Eigenstate Thermalization Hypothesis and Beyond: Exercises

GGI Winter School 2025

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Exercise 1. *Probability distribution of the Gaussian Orthogonal Ensemble (GOE).* A GEO $D \times D$ matrix H can written as $H = (A + A^T)/2$, where the entries of A are independent normal gaussian random variables, i.e $P(A_{ij}) \propto e^{-H_{ij}^2/2}$. Show that the probability distribution of H can be written as

$$P(H) = \prod_i \frac{e^{-\frac{H_{ii}^2}{2}}}{\sqrt{2\pi}} \prod_{j>i} \frac{e^{-H_{ij}^2}}{\sqrt{\pi}} \propto \exp(-\frac{1}{2}\mathrm{Tr}(H^2)) \ . \tag{1}$$

Note that this distribution is characterized by $\mathrm{Var}H_{ij} = \frac{1}{2}\mathrm{Var}(H_{ii})$.

Exercise 2. *The Wigner Surmise.* Consider from a 2×2 GOE matrix:

$$H = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_1 \end{pmatrix} \ , \quad x_1, x_2 = \mathcal{N}(0,1), \quad x_2 = \mathcal{N}(0,1/2) \ , \tag{2}$$

where $\mathcal{N}(a,b)$ indicates the normal distribution of average a and variance b. Write the eigenvalues $\lambda_{1,2}$ and show that the probability distribution of the level spacing $s = \lambda_1 - \lambda_2$ reads:

$$p(s) = \frac{s}{2} e^{-s^2/4} \ . \tag{3}$$

(Hint: for $x = f(y)$ the probability distribution of x reads $p(x) = \int dy p(y) \delta(x - f(y))$). Rescale the level spacing by its average $\bar{s} = s/\langle s \rangle$, such that $\langle \bar{s} \rangle = 1$ and show that the previous equation leads to the so-called Wigner Surmise for the GOE:

$$\bar{p}(s) = \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2} \ . \tag{4}$$

Exercise 3. *Poisson distribution for independent random variables.* Consider D iid random variables $X_1, \ldots X_D$ from a common probability distribution $p_X(x)$. Show that the distribution of the local level spacins is given by the exponential law

$$\lim_{D \to \infty} p_D(\bar{s}) = e^{-\bar{s}} \tag{5}$$

which is the law for the spacing in a Poisson process, where

$$\bar{s} = s \ D p_X(x) \ . \tag{6}$$

Exercise 4. *Average level density.* Given the Wigner semi-circle law

$$\bar{\rho}(\lambda) = \frac{1}{\pi} \frac{1}{\sqrt{\beta D}} \sqrt{2 - \left(\frac{\lambda}{\sqrt{\beta D}} \right)^2} \ , \tag{7}$$

write it for the variables $x = \lambda/\sqrt{\beta D}$ and compute its Fourier transform:

$$\mathrm{SFF}_1(t) \equiv \int e^{ixt}\bar{\rho}(x)d\lambda \equiv z_1(t)\ . \tag{8}$$

What is the asymptotic behaviour at large t?

Exercise 5. *The spectral form factor for the GUE.* Given a $D \times D$ GUE matrix H with variance $\frac{1}{2D}$, compute the spectral form factor defined as

$$\mathrm{SFF}_2(t) = \overline{\left|\frac{\mathrm{Tr}\left(e^{-iHt}\right)}{D}\right|^2}\ , \tag{9}$$

using that the eigenvalues x_i and x_j for $i \neq j$ are correlated as

$$\bar{\rho}^{(2)}(x_i, x_j) = \frac{D^2}{D(D-1)}\left[\bar{\rho}(x_i)\bar{\rho}(x_j) - \left(\frac{D}{\pi}\frac{\sin(2\beta D(x_i - x_j))}{2\beta D(x_i - x_j)}\right)^2\right]\ . \tag{10}$$

Exercise 6. *Porter-Thomas distribution.* Given the distribution of the components of unitary vectors

$$\rho_{\mathrm{UE}}(\mathbf{c}^{(i)}) = c_D\,\delta\left(1 - \sum_{\alpha=1}^{D}|c_\alpha^{(i)}|^2\right) \quad \forall i\ , \tag{11}$$

consider the following marginal distribution of a single component

$$\rho_{\mathrm{UE}}(y) = \int d^2c_1 \dots d^2c_D\,\delta(y - |c_1|^2)\rho_{\mathrm{UE}}(\mathbf{c})\ . \tag{12}$$

Show that the re-scaled single component $\eta = \frac{y}{\bar{y}}$ follows the so-called Porter-Thomas distribution in the large D limit:

$$\rho_{\mathrm{UE}}(\eta) = \lim_{D\to\infty}\frac{1}{D}\rho_{\mathrm{UE}}(\eta/D) = e^{-\eta}\ . \tag{13}$$

Exercise 7. *Toy model for ETH 1.* Given Haar expectation values $\overline{U_{i\alpha}U^\dagger_{\beta j}} = \delta_{ij}\delta_{\alpha\beta}/D$ and

$$\overline{U_{i\alpha}U^\dagger_{\alpha j}U_{j\beta}U^\dagger_{\beta i}} = \frac{1}{D^2-1}\left(1 + \delta_{\alpha\beta} - \frac{1}{D} - \frac{\delta_{\alpha\beta}\delta_{ij}}{D}\right)\ , \tag{14}$$

show that the statistics of the matrix elements of an observable in the energy eigenbasis, $A_{ij} = \sum_\alpha A_\alpha U_{i\alpha}U^\dagger_{\alpha j}$ with $U_{i\alpha} \equiv \langle i|\alpha\rangle$, at the leading order in D reads

$$\overline{A_{ij}} = \langle A\rangle\delta_{ij}\ , \quad \overline{A_{ii}^2} - \overline{A_{ii}}^2 = \frac{\langle A^2\rangle - \langle A\rangle^2}{D} + \mathcal{O}(D^{-2}) \tag{15}$$

while for $i \neq j$:

$$\overline{|A_{ij}|^2} = \frac{\langle A^2\rangle - \langle A\rangle^2}{D} + \mathcal{O}(D^{-2})\ , \tag{16}$$

with $\langle\cdot\rangle = \mathrm{Tr}(\cdot)/D$.

Exercise 8. *Toy model for ETH 2.* Show that the summary ansatz

$$A_{ij} = \langle A\rangle \delta_{ij} + \frac{R_{ij}}{\sqrt{D}}\sqrt{\kappa_2(A)}\,, \tag{17}$$

with $\overline{R_{ij}} = 0$ and $\overline{|R_{ij}|^2} = 1$ is equivalent to $\overline{A_{ii}} = \langle A\rangle$, Eq.(15) and Eq.(16).

Exercise 9. *Thermalization via ETH. 1)* Consider a quantum quench from an initial state $|\psi_0\rangle$ with extensive initial energy $\langle\psi_0|\hat{H}|\psi_0\rangle = E_0 = Ne_0$ and sub-extensive energy fluctuations $\langle\psi_0|(\hat{H}-E_0)|\psi_0\rangle = \Delta^2_{E_0} = N\delta^2_{e_0}$. Assuming absence of degeneracies and using the ETH ansatz, show that the infinite time-average of a local operator $\hat{A}(t)$ is given by

$$[A]_\infty \equiv \lim_{T\to\infty}\int_0^T \langle\psi_0|\hat{A}(t)|\psi_0\rangle \simeq \mathcal{A}(E_0) + \frac{1}{2}\left(\frac{\partial^2\mathcal{A}}{\partial E^2}\right)\Delta^2_{E_0} = \langle E_0|\hat{A}|E_0\rangle + O(1/N)\,. \tag{18}$$

Exercise 10. *Thermalization via ETH. 2)* In the same quench scenario as the previous exercise, show how ETH implies thermalization: not only local observables equilibrate to the micro-canonical expectation value [cf. Eq.(18)], but there are only exponentially small fluctuations on top. In other words, show the following scaling

$$\sigma^2_{\hat{A}} \equiv [A^2]_\infty - [A]^2_\infty = \sum_{mn,m\neq n} |c_i|^2|c_m|^2\,|A_{nm}|^2 \leq \max|A_{nm}|^2 \propto e^{-S(\overline{E})}\,. \tag{19}$$

Exercise 11. *ETH factorization of observables with repeated indices.* Using ETH and saddle-point integration, show that for local observables $\hat{A}$ the following holds:

$$\sum_i \frac{e^{-\beta E_i}}{Z} A^2_{ii} \simeq \mathcal{A}(e_\beta)^2 + \mathcal{O}(N^{-1}) \simeq [\langle\hat{A}\rangle_\beta]^2 + \mathcal{O}(N^{-1})\,. \tag{20}$$

Exercise 12. *Corrections to ETH factorization.* Using the ETH ansatz, compute the correction to the Eq.(20) above. Namely show that

$$\sum_i \frac{e^{-\beta E_i}}{Z} A^2_{ii} \simeq [\langle A\rangle_\beta]^2 + \mathcal{A}'(e_\beta)^2\Delta^2_{E_\beta} + \mathcal{O}(N^{-2})\,, \tag{21}$$

where $\mathcal{A}'(e^*) = \frac{\mathcal{A}(E)}{\partial E}|_{e=e^*}$ and $\Delta^2_{E_\beta} = \langle(\hat{H}-E_\beta)^2\rangle_\beta = \frac{1}{N|S''|}$ is the energy variance of the canonical state. (Hint: use the saddle point to compute corrections to both $\sum_i e^{-\beta E_i}/Z\,A_{ii}$ and $\sum_i e^{-\beta E_i}/Z\,A^2_{ii}$.)
This diagram enters in the calculation of the two point function $\kappa_2(t) \equiv \langle A(t)A\rangle_\beta - \langle A\rangle^2_\beta$. For which class of observables can this correction become of the same order of $\kappa_2(t)$?

Exercise 13. Using the Eigenstate Thermalization Hypothesis Ansatz, show that the Fourier transform of the following regularized correlator

$$F_2(t) = \mathrm{Tr}\left(\rho^{1/2}\hat{A}(t)\rho^{1/2}\hat{A}\right) - \mathrm{Tr}\left(\rho\hat{A}\right)^2 \quad \text{with } \rho = \frac{e^{-\beta H}}{Z} \tag{22}$$

is the bare off-diagonal smooth function appearing in ETH evaluated at the thermal energy, i.e. $\tilde{F}(\omega) = |f(e_\beta, \omega)|^2$.

Exercise 14. *Classical and Free cumulants.* Using the moment-cumulant implicit definitiions

$$\mathbb{E}(x_1 x_2 \dots x_n) = \sum_{\pi \in P(n)} \prod_{b \in \pi} c_{|b|}(x_{b(1)} x_{b(2)} \dots x_{b(n)}) \,, \quad \langle A_1 A_2 .. A_n \rangle = \sum_{\pi \in NC(n)} \prod_{b \in \pi} \kappa_{|b|}(A_{b(1)} A_{b(2)} .. A_{b(n)}) \,, \tag{23}$$

where $b = (b(1), b(2), \dots, b(n))$ denotes the element of the block of the partition and $|b|$ its length, compute the classical and free cumulants

$$\begin{aligned} &c_3(x_1, x_2, x_3) \,, \quad c_4(x_1, x_2, x_3, x_4) \,, \\ &\kappa_3(A_1, A_2, A_3) \,, \quad \kappa_4(A_1, A_2, A_3, A_4) \,. \end{aligned}$$

Then compare $\kappa_4(A_1, A_2, A_3, A_4)$ and $c_4(A_1, A_2, A_3, A_4)$, by writing the last one to respect to $\langle \cdot \rangle$ and not $\mathbb{E}(\cdot)$.

Exercise 15. *Free-cumulants of GUE.* Compute $\kappa_4(H, H, H, H)$ for a GUE random matrix H in the large D limit.

Exercise 16. *Freeness as a rule for mixed moments.* In the case of $D \times D$ matrices A, B and , show that $\kappa_2(AB) = 0$, $\kappa_3(ABA) = 0$ and $\kappa_4(ABAB) = 0$ imply

$$\langle ABAB \rangle = \langle A^2 \rangle \langle B \rangle^2 + \langle A \rangle^2 \langle B^2 \rangle - \langle A \rangle^2 \langle B \rangle^2 \,. \tag{24}$$

Show that this correspond to the $n = 2$ of the more general expression

$$\langle ABAB \dots AB \rangle = \sum_{\pi \in NC(n)} \kappa_\pi(A, \dots, A) \, \langle B, \dots, B \rangle_{\pi^*} \,, \tag{25}$$

where π^* is the dual of the partition.

Exercise 17. *Factorization of non-crossing ETH diagrams.* Show that in the large N limit the following factorizations hold:

$$\kappa_{(b)}(t_1, t_2, t_3) = \sum_{i \neq j \neq m} \frac{e^{-\beta E_i}}{Z} A(t_1)_{ij} A(t_2)_{ji} A(t_3)_{im} A_{mi} = k_2^{\mathrm{ETH}}(t_1, t_2) \, k_2^{\mathrm{ETH}}(t_3, 0) + \mathcal{O}(N^{-1}) \tag{26a}$$

$$\kappa_{(c)}(t_1, t_2, t_3) = \sum_{i \neq j \neq k} \frac{e^{-\beta E_i}}{Z} A(t_1)_{ii} A(t_2)_{ik} A(t_3)_{km} A_{mi} = k_1^{\mathrm{ETH}} \, k_3^{\mathrm{ETH}}(t_2, t_3, 0) + \mathcal{O}(N^{-1}) \tag{26b}$$

$$\kappa_{(e)}(t_1, t_2, t_3) = \sum_{i \neq j} \frac{e^{-\beta E_i}}{Z} A(t_1)_{ii} A(t_2)_{ii} A(t_3)_{ij} A_{ji} = [k_1^{\mathrm{ETH}}]^2 \, k_2^{\mathrm{ETH}}(t_3, 0) + \mathcal{O}(N^{-1}) \,, \tag{26c}$$

where

$$k_n^{\mathrm{ETH}}(A(t_1) A(t_2) \dots A(t_n)) = \frac{1}{Z} \sum_{i_1 \neq i_2 \neq \dots \neq i_n} e^{-\beta E_i} A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_n i_1} e^{i t_1 \omega_{i_1 i_2} + \dots i t_n \omega_{i_n i_1}} \,. \tag{27}$$

Exercise 18. Given n energy eigenvalues $E_{i_1}, \dots E_{i_n}$, the average energy $E^+ = (E_{i_1} + E_{i_2} + \dots + E_{i_n})/n$ and the energy differences $\vec{\omega} = (E_{i_1} - E_{i_2}, E_{i_2} - E_{i_3}, \dots, E_{i_n} - E_{i_1})$, show the following identity

$$E_{i_1} = E^+ + \vec{\ell}_n \cdot \vec{\omega} \quad \text{with} \quad \vec{\ell}_n = \left(\frac{n-1}{n}, \dots, \frac{1}{n}, 0 \right) \,. \tag{28}$$