a quantum jump (detection of an emission)** *prepares the atom in the ground state*

$$|\psi(t)\rangle \;\longrightarrow\; |\psi(t+dt)\rangle \;\sim\; \sigma_-|\psi(t)\rangle \;\equiv\; |g\rangle$$

probability for click in time interval *(t,t+dt]* $\quad p_{(t,t+dt]} = \gamma |\langle e|\psi(t)\rangle|^2 dt$

- **master equation (Optical Bloch Equations)**

$$\frac{d}{dt}\rho = -i\left[H_{\rm sys}, \rho\right] + \tfrac{1}{2}\gamma(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-)$$

Optical Bloch Equations

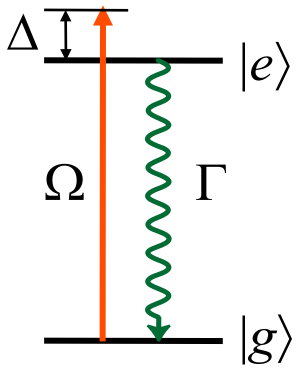
- **Writing out the optical Bloch equations**

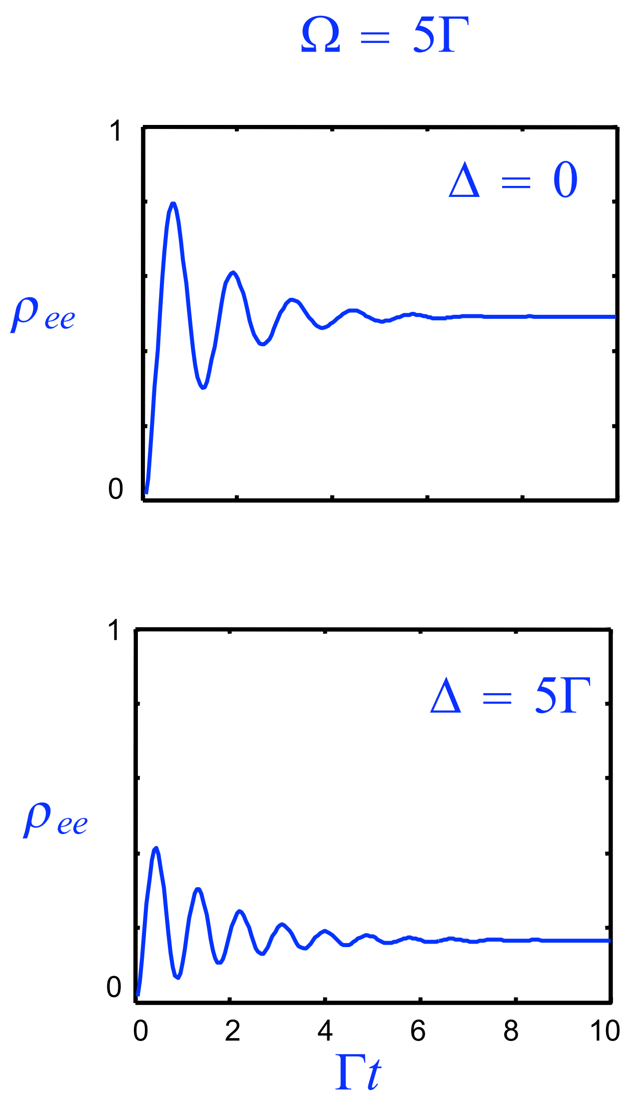
$$\begin{aligned}
\frac{d}{dt}\bar{\rho}_{eg} &= (i\Delta - \frac{1}{2}\Gamma)\bar{\rho}_{eg} - i\frac{1}{2}\Omega\left(\rho_{ee} - \rho_{gg}\right), \\
\frac{d}{dt}\bar{\rho}_{ge} &= (-i\Delta - \frac{1}{2}\Gamma)\bar{\rho}_{eg} + i\frac{1}{2}\Omega^*\left(\rho_{ee} - \rho_{gg}\right), \\
\frac{d}{dt}\rho_{ee} &= -\Gamma\rho_{ee} - i\frac{1}{2}\Omega^*\bar{\rho}_{eg} + i\frac{1}{2}\Omega\bar{\rho}_{ge}, \\
\frac{d}{dt}\rho_{gg} &= +\Gamma\rho_{ee} + i\frac{1}{2}\Omega^*\bar{\rho}_{eg} - i\frac{1}{2}\Omega\bar{\rho}_{ge}
\end{aligned}$$

or

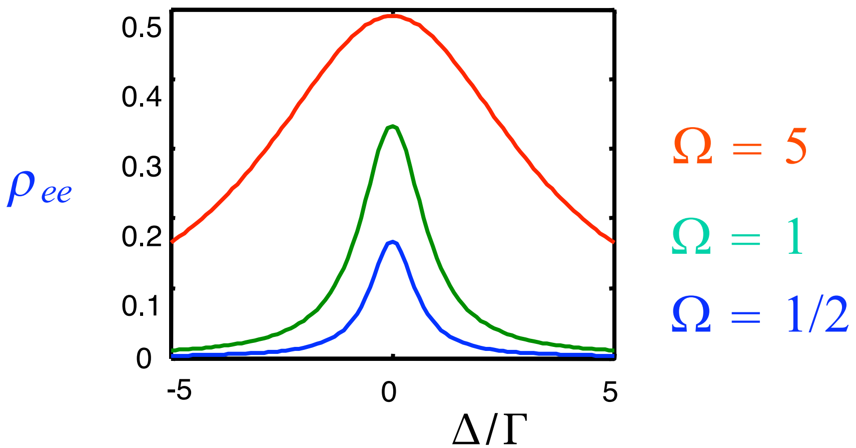
$$\frac{d}{dt}\begin{bmatrix}\bar{\rho}_{eg}\\ \bar{\rho}_{ge}\\ \rho_{ee}\\ \rho_{gg}\end{bmatrix} = \begin{bmatrix} i\Delta - \frac{1}{2}\Gamma & 0 & -i\frac{1}{2}\Omega & i\frac{1}{2}\Omega \\ 0 & -i\Delta - \frac{1}{2}\Gamma & i\frac{1}{2}\Omega^* & -i\frac{1}{2}\Omega^* \\ -i\frac{1}{2}\Omega^* & i\frac{1}{2}\Omega & -\Gamma & 0 \\ +i\frac{1}{2}\Omega^* & -i\frac{1}{2}\Omega & +\Gamma & 0 \end{bmatrix}\begin{bmatrix}\bar{\rho}_{eg}\\ \bar{\rho}_{ge}\\ \rho_{ee}\\ \rho_{gg}\end{bmatrix}$$

damped Rabi oscillations



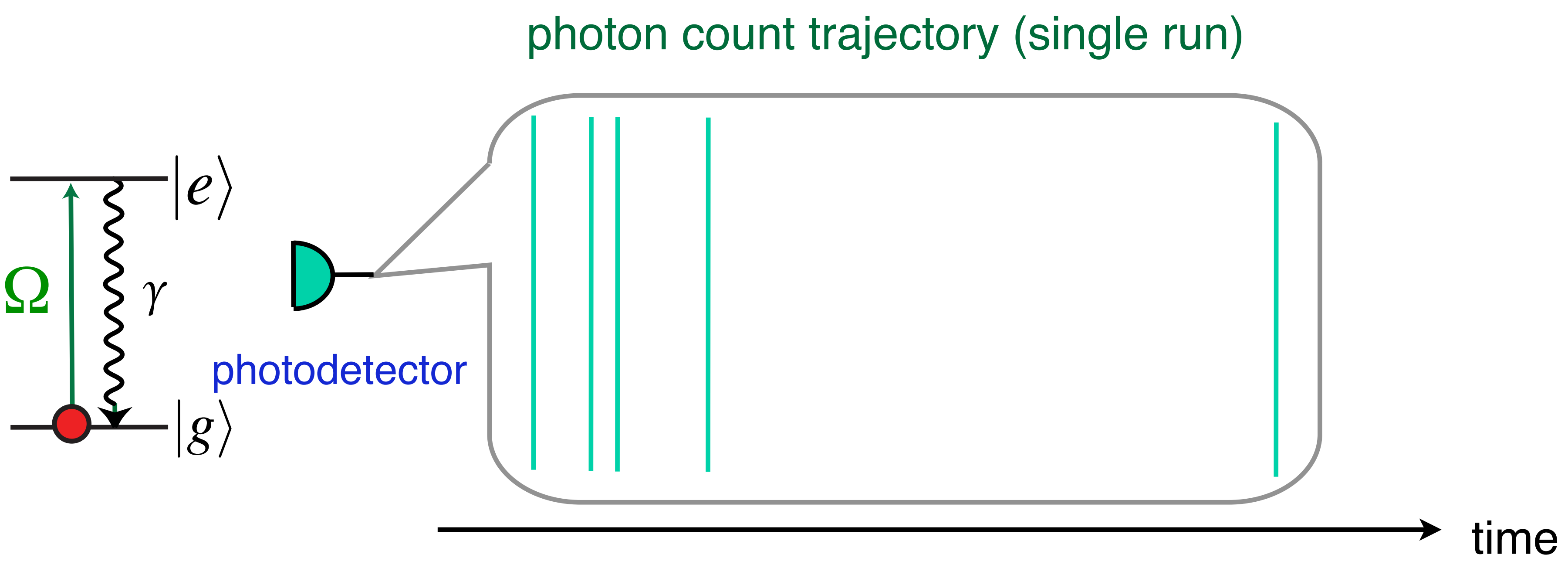


stationary solutions



2. Driven Two-Level System undergoing Spontaneous Emission

Conditional time evolution



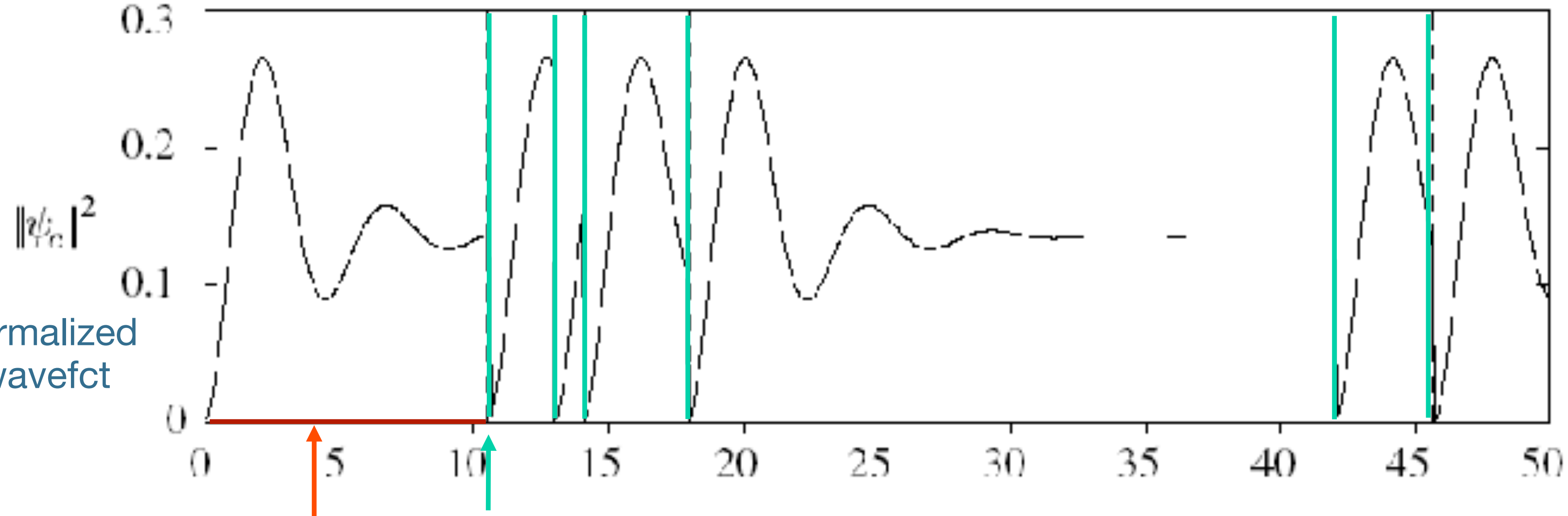
Evolution of the atom, *given* this counting trajectory?

conditional time evolution / wave function

2. Driven Two-Level System undergoing Spontaneous Emission

Conditional time evolution

Fig.: typical quantum trajectory (upper state population)



normalized wavefct

γt

click:

"quantum jump" = effect of detecting a photon on system

$$|\psi_{\text{sys}}(t)\rangle \to \sqrt{\gamma}\sigma^- \, |\psi_{\text{sys}}(t)\rangle$$

no click:

$$|\psi_{\text{sys}}(t)\rangle = e^{-iH_{\text{eff}}t} \, |\psi_{\text{sys}}(0)\rangle$$

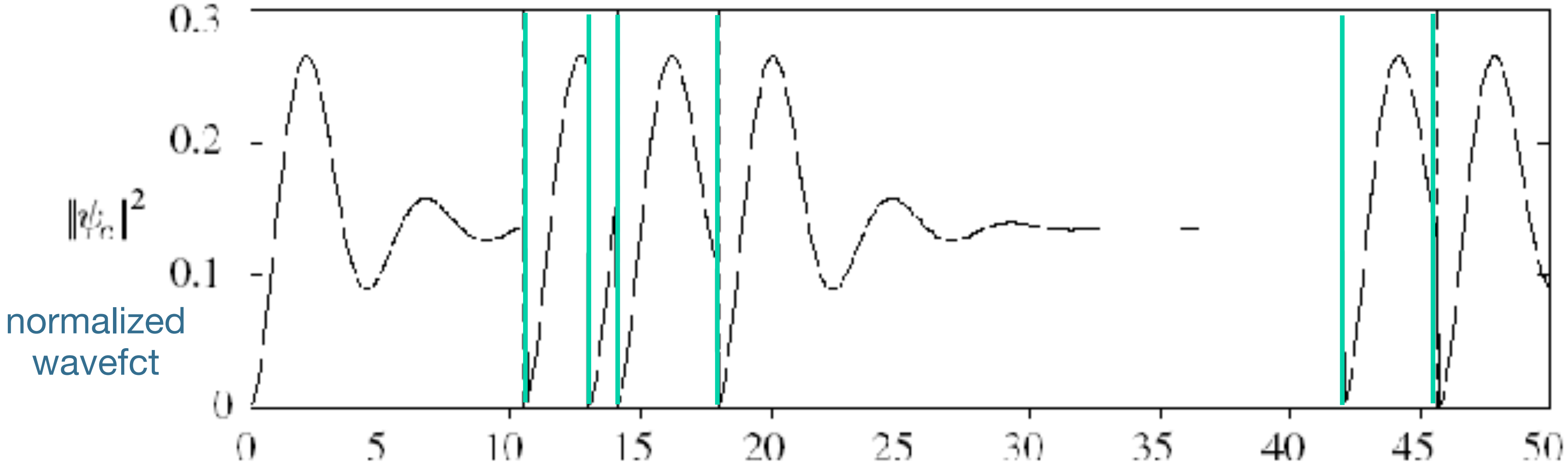
with Wigner -Weisskopf Hamiltonian

$$H_{\text{eff}} = \left(\omega_{eg} - i\frac{1}{2}\gamma\right)\sigma_{ee} + \ldots$$

2. Driven Two-Level System undergoing Spontaneous Emission

Conditional time evolution

Fig.: typical quantum trajectory (upper state population)



- Monte Carlo wave function simulation

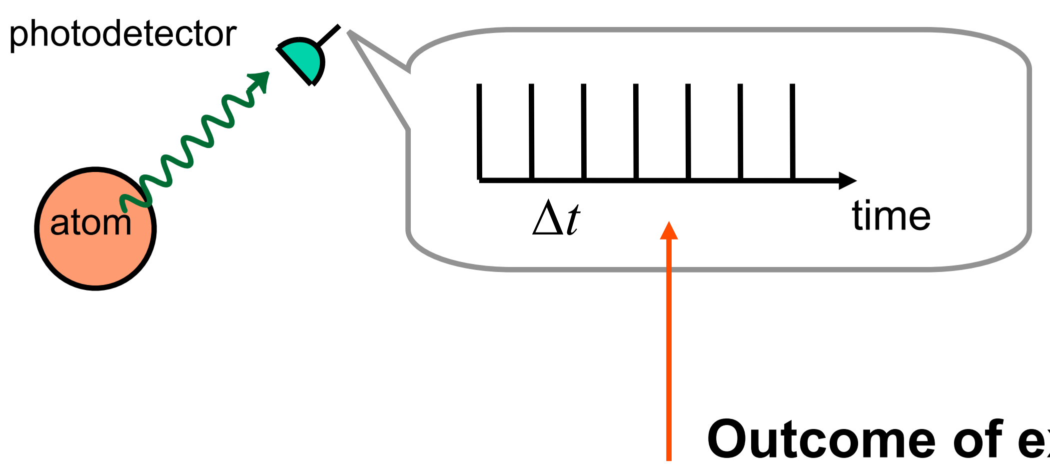
stochastic wavefunction $|\psi(t)\rangle_{\text{sys}}$ (dim d)

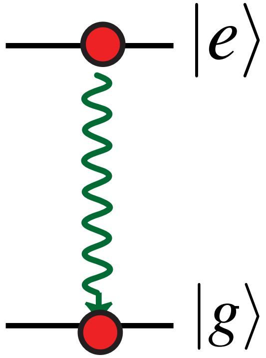
reduced density matrix $\rho(t) = \langle\, |\psi_{\text{sys}}(t)\rangle \langle\psi_{\text{sys}}(t)|\, \rangle_{\text{st}}$

DMRG + wave function simulation $\longleftrightarrow$ density matrix $\rho_{\text{sys}}(t)$ (dim $d \times d$)

3. Decaying Two-Level Atom

Conditional time evolution





$$|\psi_c(0)\rangle = c_g|g\rangle + c_e|e\rangle$$

Outcome of experiment:
We observe NO photon up to time t

Question: what is the state of the atom conditional to this observation after time t?

Answer: $$|\psi_c(t)\rangle = \frac{e^{-iH_{\text{eff}}t/\hbar}|\psi_c(0)\rangle}{\|\ldots\|} = \frac{c_g|g\rangle + c_e e^{-\gamma t/2}|e\rangle}{\|\ldots\|}$$

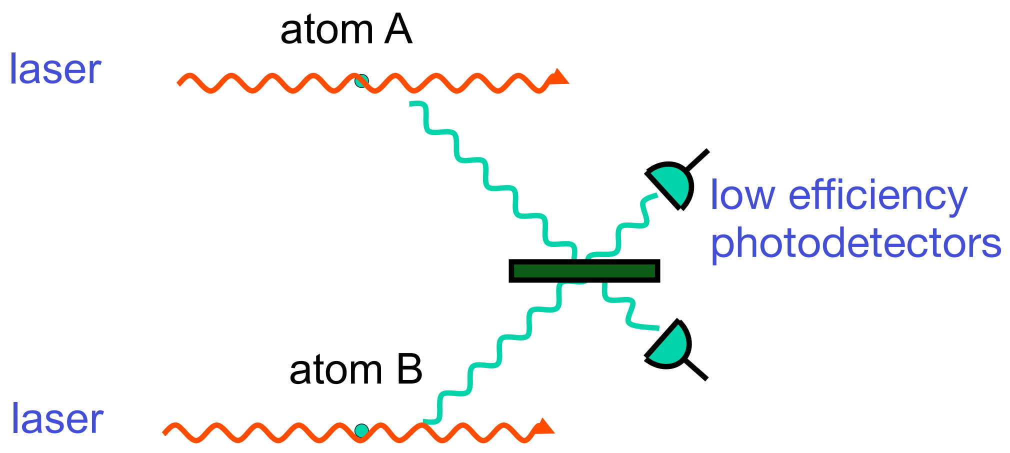
$\longrightarrow |g\rangle$ for $t \to \infty$

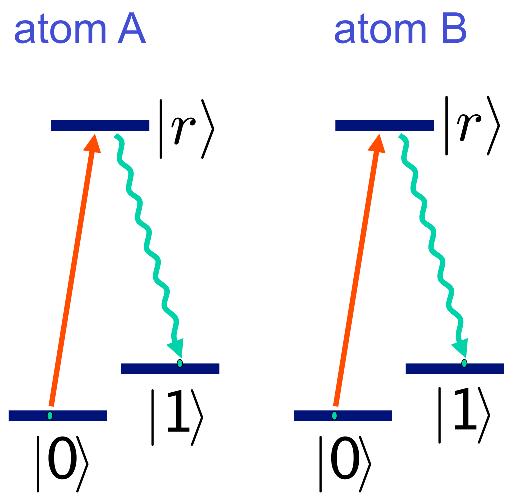
We *learn* that the system is in the ground state

4. Entanglement of Distant Atoms from Observation (Detector Click)

Preparation of EPR / Bell pairs of entangled atoms from conditional evolution

- **System:** two atoms with ground states $|0\rangle$, $|1\rangle$ and excited state $|r\rangle$





- Weak (short) laser pulse, so that the excitation probability is small.

- If no detection, pump back and start again.

- If detection, an entangled state is created.

$$\sim |0,1\rangle + |1,0\rangle$$

Process:

- preparation (by optical pumping)

$$|\Psi(t=0)\rangle \;=\; |\text{vac}\rangle|0\rangle_1|0\rangle_2$$

- excitation by a weak short laser pulse

$$\begin{aligned}|\Psi(t=0^+)\rangle &= |\text{vac}\rangle\,(|0\rangle_2+\epsilon|r\rangle_2)(|0\rangle_2+\epsilon|r\rangle_2)\\ &= |\text{vac}\rangle\,[|0\rangle_1|0\rangle_2+\epsilon(|r\rangle_1|0\rangle_2+|0\rangle_1|r\rangle_2)+O(\epsilon^2)]\end{aligned}$$

- spontaneous emission

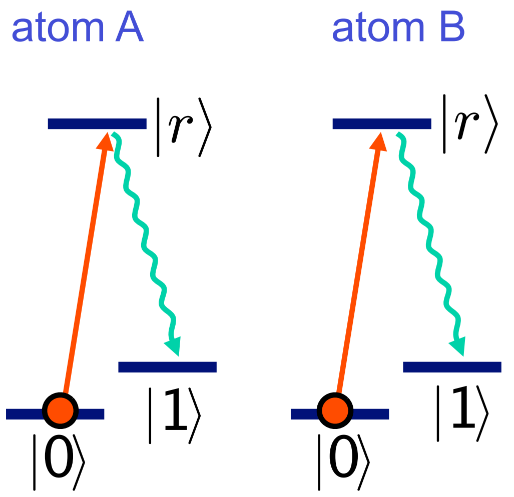
$$\begin{aligned}|\Psi(t>0^+)\rangle &= [|0\rangle_1|0\rangle_2+\epsilon e^{-\gamma t/2}(|r\rangle_1|0\rangle_2+|0\rangle_1|r\rangle_2)]\otimes|\text{vac}\rangle\\ &+\sum_{t_1}\Delta B_1^\dagger(t_1)|\text{vac}\rangle\otimes\epsilon\sqrt{\gamma}\,e^{-\gamma t_1/2}|1\rangle_1|0\rangle_2\\ &+\Delta B_2^\dagger(t_1)|\text{vac}\rangle\otimes\epsilon\sqrt{\gamma}\,e^{-\gamma t_1/2}|0\rangle_1|1\rangle_2+O(\epsilon^2)\end{aligned}$$

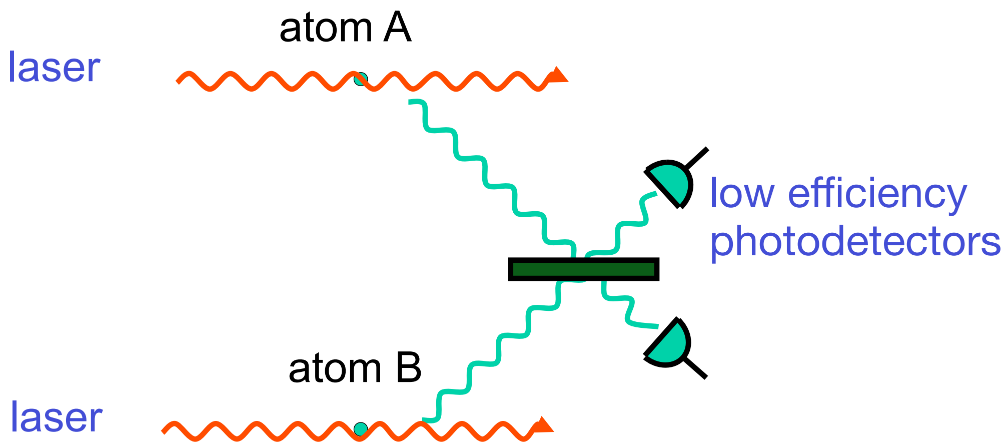
- We observe the fluorescence through a beam splitter

$$\Delta B_{1,2}^\dagger \to \frac{1}{\sqrt{2}}(\Delta B_1^\dagger \pm \Delta B_2^\dagger)$$

- Observation of a click prepares Bell state

$$|1\rangle_1|0\rangle_2+|0\rangle_1|1\rangle_2$$

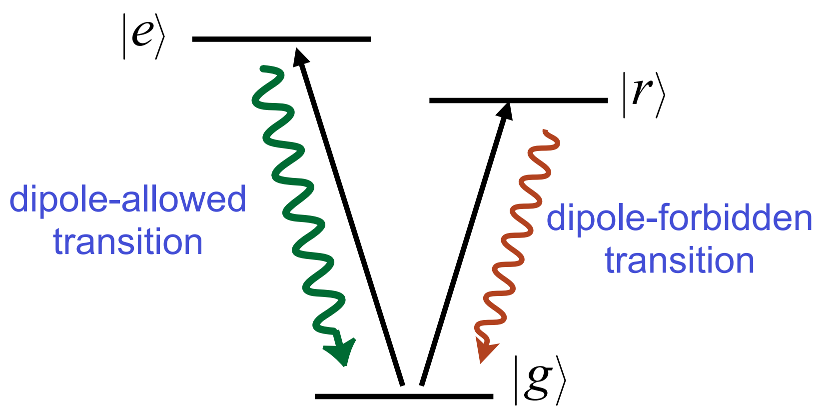




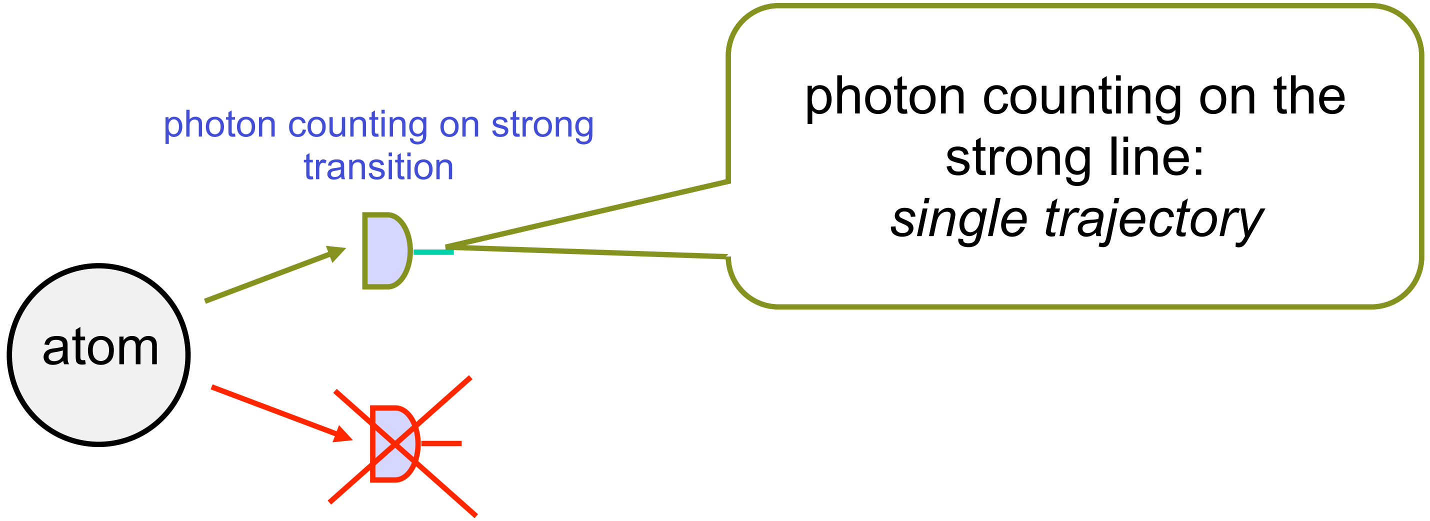
5. Quantum jumps, or reading out qubits

Three level atom

- three level atom

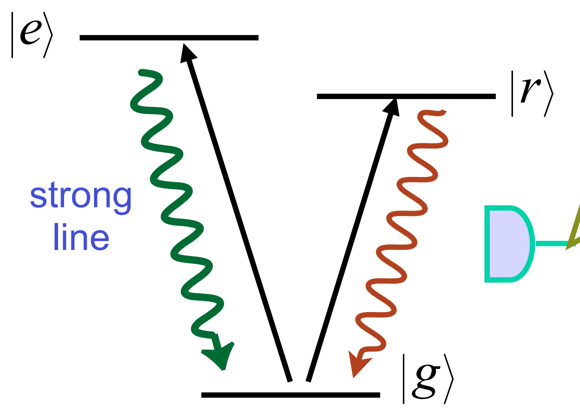


- single atom photon counting

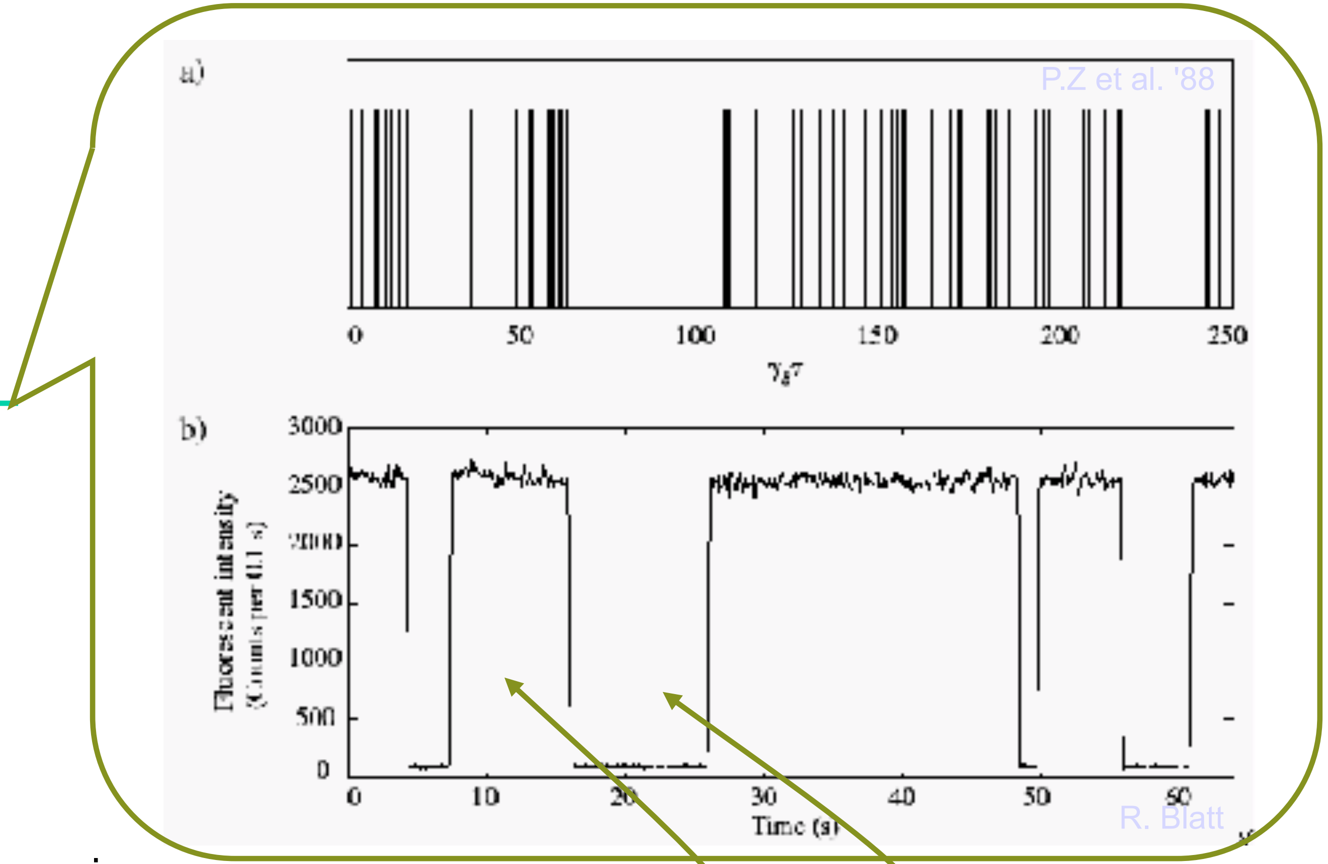


5. Quantum jumps, or reading out qubits

Three level atom



photon counting on strong transition



simulation of photon statistics

measurement

- ✓ atomic density matrix conditional to observing an emission window

$$\rho_c(t) \longrightarrow |r\rangle\langle r|$$

preparation in metastable state

- ✓ state measurement with 100% efficiency

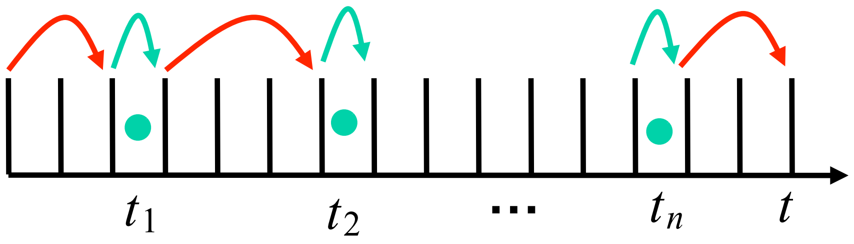
$$\psi = \alpha|g\rangle + \beta|r\rangle$$

$|\alpha|^2$ ... probability **NO** window

$|\beta|^2$ ... probability window

... reading out qubits

Unravelling of the master equation



Unraveling of the Master Equation

***n*-photon contributions to the reduced density matrix:** the total system wave function $|\Psi(t)\rangle$ can be written as

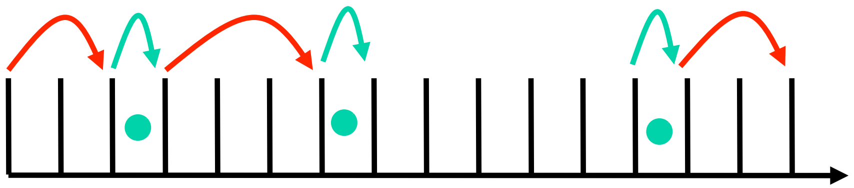
$$|\Psi(t)\rangle = |\psi(t|)\rangle \otimes |\mathrm{vac}\rangle + \ldots + \sum_{t_n > \ldots > t_1} |\psi(t|t_n, \ldots, t_1)\rangle \otimes \Delta B^\dagger(t_n) \ldots \Delta B^\dagger(t_1)|\mathrm{vac}\rangle + \ldots$$

which is a sum of $n = 0, 1, 2, \ldots$ photon contributions.

In a similar way we can decompose the reduced density operator of the system as sum over n-photon contributions,

$$\rho(t) = \mathrm{Tr}_B |\Psi(t)\rangle\langle\Psi(t)| = \sum_{n=0}^{\infty} \rho^{(n)}(t),$$

with



$$\rho^{(0)}(t) = |\psi(t|)\rangle\langle\psi(t|)|$$

$$\rho^{(n)}(t) = \int_0^t dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} dt_1 |\psi(t|t_n,\ldots,t_1)\rangle\langle\psi(t|t_n,\ldots,t_1)| \quad (n=1,2,\ldots)$$

where $\rho^{(n)}(t) := \mathrm{Tr}_B \hat{P}^{(n)}|\Psi(t)\rangle\langle\Psi(t)|$ with $\hat{P}^{(n)}$ the projector on the n-photon subspace.

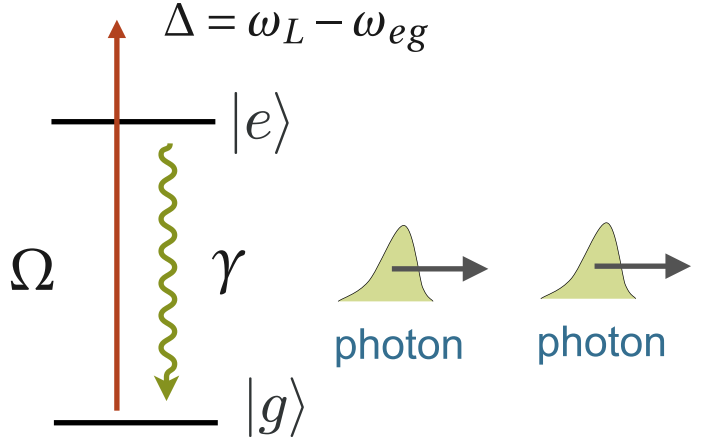
Interpretation: We have achieved here a decomposition of the reduced density operator into *pure state system wave functions* $|\psi(t|t_n,\ldots,t_1)\rangle$. Our earlier discussion showed that $|\psi(t|t_n,\ldots,t_1)\rangle$ describes the evolution of the system in the time interval $(0,t]$ with (exactly) n photons emitted (quantum jumps) at times $t_1 < \ldots < t_n$. The n-photon contribution to the density matrix $\rho^{(n)}(t)$ is obtained by integrating over these emission times, and the total density matrix is obtained by summing over these n-photon contributions, $\rho(t) = \sum_{n=0}^{\infty} \rho^{(n)}(t)$. We call the above construction unraveling of the master equation.

Equations of motion for $\rho^{(n)}(t)$: we have the hierarchy of equations

$$\dot{\rho}^{(0)}(t) = -\frac{i}{\hbar} H_{\text{eff}} \rho^{(0)}(t) + \rho^{(0)}(t) \frac{i}{\hbar} H_{\text{eff}}^{\dagger}$$

$$\dot{\rho}^{(n)}(t) = -\frac{i}{\hbar} H_{\text{eff}} \rho^{(n)}(t) + \rho)^{(n)}(t) \frac{i}{\hbar} H_{\text{eff}}^{\dagger} + \gamma c \rho^{(n-1)}(t) c^{\dagger} \quad (n = 1, 2, \ldots)$$

Proof: Take time derivative $\rho^{(n)}(t) = \ldots$ and use $\frac{d}{dt}|\psi(t|\ldots\rangle = -\frac{i}{\hbar} H_{\text{eff}}|\psi(t|\ldots\rangle$



$$c \equiv \sigma_- = |g\rangle\langle e|$$

Master equation: $\rho(t) \equiv \text{tr}_B |\Psi(t)\rangle\langle\Psi(t)|$

recycling term

$$\dot{\rho}(t) = -i H_{\text{eff}} \rho(t) + \rho(t) i H_{\text{eff}}^{\dagger} + \gamma c \rho(t) c^{\dagger}$$

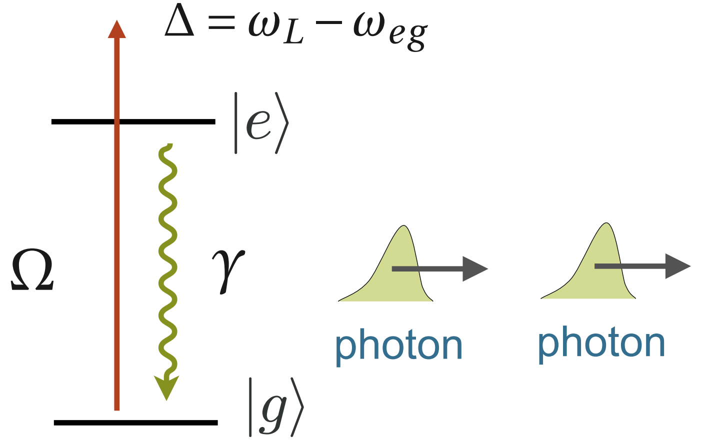
$$= -i \left[H_{\text{sys}}, \rho(t) \right] + \frac{1}{2} \gamma \left(2 c \rho(t) c^{\dagger} - c^{\dagger} c \rho(t) - \rho(t) c^{\dagger} c \right)$$

$$\equiv \mathscr{L} \rho(t)$$

quantum jump operator c

Equations of motion for $\rho^{(n)}(t)$: we have the hierarchy of equations

$$\dot{\rho}^{(0)}(t) = -\frac{i}{\hbar} H_{\text{eff}}\rho^{(0)}(t) + \rho^{(0)}(t)\frac{i}{\hbar}H_{\text{eff}}^{\dagger}$$
$$\dot{\rho}^{(n)}(t) = -\frac{i}{\hbar} H_{\text{eff}}\rho^{(n)}(t) + \rho)^{(n)}(t)\frac{i}{\hbar}H_{\text{eff}}^{\dagger} + \gamma c \rho^{(n-1)}(t)c^{\dagger} \quad (n = 1, 2, \ldots)$$



$c \equiv \sigma_- = |g\rangle\langle e|$

Discussion:

- This is a hierarchy of equations for $\rho^{(n)}(t)$ where the recycling term plays the role of a feeding term connecting the n and $n-1$ photon contributions.
- Summing over n we obtain the master equation back: $\dot{\rho} = \mathscr{L}\rho$.
- n-photon probabilities

$$P^{(0)}(t) = \text{Tr}_{\text{sys}}\rho^{(0)}(t) = p_0^t,$$
$$P^{(n)}(t) = \text{Tr}_{\text{sys}}\rho^{(n)}(t) = \int_0^t dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} dt_1 \, p_0^t(t_1, \ldots, t_n) \quad (n = 1, 2, \ldots$$

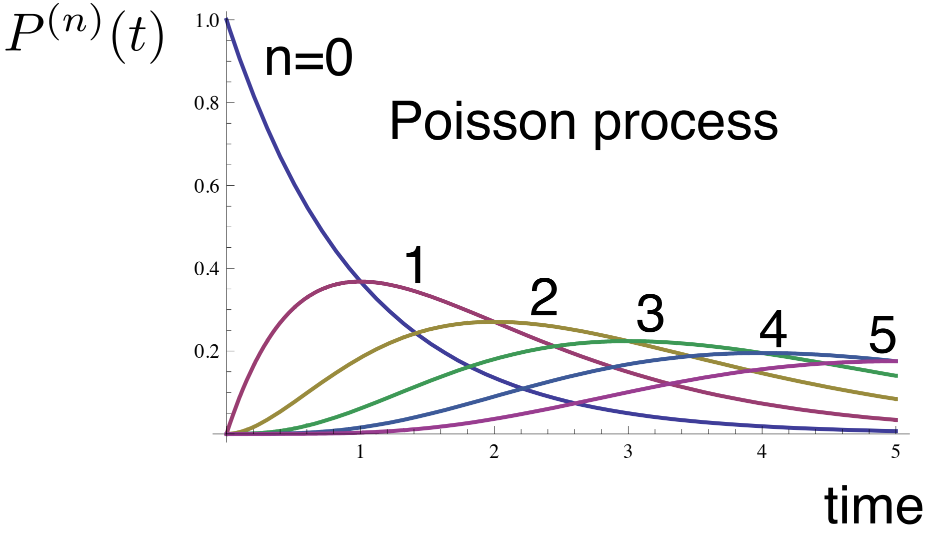
where from earlier $p_0^t \equiv \left\| \psi(t|) \right\|^2$ and $p_0^t(t_1, \ldots, t_n) \equiv \left\| \psi(t|t_n, \ldots, t_1) \right\|^2$.

Poisson process

$$\dot{P}^{(0)}(t) = -\kappa P^{(0)}(t)$$
$$\dot{P}^{(n)}(t) = -\kappa P^{(n)}(t) + \kappa P^{(n-1)}(t) \quad (n = 1, 2, \ldots)$$

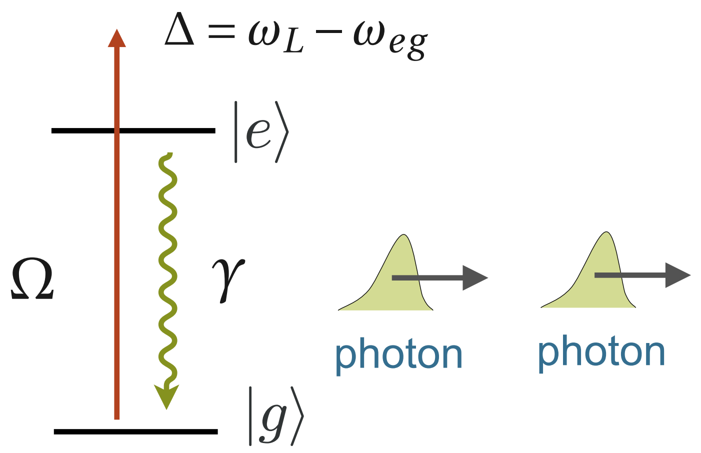
with solution

$$P^{(n)}(t) = \frac{(\kappa t)^n}{n!} e^{-\kappa t}$$



Remark: the light statistics from a driven two-level atom is *not* Poissonian

Photon Statistics: Characteristic Functions & Density Operators etc.



$c \equiv \sigma_- = |g\rangle\langle e|$

Example: Poisson process

- characteristic function:

$$\chi(s) := \sum_{n=0}^{\infty} e^{ns} P^{(n)}(t) = e^{\kappa t (e^s - 1)}$$

- normalization:

$$1 = \sum_{n=0}^{\infty} P^{(n)}(t) = \chi(s=0)$$

- mean number of jumps:

$$\langle n \rangle_t = \sum_{n=0}^{\infty} n P^{(n)}(t) = \frac{\partial \chi(0)}{\partial s} = \kappa t$$

- variance: with $\langle n^2 \rangle = \frac{\partial^2 \chi(0)}{\partial s^2}$

$$\begin{aligned} \Delta n^2 &= \langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle \\ &= \kappa t \end{aligned}$$

characteristic density operator: We definee

$$\hat{\chi}(s,t) := \sum_{n=0}^{\infty} e^{ns}\rho^{(n)}(t)$$

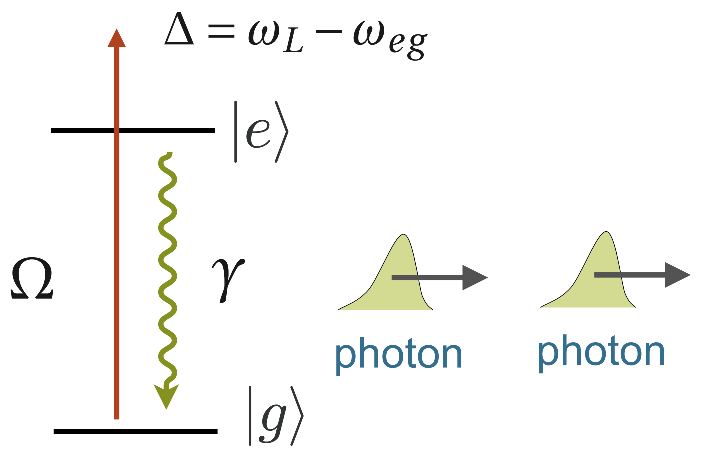
which obeys the equation

$$\frac{\partial}{\partial t}\hat{\chi}(s,t) = \mathscr{L}\hat{\chi}(s,t) + (e^{s}-1)\mathscr{J}\hat{\chi}(s,t) \quad (\hat{\chi}(s,0) = \rho(0))$$

Note: to solve for the photon statistics we must solve this equation to compute the

characteristic functional for the photon statistics

$$\chi(s,t) = \sum_{n=0}^{\infty} e^{ns}P^{(n)}(t) = \mathrm{Tr}\hat{\chi}(s,t)$$



$$c \equiv \sigma_{-} = |g\rangle\langle e|$$

Example: Photon statistics of the driven two-level atom

- mean number of photons

$$\frac{d}{dt}\langle n\rangle = \Gamma \rho_{ee}(t)$$

  i.e. for long times $\langle n\rangle \to \Gamma \rho_{ee} t$

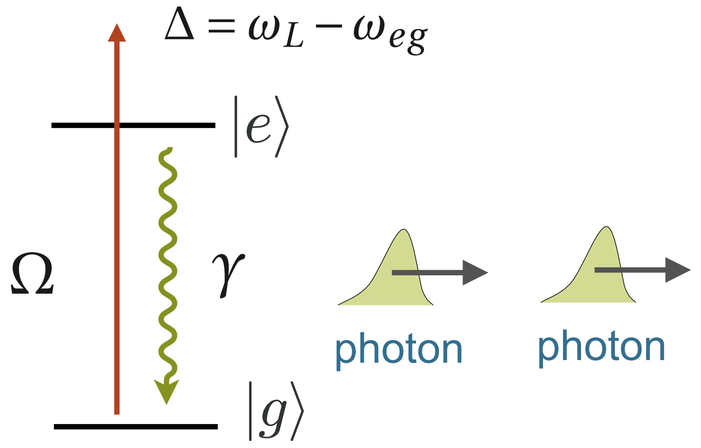
- photon number fluctuations

$$\frac{\Delta n^2}{\langle n\rangle} = \frac{\langle n^2\rangle - \langle n\rangle^2}{\langle n\rangle} = 1 + Q$$

  with Mandel Q-parameter (which measures the deviation from Poisson statistics)

$$Q = \frac{\frac{1}{2}\Omega^2\left(\Delta^2 - \frac{3}{4}\Gamma^2\right)}{\left(\Delta^2 + \frac{1}{4}\Gamma^2 + \frac{1}{2}\Omega^2\right)^2}$$

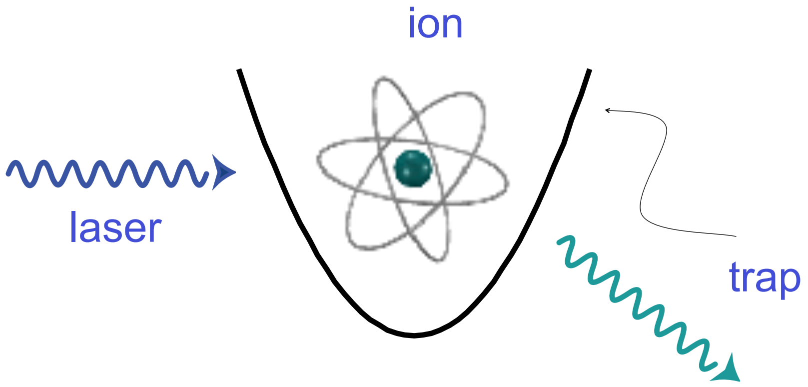
  Discussion: For weak fields and strong driving the photon statistics is Poissonian ($Q \approx 0$), on resonance and for medium driving we have sup-Poissonian fluctuations.

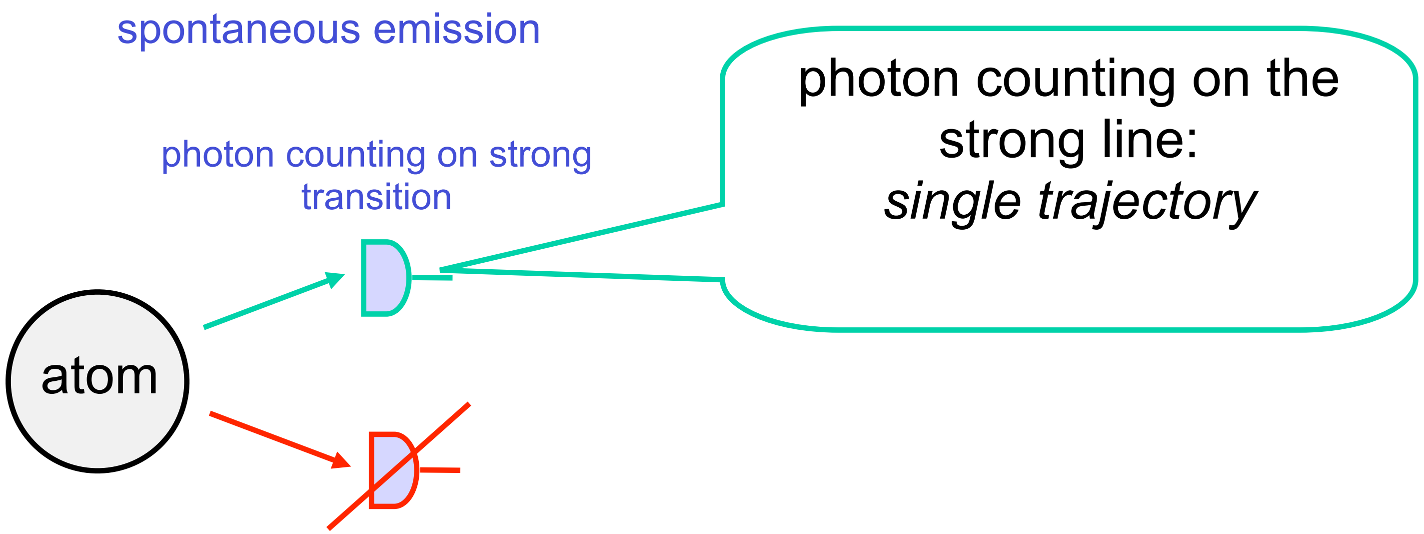


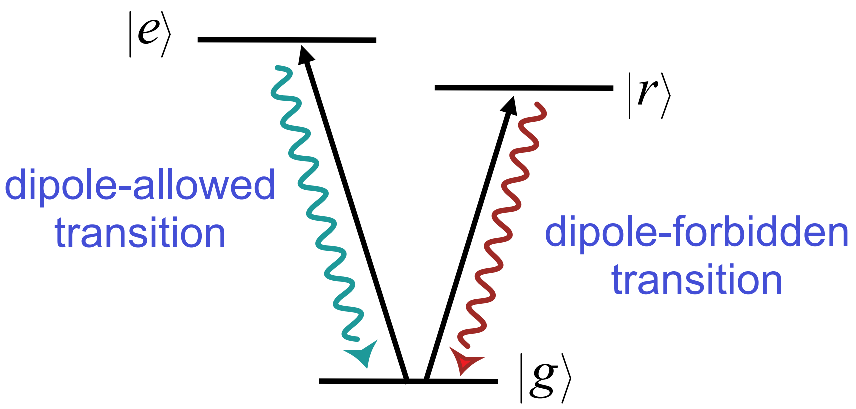
$c \equiv \sigma_- = |g\rangle\langle e|$

Example: Quantum Jumps in Three-Level Atoms

Background: Experiments with single trapped ions represent an *experimental realization of continuous observation of a single quantum system* in the context of quantum optics. The observation of quantum jumps in a three level system is probably one of the best-known examples in quantum optics where quantum jumps are “seen” in experiments where fluorescence from single ions is observed by continuously monitoring the atom with a photodetector.

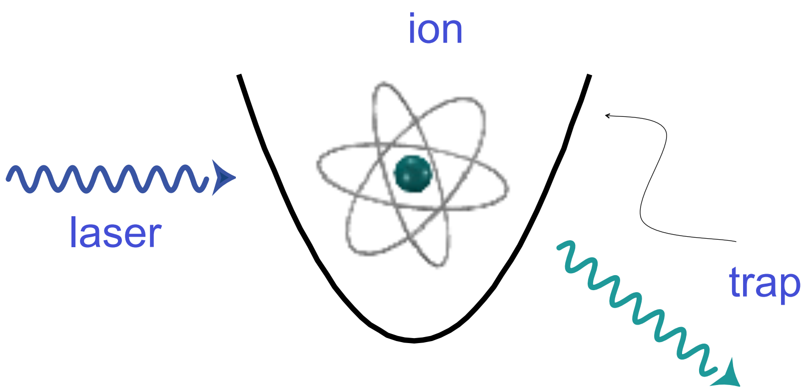


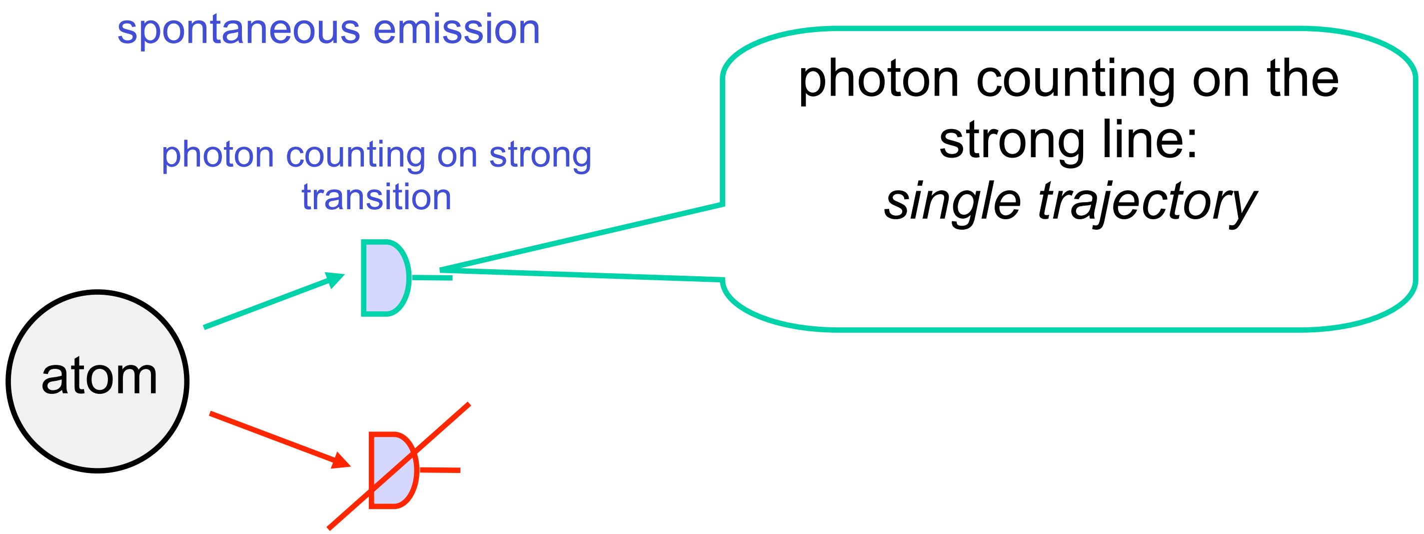


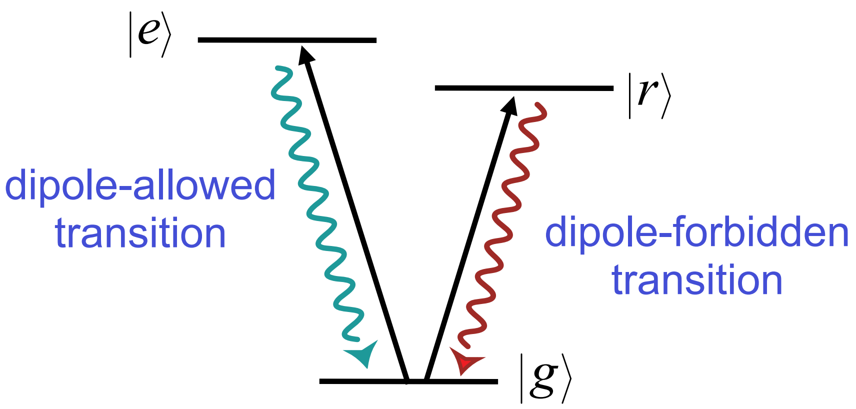


System of interest: double resonance with two excited states $|e\rangle$ and $|r\rangle$ are connected to a common lower level $|g\rangle$ via a *strong* and *weak* transition.

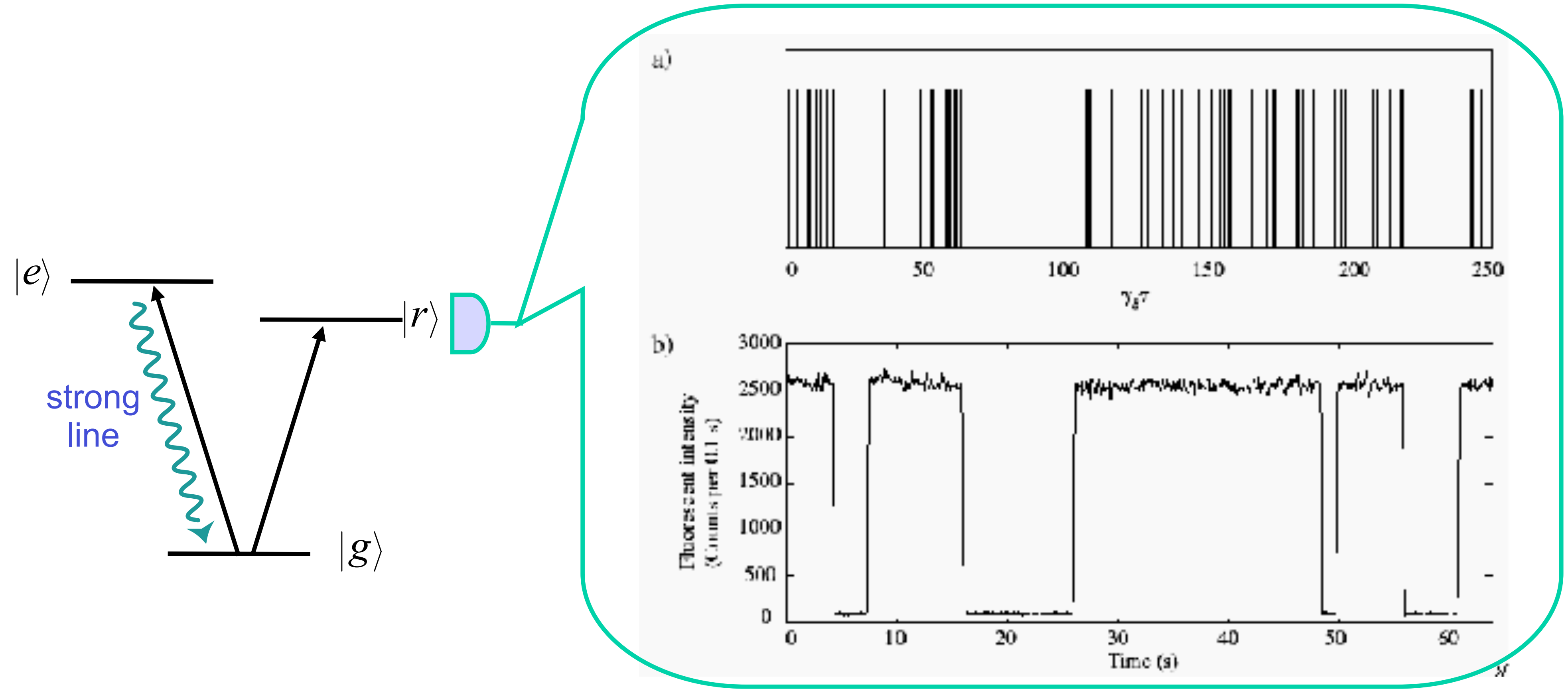
Continuous observation: fluorescence photons from the strong transition are observed in a photon counting experiment. Excitation of the weak transition with a laser will induce a quantum jump to the metastable state $|r\rangle$, or will temporarily *shelve* the atomic electron in $|r\rangle$. This will cause the fluorescence from the strong transition to be turned off. Quantum jumps of the weak transition are thus monitored via emission windows in the signal provided by the fluorescence of the strong transition.







photon counting on strong transition



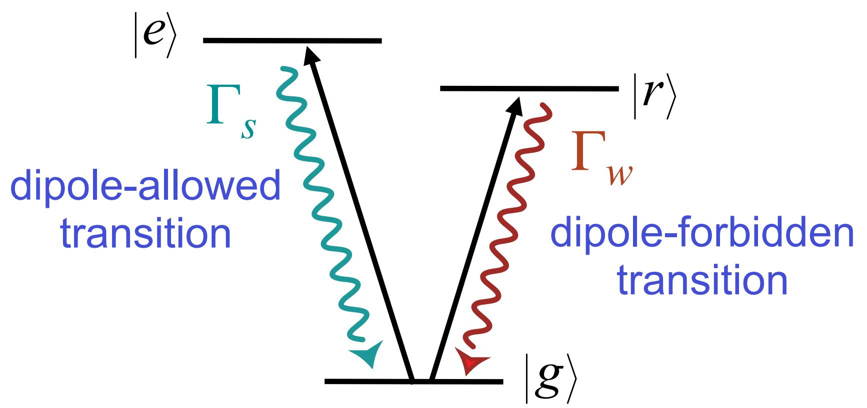
simulation (theory)

experimental run

Continuous observation: fluorescence photons from the strong transition are observed in a photon counting experiment. Excitation of the weak transition with a laser will induce a quantum jump to the metastable state $|r\rangle$, or will temporarily *shelve* the atomic electron in $|r\rangle$. This will cause the fluorescence from the strong transition to be turned off. Quantum jumps of the weak transition are thus monitored via emission windows in the signal provided by the fluorescence of the strong transition.

master equation: three-level system

$$\dot{\rho} = -\frac{i}{\hbar}(H_{\text{eff}}\rho - \rho H_{\text{eff}}^{\dagger}) + \mathcal{J}_{\text{s}}\rho + \mathcal{J}_{\text{w}}\rho$$
$$\equiv \mathcal{L}\rho$$



effective Hamiltonian

$$H_{\text{eff}} = \hbar\left(-\Delta_s - i\frac{\Gamma_s}{2}\right)|e\rangle\langle e| + \hbar\left(-\Delta_w - i\frac{\Gamma_w}{2}\right)|r\rangle\langle r|$$
$$-\frac{1}{2}\hbar\Omega_s\left(|g\rangle\langle e| + |e\rangle\langle g|\right) - \frac{1}{2}\hbar\Omega_w\left(|g\rangle\langle r| + |r\rangle\langle g|\right)$$

radiative decay terms Γ_s and Γ_w ($\Gamma_s \gg \Gamma_w$), $\Delta_{s,w}$ detunings, $\Omega_{s,w}$ Rabi frequen-cies

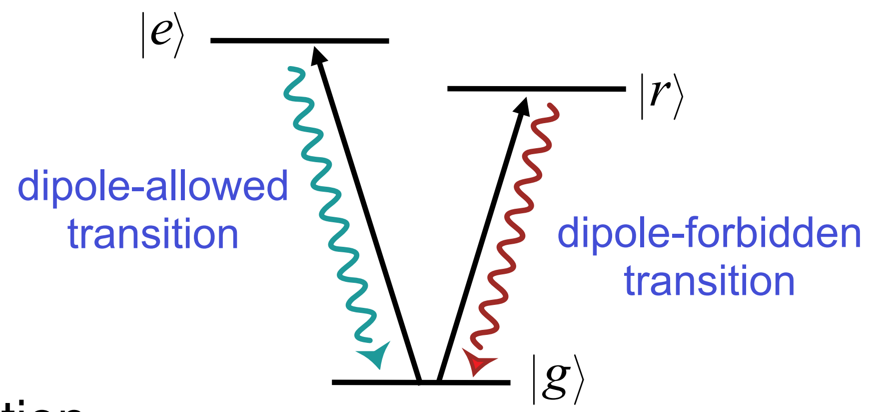
recycling operators

$$\mathcal{J}_{\text{s}}\rho = |g\rangle\langle g|\Gamma_s\rho_{ee},$$ … on the strong line

$$\mathcal{J}_{\text{w}}\rho = |g\rangle\langle g|\Gamma_w\rho_{rr}.$$ … on the weak line

$\Gamma_s \gg \Gamma_w$

Master equation with photon counter n on the strong transition



$$\dot{\rho}^{(n)} = -\frac{i}{\hbar}(H_{\text{eff}}\rho^{(n)} - \rho^{(n)}H_{\text{eff}}^{\dagger}) + \mathcal{J}_{\text{s}}\rho^{(n-1)} + \mathcal{J}_{\text{w}}\rho^{(n)}$$
$$\equiv (\mathcal{L} - \mathcal{J}_{\text{s}})\rho^{(n)} + \mathcal{J}_{\text{s}}\rho^{(n-1)}$$

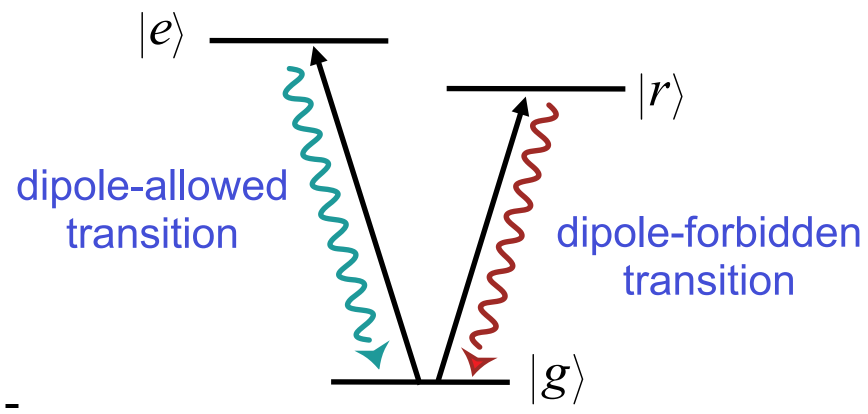
[Note: by summing over n, $\rho = \sum_{n=0}^{\infty}\rho^{(n)}$ we obtain again the master equation $\dot{\rho} = \mathcal{L}\rho$]

formal solution with initial condition $\rho(0) = |g\rangle\langle g|$

$$\rho(t) = \sum_{n=0}^{\infty}\rho^{(n)}(t)$$
$$= e^{(\mathcal{L}-\mathcal{J}_{\text{s}})t}\rho(0)$$
$$+ \sum_{n=1}^{\infty}\int_0^t dt_n \int_0^{t_n} dt_{n-1}\ldots\int_0^{t_2} dt_1 e^{(\mathcal{L}-\mathcal{J}_{\text{s}})(t-t_n)}\mathcal{J}_{\text{s}}e^{(\mathcal{L}-\mathcal{J}_{\text{s}})(t_n-t_{n-1})}\ldots$$
$$e^{(\mathcal{L}-\mathcal{J}_{\text{s}})(t_2-t_1)}\mathcal{J}_{\text{s}}e^{(\mathcal{L}-\mathcal{J}_{\text{s}})t_1}\rho(0)$$

unraveling of the master equation with respect to photons on the strong (green) line

We read off the photon statistics:



- **probability that no photon is emitted** during $(0, t]$ on the strong transition

$$p_0^t = \mathrm{Tr}_{\mathrm{sys}}\{e^{(\mathcal{L}-\mathcal{J}_s)t}|g\rangle\langle g|\}$$

- **probability density that exactly n photons are emitted** on the strong transition at times $t_1, \ldots, t_n$ in the time interval $(0, t]$

$$\begin{aligned} p_0^t(t_1, \ldots, t_n) &= \mathrm{Tr}_{\mathrm{sys}}\Big\{e^{(\mathcal{L}-\mathcal{J}_s)(t-t_n)}\mathcal{J}_s e^{(\mathcal{L}-\mathcal{J}_s)(t_n-t_{n-1})}\ldots \\ &\quad e^{(\mathcal{L}-\mathcal{J}_s)(t_2-t_1)}\mathcal{J}_s e^{(\mathcal{L}-\mathcal{J}_s)t_1}|g\rangle\langle g|\Big\} \end{aligned}$$

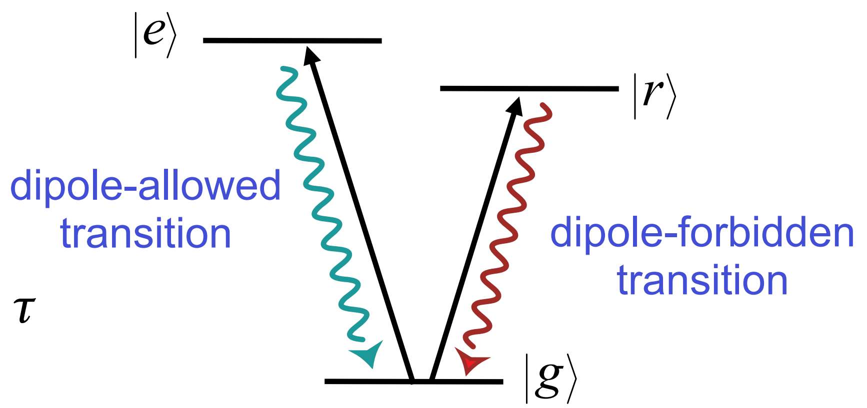
which *factorizes*

$$p_0^t(t_1, \ldots, t_n) = p_0^{t-t_n}\ \tilde{c}(t_n - t_{n-1})\ldots\tilde{c}(t_1 - 0)$$

Factorization is due to the fact that a quantum jump always prepares the atom in the ground state $|g\rangle$, and thus there is no dependence on the previous history of the wave function.

- probability density for "**next photon emission**" or "**delay function**"

$$\tilde{c}(\tau) = \mathrm{Tr_{sys}}\{\mathscr{J}_s e^{(\mathscr{L}-\mathscr{J}_s)\tau}|g\rangle\langle g|\}$$



Interpretation: $\tilde{c}(\tau)d\tau$ = the probability that a photon is emitted at at time τ when the previous photon was emitted at time $\tau = 0$.

Properties:

$$\int_0^t \tilde{c}(\tau)d\tau = 1 - p_0^t$$

Proof:

$$\frac{d}{dt}p_0^t = \frac{d}{dt}\mathrm{Tr_{sys}}\{e^{(\mathscr{L}-\mathscr{J}_s)t}|g\rangle\langle g|\}$$

$$= \mathrm{Tr_{sys}}\{(\mathscr{L}-\mathscr{J}_s)e^{(\mathscr{L}-\mathscr{J}_s)t}|g\rangle\langle g|\} \quad \text{(we use } \mathrm{Tr_{sys}}\mathscr{L}(\ldots) = 0\text{)}$$

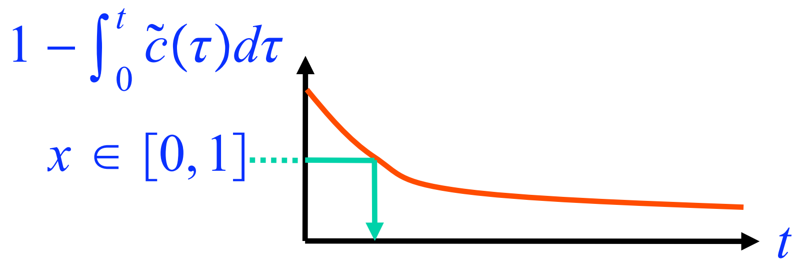
$$= -\mathrm{Tr_{sys}}\{\mathscr{J}_s e^{(\mathscr{L}-\mathscr{J}_s)t}|g\rangle\langle g|\} = -\tilde{c}(t)$$

The probability density is normalized

$$\int_0^\infty \tilde{c}(\tau)d\tau = 1$$

- **How to simulate $\tilde{c}(\tau)$:** take a random number generator which produces equally distributed numbers $x \in [0,1]$. We can simulate the decay time t of "emission of the next photon" according to

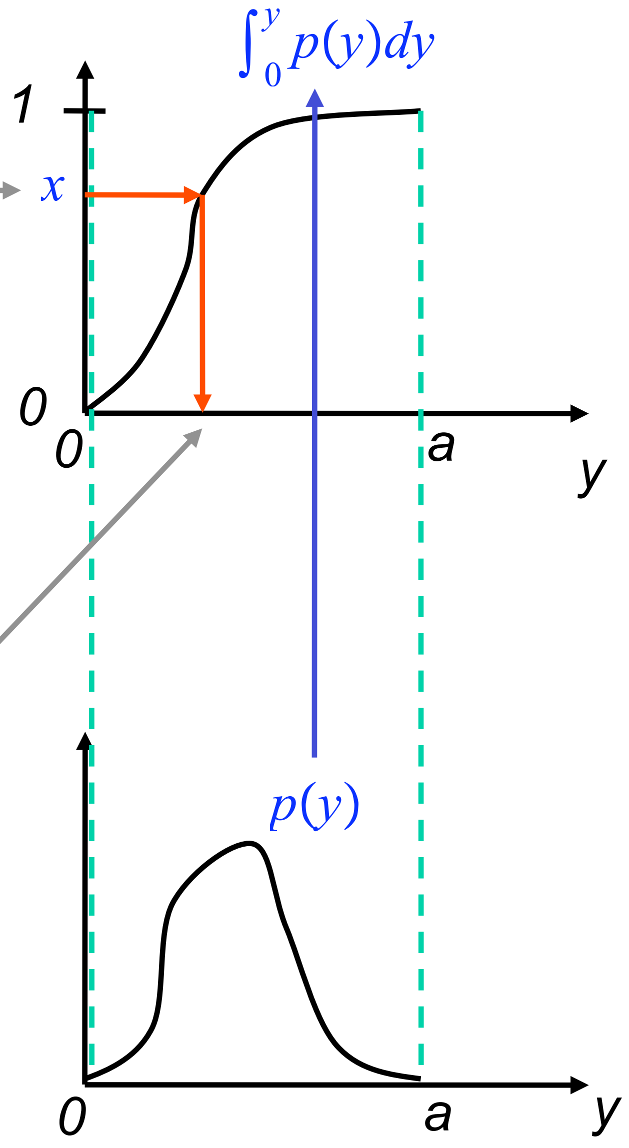
$$t = t(x) \quad \text{with } t \text{ solution of } 1 - \int_0^t \tilde{c}(\tau)d\tau = x \text{ (given } x\text{)}$$



complement: simulating a distribution

How to simulate random numbers $y \in [0,a]$ according to the distribution *p(y):*

choose a random number *x* equally distributed in *[0,1]*



we find a random $y \in [0,a]$ distributed according to $p(y)$

- **Discussion of $\tilde{c}(\tau)$:** $\tilde{c}(\tau)$ consists of a sum of exponentials with different decay constants, reflecting the different time scales for the excitation and decay on the strong and metastable transition.

delay function

$\tilde{c}(\tau)$

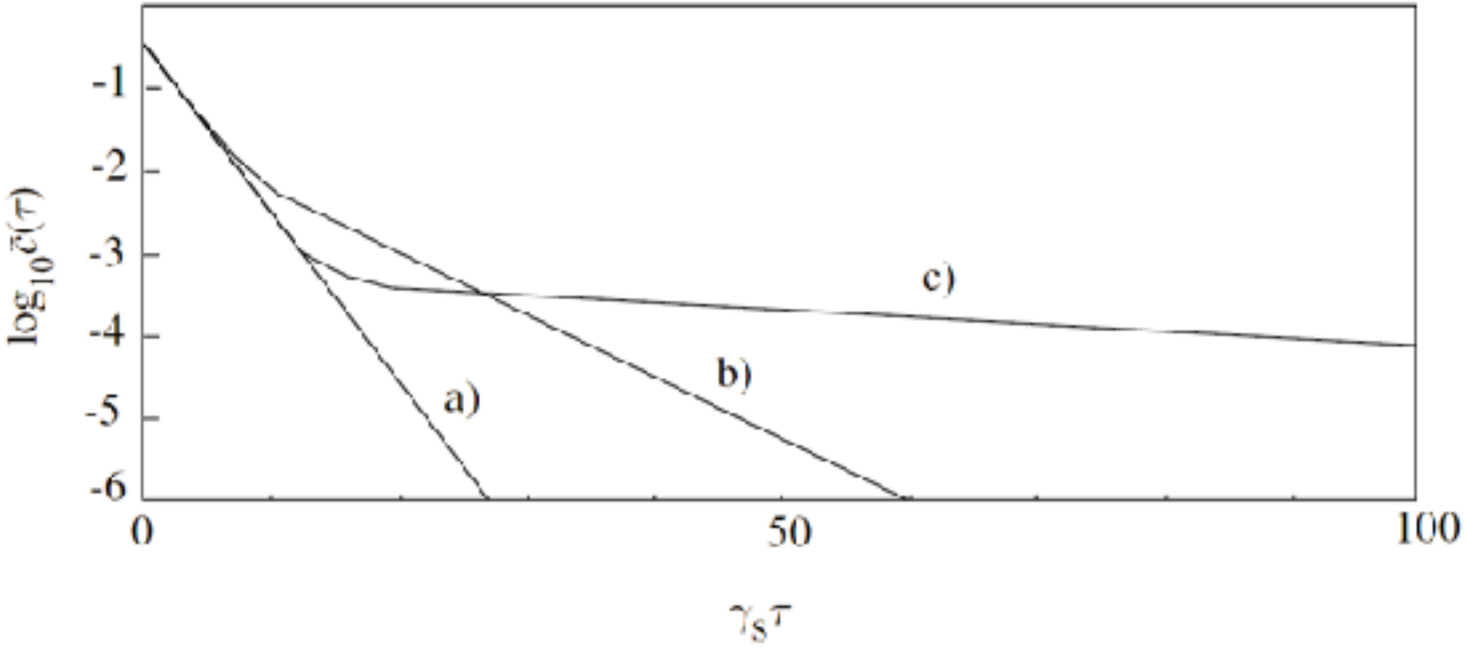
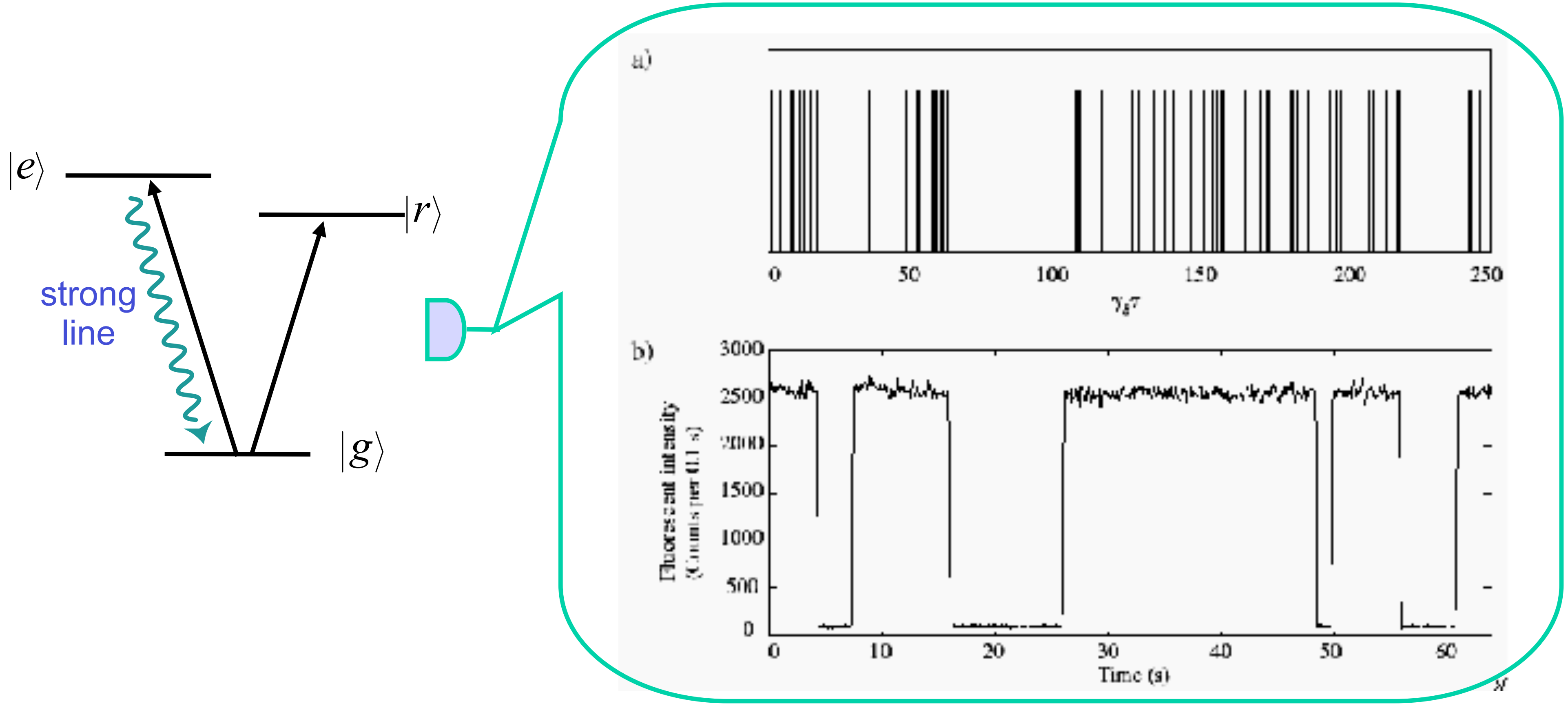


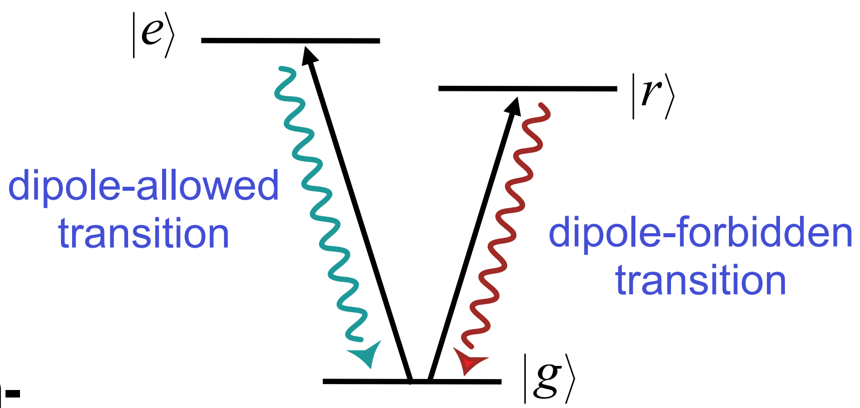
Fig.11.6 The conditional probability density $\tilde{c}(\tau)$ as a function of $\gamma_s\tau$ according to [11.4]. $\tilde{c}(\tau)$ is the conditional probability density that; given a count has occurred on the strong transition $|g\rangle \leftrightarrow |e\rangle$ at time $\tau = 0$—the *next* count on the strong transition will occur at time τ. The three curves show: a) the metastable state is not excited, $\Omega_w = 0$, and $\tilde{c}(\tau)$ decays essentially exponentially; b) the lifetime of the metastable state $|r\rangle$ is ten times longer than the lifetime of the rapidly decaying state $|e\rangle$, $W_{gr} = \gamma_w = 10^{-1}\gamma_s$. c) the lifetime of the metastable state $|r\rangle$ is one hundred times longer than the lifetime of the rapidly decaying state $|e\rangle$, $W_{gr} = \gamma_w = 10^{-2}\gamma_s$. W_{gr} is an incoherent excitation rate on the weak transition.

The existence of a weak transition leads to a *long time tail* in the delay function.

In a simulation of a photon emission sequence on the strong line this will manifest itself in the *periods of brightness* and *darkness*.

photon counting on strong transition





- **The atomic density matrix conditional on observing an emission window** on the strong line when the *last photon on the strong transition* was emitted at time t_r is

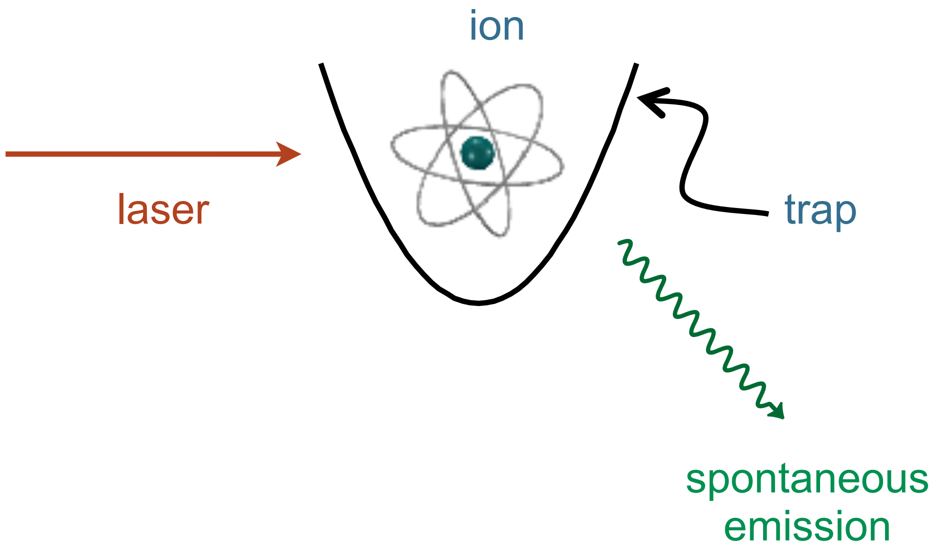
$$\rho_c(t) = \frac{e^{(\mathcal{L}-\mathcal{J}_{\mathrm{s}})(t-t_r)}|g\rangle\langle g|}{\mathrm{Tr}_{\mathrm{sys}}\{\ldots\}}$$
$$\rightarrow \ |r\rangle\langle r| \qquad (\Gamma_s t \gg 1)$$

That is, *observation of a window* in a single trajectory of counts corresponds to a *preparation of the electron in the metastable state* $|r\rangle$. This state preparation is what is usually referred to as *shelving of the electron*.

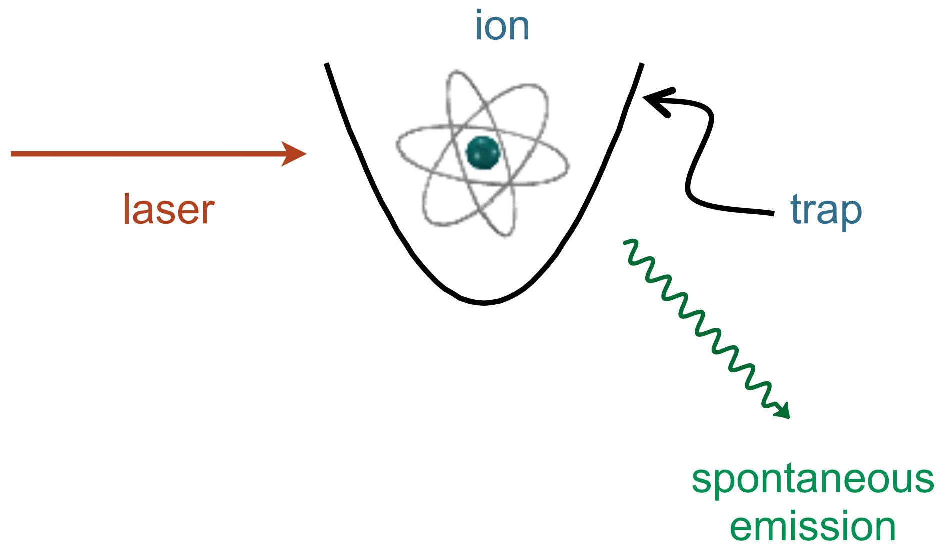
application: projective state measurement in ion trap quantum computer and simulator

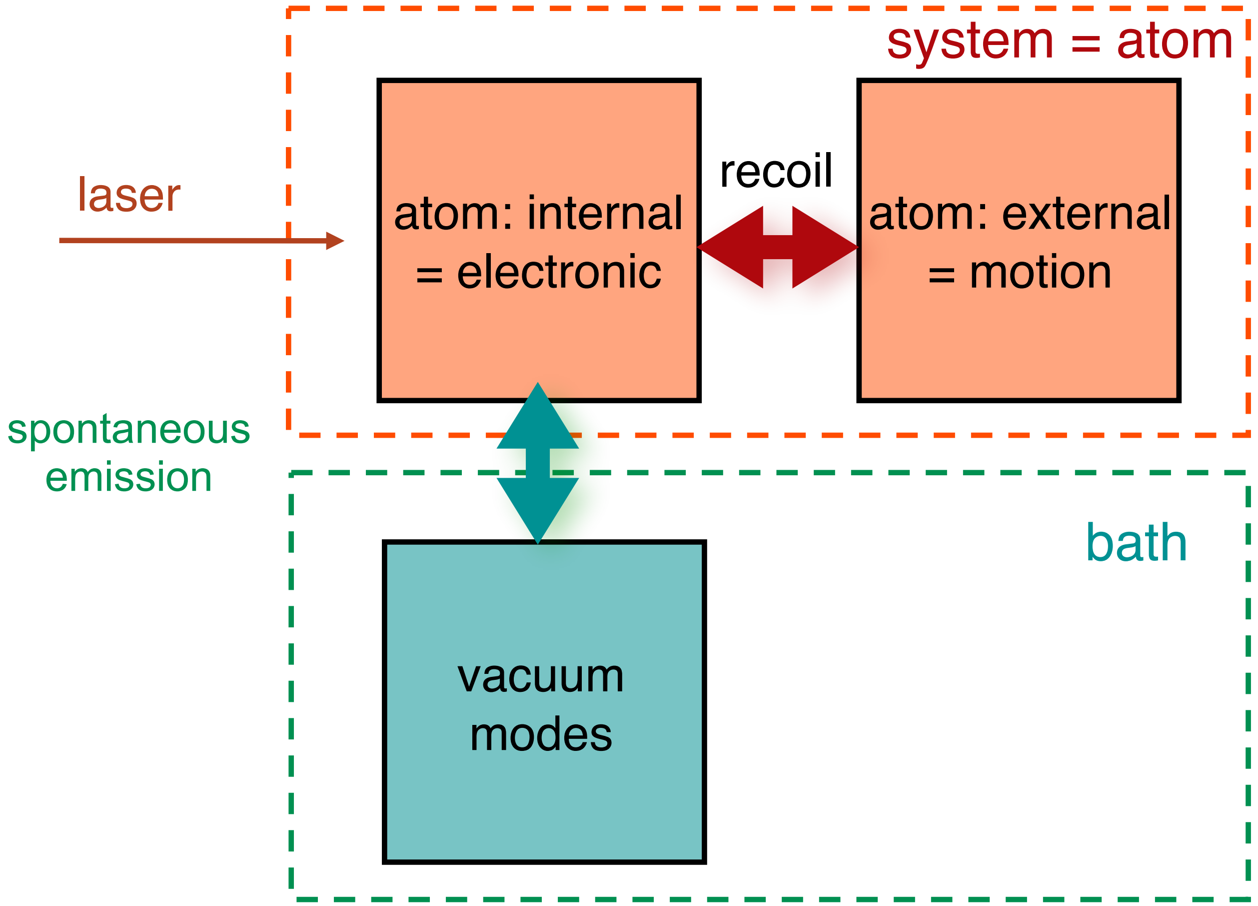
END Unravelling of the master equation

6. Laser Cooling Master Equation

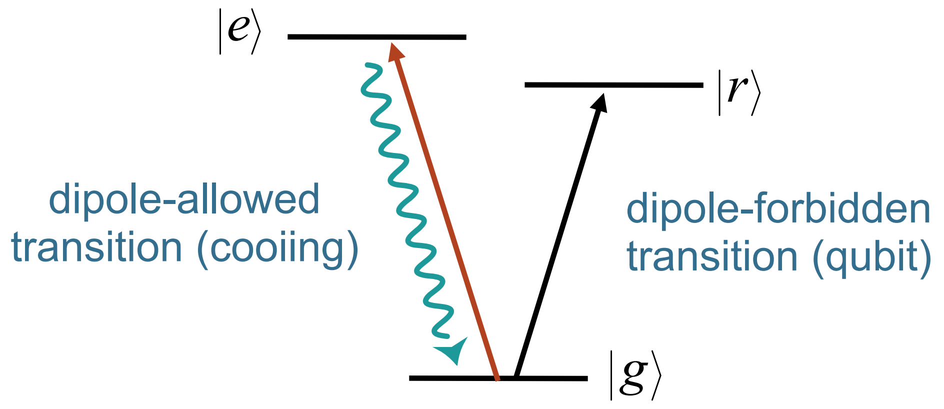


System: We consider a radiatively damped two-level atom driven by a classical light field. We wish to include the motion of the atom and mechanical effects due to induced and spontaneous radiative processes.

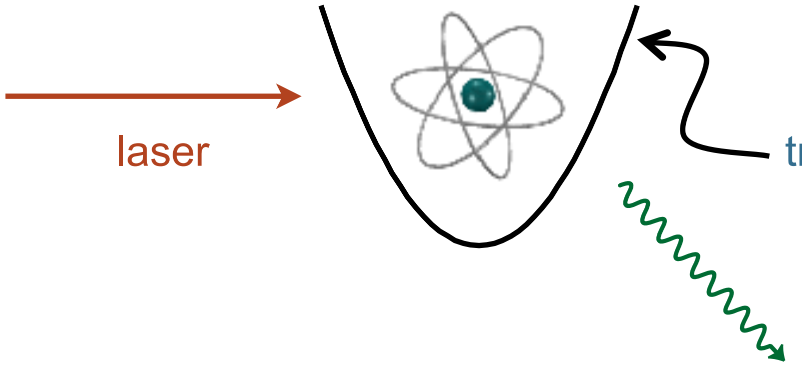




Compare Chapter on Quantum Computing with Trapped Ions.



Model



Hamiltonian of the driven atom + radiation field

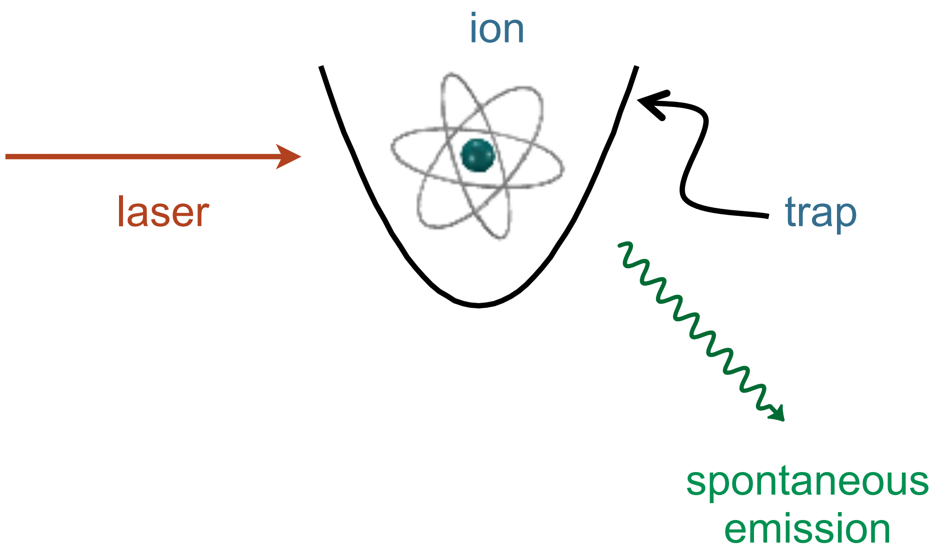
$$H = H_{0A} + H_{0F} + H_{\text{int}}$$

- **Atom:** The atomic degrees of freedom include the center-of-mass degrees of freedom (motion) of the atom, and the internal (electronic) degrees of freedom.
  The atomic part of the Hamiltonian includes the kinetic energy, and possibly a trapping potential

$$H_{0A} = \frac{\hat{\vec{p}}^2}{2M} + V(\hat{\vec{x}}) + \hbar\omega_{eg}|e\rangle\langle e| - \left(\vec{\mu}_{eg}\vec{E}_{\text{cl}}^{(+)}(\hat{\vec{x}},t)\sigma_+ + \text{h.c.}\right)$$

with e.g. a travelling wave $\vec{E}_{\text{cl}}^{(+)}(\vec{x},t) = \mathcal{E}\vec{e}e^{i(\vec{k}_L\vec{x}-\omega_L t)}$.

Example: harmonic trapping $V(\hat{\vec{x}}) = \frac{1}{2}M\left(\nu_x^2\hat{x}^2 + \nu_y^2\hat{y}^2 + \nu_z^2\hat{z}^2\right)$



Optical Bloch equations for laser cooling

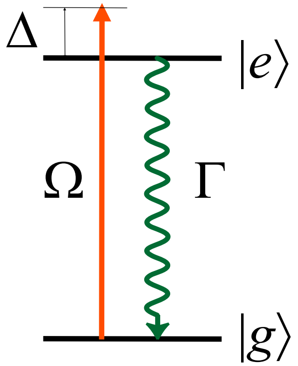
The reduced atomic density matrix obeys the OBE (including the COM motion)

Lindblad master equation

$$\dot{\rho}_A = -\frac{i}{\hbar}[H_{0A}, \rho_A] + \frac{1}{2}\Gamma\left(2\int d\Omega_{\vec{n}}\ \Phi(\vec{n})\left(\sigma_- e^{-ik_{eg}\vec{n}\cdot\hat{\vec{x}}}\right)\rho_A\left(\sigma_+ e^{ik_{eg}\vec{n}\cdot\hat{\vec{x}}}\right) - \sigma_+\sigma_-\rho_A - \rho_A\sigma_+\sigma_-\right)$$

angular distribution of spontaneous emission

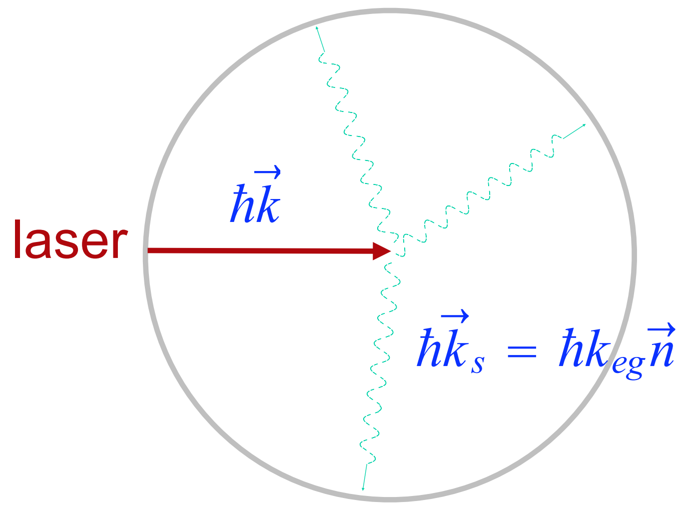
quantum jump operator for $|e\rangle \to |g\rangle$ + momentum recoil



or

$$\dot{\rho}_A = -\frac{i}{\hbar}(H_{\text{eff}}\rho_A - \rho_A H_{\text{eff}}^\dagger) + \Gamma\int d\Omega_{\vec{n}}\ \Phi(\vec{n})\left(\sigma_- e^{-ik_{eg}\vec{n}\cdot\hat{\vec{x}}}\right)\rho_A\left(\sigma_+ e^{ik_{eg}\vec{n}\cdot\hat{\vec{x}}}\right)$$

spontaneous photon recoil



Properties:

- The master equation has Lindblad form.
- recycling term

$$\mathcal{J}\rho_A(t) = \Gamma \int d\Omega_{\vec{n}}\; \Phi(\vec{n}) \left(\sigma_- e^{-ik_{eg}\vec{n}\cdot\hat{\vec{x}}}\right) \rho_A(t) \left(e^{ik_{eg}\vec{n}\cdot\hat{\vec{x}}}\sigma_+\right)$$

The quantum jump operator $\sigma_- e^{-ik_{eg}\vec{n}\cdot\hat{\vec{x}}}$ describes the return of the electron to the ground state and the momentum kick to the center-of-mass motion associated with the emission of a spontaneous photon. Here $\hbar\vec{k}_s = \hbar k_{eg}\vec{n}$ is the momentum kick to the center-of-mass with $k_{eg} = \omega_{eg}/c$ in the direction $\vec{n}$, as determined by the angular distribution

$$\Phi(\vec{n}) = \frac{1}{\Gamma}\frac{d\Gamma}{d\Omega_{\vec{n}}}.$$

angular distribution of the emitted light