Lecture Notes

Chaotic dynamics: the Eigenstate Thermalization Hypothesis and beyond

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Contents

The notes aim at introducing a personal understanding of the *Eigenstate Thermalization Hypothesis* and at discussing recent advances in the field, which include the role of correlations and the relation to Free Probability.

Lecture Notes

These notes are chaotic as well: if you find typos or mistakes, you are encouraged to write me an email.

1 The Eigenstate Thermalization Hypothesis

> "According to the fundamental principles of statistical physics, the result of statistical averaging does not depend on whether it is with respect to the exact wave function of a stationary state of a closed system or by means of the Gibbs distribution. "
>
> — L.D. LANDAU AND E.M. LIFSCHITZ, Volume 9

Selected bibliography:

1. (The first and still the best): M. Srednicki, *The approach to thermal equilibrium in quantized chaotic systems*, J. Phys. A: Math. Gen. 32 1163 (1999).
2. (*The* review): L. D'Alessio, Y. Kafri, A. Polkovnikov and M. Rigol, *From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics*, Advances in Physics, 65, (2016).

1.1 Equilibration and thermalization

To study out-of-equilibrium dynamics, the system is initialized at time $t = 0$ in a **pure initial state** $|\psi_0\rangle$, which could be prepared for example as the ground state of a different Hamiltonian $\hat{H}_0$. Since the system is isolated, its time evolution is governed by the Schrödinger equation. Namely, the initial state will evolve unitarily in time as

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\,|\psi_0\rangle = \sum_i c_i\, e^{-iE_i t/\hbar}\,|E_i\rangle\ , \tag{1}$$

where on the right-hand side $c_i = \langle E_i|\,\psi_0\rangle$ is the overlap between the initial state and a single energy eigenstate. Unless $\hat{H}$ commutes with $\hat{H}_0$, the resulting evolution is generally nontrivial, with local observables and their quantum correlations changing in time as the state of the system evolves. This protocol goes under the name of *quantum quench* and represents the simplest example to drive the system out-of-equilibrium. Typically, *global* and *sudden* quenches refer to protocols where one parameter h_0 of the pre-quench Hamiltonian $\hat{H}_0 = \hat{H}(h_0)$ is changed uniformly and instantaneously to h of the final one $\hat{H} = \hat{H}(h)$. One can also consider inhomogeneous or local quenches, where $\hat{H}_0$ differs from $\hat{H}$ only by a local impurity. In a global quench, a finite amount of energy $E = \langle\psi_0|\,\hat{H}\,|\psi_0\rangle$ is introduced into the system, which corresponds, in jargon, to an increase in temperature $1/\beta_E$, usually identified by the following relation between the energies

$$\langle\psi_0|\,\hat{H}\,|\psi_0\rangle = E = \mathrm{Tr}\left(\hat{H}\,\frac{e^{-\beta_E\hat{H}}}{Z}\right) \tag{2}$$

We consider typical initial states, with an extensive energy and sub-extensive energy fluctuations, i.e.,

$$E \propto N\ ; \qquad \Delta_E^2/E^2 \propto \frac{1}{N^a}\ , \quad a > 0 \tag{3}$$

with $\Delta_E^2 = \langle\psi_0|\,\hat{H}^2\,|\psi_0\rangle - \langle\psi_0|\,\hat{H}\,|\psi_0\rangle^2$.

Quantum equilibration is nowadays understood from the dynamics of physical observables. In particular, the primary quantities are *local* observables $\hat{A}$, with support in a finite region A of the physical space. The time evolution can be expressed in terms of the matrix element of the operator in the energy eigenbasis as

$$\langle\hat{A}(t)\rangle \equiv \langle\psi(t)|\hat{A}|\psi(t)\rangle = \sum_i |c_i|^2 A_{ii} + \sum_{n\neq j} c_i\, c_j^*\, e^{-i(E_i-E_j)t/\hbar} A_{ij}\ . \tag{4}$$

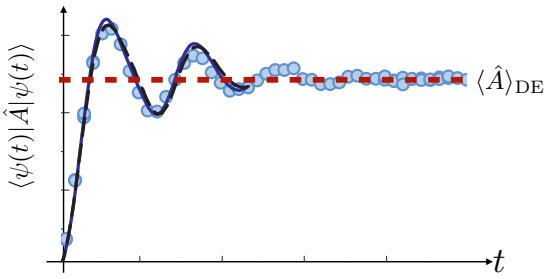

Figure 1: Illustration of thermalization dynamics of a local observable in a many-body system. Image adapted from experimental data in Ref. [1], that probes the thermalizing dynamics of ultra-cold atoms in optical lattices.

A system is said to **equilibrate** if the expectation value of all local observables converge to a finite value at infinite times, i.e.,

$$\lim_{t\to\infty}\langle\hat{A}(t)\rangle = A_{\rm eq}\ . \tag{5}$$

This is formally studied via the *infinite-time average* of $\langle\hat{A}(t)\rangle$, defined as

$$[A]_\infty \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\,\langle\hat{A}(t)\rangle\ . \tag{6}$$

Notice that the infinite-time average must be regarded as a purely mathematical tool [1] [2]. In the absence of degeneracies in the spectrum, a mild assumption for non-integrable systems without global symmetries, one obtains

$$[A]_\infty = \sum_i |c_i|^2 A_{ii} = {\rm Tr}\left(\hat{\rho}_{\rm DE}\hat{A}\right) \equiv \langle\hat{A}\rangle_{\rm DE}\ , \tag{7}$$

where $\hat{\rho}_{\rm DE}$ is the density matrix of the so-called *diagonal ensemble* defined by

$$\hat{\rho}_{\rm DE} = \sum_i |c_i|^2\ |E_i\rangle\langle E_i|\ , \tag{8}$$

which, in principle, contains all the classical information about the initial state $|\psi_0\rangle$.

A system is said to **thermalize** if it equilibrates to the value predicted by statistical mechanics and it remains close to it at most later times. In other words, a quantum system thermalizes if the equilibrium value in Eq.(5) corresponds to the microcanonical prediction at energy E.

Let us remark here that this definition of thermalization is fully analogous to what is requested by Khinchin's approach to statistical mechanics [2]: one only cares about the relaxation of certain (physical) observables in the thermodynamic limit.

[1] In fact, the averaging time T has to be larger than all the relevant time-scales and in particular then the Heisenberg time $T_{\rm Heis} = 2\pi\hbar/\delta \sim e^N$, with δ the mean spacing between energy eigenvalues. Hence, this time scale is too large to be physically relevant.

[2] In classical ergodicity, a different approach to statistical mechanics is propesed by A. Khinchin in *Mathematical Foundations of Statistical Mechanics* 1949. His approach is a practical one and it is based on the following observations:

- When one looks at *macroscopic systems*, one studies problems where $N \gg 1$ is very large;
- One is not interested in the validity of statistical mechanics $\overline{F} = [F]_{\rm mc}$ for arbitrary functions, but for *physical ones* (relevant for instance in thermodynamics, such as kinetic energy, densities, pressure) thus one shall focus rather on these;
- we do *not* need ergodicity to be valid for *every initial condition*, so one can accept that it violations $\overline{F} \neq [F]_{\rm mc}$ for a small portion of initial conditions (vanishing when $N \to \infty$).

Local perspective Let us conclude this section by adopting a "local" perspective which allows us to mention the **entanglement of eigenstates**. In general, the expectation values $\langle\psi|A|\psi\rangle$ of local observables A are encoded in the *reduced density matrix* $\hat{\rho}_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$, where $\mathrm{Tr}_{\bar{A}}$ indicates the partial trace performed over $\bar{A}$, the complement of A. This follows from $\langle\psi|\hat{A}|\psi\rangle = \mathrm{Tr}\left(\hat{A}\hat{\rho}_A\right)$. Therefore, thermalization on observables [cf. Eq.(39)] implies that the stationary state of the system will be locally equivalent to a thermal state

$$\lim_{t\to\infty}\lim_{N\to\infty}\hat{\rho}_A(t) = \mathrm{Tr}_{\bar{A}}\left(\frac{e^{-\beta_E\hat{H}}}{Z}\right), \tag{9}$$

with the temperature fixed by the energy (2). So, despite the global evolution being unitary (the total information is conserved), thermalization implies a local loss of information: Independently of the initial states, all the stationary density matrices locally "look" thermal [cf. Eq.(9)]. It is now well known that this apparent paradox is explained by the non-local growth in time of quantum correlations and in particular of the entanglement entropy S_A. The latter is defined as the von Neumann entropy of the reduced density matrix, i.e.

$$S_A \equiv -\mathrm{Tr}\left(\hat{\rho}_A \ln \hat{\rho}_A\right) \tag{10}$$

and it describes the quantum correlations of the state $|\psi\rangle$ shared among A and its complement.

In the long-time saturation value of the entanglement entropy is expected to correspond to the value in the thermal ensemble. Using the convergence of the reduced density matrix in Eq.(9), one has

$$\lim_{t\to\infty}\lim_{N\to\infty}\frac{S_A(t)}{N_A} \simeq \lim_{N\to\infty}\frac{S_A(|E\rangle)}{N_A} \simeq \frac{S(E)}{N} \equiv s(E), \tag{11}$$

where we denoted by N_A and N the size of the region A and of the whole system. Here $|E\rangle$ is an eigenstate satisfying ETH (32) and s is the entropy density. As opposed to the low-energy states , the entanglement entropy of chaotic eigenstates $S_A(|E\rangle)$ obeys volume law, i.e. it scales with the volume of the subregion A. More importantly, $S_A(|E\rangle)$ is equal to the thermodynamic entropy $S(E)$ at the same energy, at the leading order in the system size [3, 4]. The leading correction to Eq.(11) is universal and proportional to the square root of the system's heat capacity [5]. These observations provide a profound bridge between quantum entanglement and standard thermodynamics: The thermalization of a subsystem is identified as the accumulation in time of quantum entanglement between the subsystem and the rest.

This first observation already shows how a quantum information perspective can be very insightful for the understanding of the thermalization of quantum systems.

1.2 Eigenenergies statistics of many-body hamiltonians

It is clear from Eq.(4) that the information about thermalizing dynamics is contained in the eigenvalues and eigenvectors of the Hamiltonian which dictates the dynamics. The spectrum of a many-body Hamiltonian $\hat{H}$ is characterized by the following dimensionalities:

- $N = \#$ degrees of freedom[3];
- $D = \dim\mathcal{H} \propto \exp(N)$ Hilbert space dimension;

Thus, by relaxing the requirements he focuses on *extensive observables* (given by the sum of N components, each of them depending on a single degree of freedom: $F(\mathbf{x}) = \sum_{i=1}^{N} f_i(x_i, p_i)$, and on separable Hamiltonians: $H = \sum_{i=1}^{N} h_i(x_i, p_i)$. This last requirement may seem quite restrictive (the system is integrable $\{H, h_i\} = 0$) but the idea is that small short-range interactions would not change qualitatively the properties. Note that the essence of this approach is the shift of focus from trajectories in phase-space to the properties of physical observables in the large N limit on a finite number of initial conditions.

[3] Note that in one-dimensional spin chains of length L as in the example above, one has $N = L$.

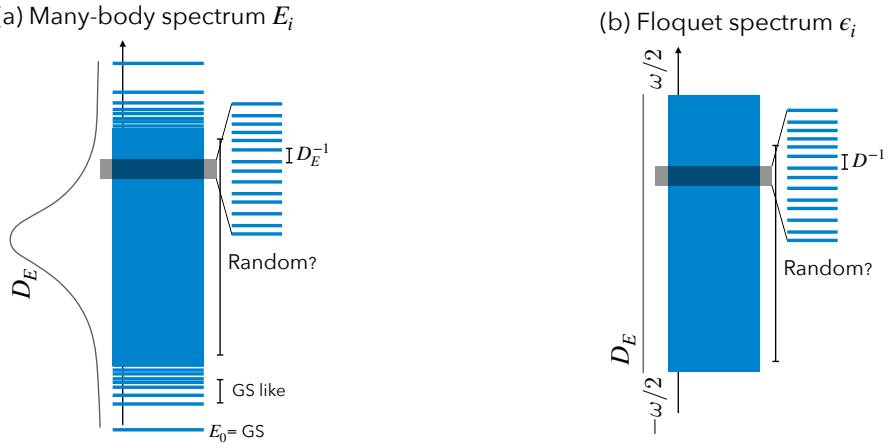

Figure 2: Illustration of the many-body spectrum features. (a) Picture of a many-body Hamiltonian spectrum, where D_E is the density of states (12), which is typically a Gaussian. (b) Floquet spectrum [cf. Eq.(**??**)].

- density of states at energy E:

$$D_E = \sum_i \delta_\tau(E - E_i) \,. \tag{12}$$

  Here $\delta_\tau(x)$ is a smoothed delta function, such that D_E is a smooth function of energy E[4]. In the many-body case, we have that $D_E = e^{S(E)}/E \propto \exp(N)$, where $S(E) = Ns(e)$ is the thermodynamic entropy[5].

The spectrum $\{E_i\}$ contains ordered eigenvalues

$$E_0 \leq E_1 \leq \cdots \leq E_D \,,$$

where E_0 is the ground state, and the first low-lying excited states can be usually treated via perturbation theory and can be thought of as simple excitations on top of the ground state[6]. On the other hand, many-body dynamics involve the whole spectrum and, in particular, its bulk of exponentially many eigenstates (which contains most of the spectrum), where the effective level spacing at energy E is D_E^{-1}, see Fig.2 for a pictorial illustration.

The first crucial insight, based on early works by Wigner and Dyson [6–8], was the understanding that the *eigenvalues* E_i of sufficiently complex Hamiltonian, when focusing on an appropriately small energy window (where the density of states is constant), essentially possess the same universal statistical properties as the eigenvalues of random matrices. The main upshot can be summarized as follows:

- Integrable Hamiltonians → independent random variables = Poisson statistics;
- Non-integrable systems → correlated eigenvalues of Random Matrices = Wigner Dyson statistics.

[4]The presence of the smoothed delta function makes it already analogous to the average level density defined in Random Matrix Theory, with a different normalizations with respect to Eq.(160) in the Appenxix, i.e. we have $D_E = D\bar{\rho}(E)$.

[5]The denominator E is an energy scale needed for dimensional analysis but inessential for thermodynamic arguments.

[6]One could say *dressed quasiparticles* in the language of Landau theory.

We refer the reader who is not familiar with Random Matrix theory (RMT) to the Appendix A, where we provide a non-exhaustive introduction to the RMT tools and notions that are useful for the study of chaotic dynamics.

Comments:

1. The previous statement concerns several probes of *local correlations*, such as:

   - *Level spacing* statistics $s_n = E_{n+1} - E_n$: Integrable systems obey the Poisson distribution (cf. Eq.(140)) while non-integrable ones the Wigner-Dyson one (cf. Eq.(125)). After rescaling the Hamiltonian such that $\bar{s} = 1$, one expects

   $$p(s) = \begin{cases} \frac{\pi}{2} s\, e^{-\pi s^2/4} & \text{Wigner-Dyson: chaotic Hamiltonian} \\ e^{-s} & \text{Poisson: integrable Hamiltonian} \end{cases} . \tag{13}$$

   - The *level spacing ratio* between nearby gaps

   $$r_n = \frac{\min\{s_n, s_{n+1}\}}{\max\{s_n, s_{n+1}\}} . \tag{14}$$

   This quantity is less sensible to finite size effects, and its distribution reproduces the properties of the spectrum [9]. Namely,

   $$p(r) = \begin{cases} \frac{27}{4} r \frac{1+r}{(1+r+r^2)^{5/2}} & \text{Wigner-Dyson: chaotic Hamiltonian} \\ \frac{2}{(1+r)^2} & \text{Poisson: integrable Hamiltonian} \end{cases} . \tag{15}$$

   The different level spacing distributions are characterized by a different value of the average *level spacing ratio* $\langle r \rangle = \langle r_n \rangle$. From the results One expects $\langle r \rangle_{\text{Poisson}} = 0.386$ if the system behaves integrably and the distribution is Poisson; on the other side, if the distribution is Wigner-Dyson and the system behaves ergodically, then $\langle r \rangle_{\text{WD}} = 0.5295$, see e.g. Ref. [9].

   - *Spectral form factor*. After an initial non-universal time-interval $t \geq t_{\text{Th}}$, the spectral form factor of complex Hamiltonian shall follow either the RMT prediction of the Poisson ones:

   $$\text{SFF}(t) = \frac{|\text{Tr}U(t)|^2}{D^2} \simeq \begin{cases} t \quad \text{for} \quad t \leq t_{\text{Heis}} \quad , \frac{1}{D} \quad \text{for} \quad t \geq t_{\text{Heis}} & \text{RMT: chaotic Hamiltonian} \\ \frac{1}{D} & \text{integrable Hamiltonian} \end{cases} , \tag{16}$$

   where one introduces the so-called *Heisenberg time* $t_{\text{Heis}} \sim D_E$, which is the time-scale at which one shall resolve the discreteness of the spectrum. Above, we have defined t_{Th} as the time scale after which the spectrum looks RMT.

2. Note that, given a fixed Hamiltonian, its eigenvalues are *non-random*[7]. This is different from the random matrix theory discussed above, where one has an ensemble from which one can average. However, one can do histograms of the above properties and find smooth statistical properties. For this reason, one refers to the statistical properties of Hamiltonian's spectra as *pseudo-random*. Is there a way to understand where this

[7] One can write the Hamiltonian on the computer and diagonalize it numerically: the eigenvalues are fixed.

emergent pseudorandomness is coming from? We have a separation of scales: at given energy $E = Ne$ (which extensive quantity), the level spacing between nearby levels is $D_E^{-1} \sim e^{-N}$ exponentially suppressed. Hence, one can imagine that the macroscopic properties on the slow scales do not depend on the details of the small one. Hence, they should be invariant under averages (reshuffling) for small energy intervals. Summarizing, *spectral properties look as the one of random matrices in a small energy window.*

3. The intuitions about the emergence of random matrix statistics and its dynamical properties came about by studying the semi-classical limit for $\hbar \to 0$: classical systems (whose classical phase-space is integrable or classically chaotic), which are then quantized. The main two conjectures can be summarized as follows:

   Berry-Tabor conjecture [10]: a quantum system whose corresponding classical counterpart is integrable has a spectrum that can be taught as a sequence of independent random variables exhibiting Poisson statistics with $P(s) = e^{-s}$
   The qualitative argument underlying the conjecture is very intuitive and can be illustrated using the simple example of N harmonic oscillators with incommensurate frequencies. The incommensurability here is precisely the condition defining that this system is "generic" integrable, and we can consider a Hamiltonian

   $$\hat{H} = \sum_{j=1}^{N} \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k \ . \tag{17}$$

   The frequencies ω_k can e.g. represent the normal modes of a harmonic chain. The energy spectrum of this model immediately follows as the sum of the energies of different modes:

   $$E_{n_1,\dots n_N} = \sum_k \hbar\omega_k n_k \tag{18}$$

   where n_k are arbitrary integers. Unless the frequencies ω_k are commensurate with each other, the high-energy states are completely uncorrelated because states with similar energy can arise from very different combinations of the occupation numbers n_k. Such a lack of correlations between nearby levels is characteristic of a Poisson process, where energy levels are randomly chosen. Poissonian statistics would also be observed in models with additional symmetries, even if Wigner-Dyson statistics appear within individual symmetry sectors, as evidenced by the possibility of level crossings between states corresponding to different symmetry sectors.

   Bohigas-Giannoni-Schmit (BGS) conjecture [13]: the level statistics of quantum systems that have a classically chaotic counterpart are described by Wigner-Dyson statistics of the energy levels.
   BGS studied a single particle placed in an infinite potential well with the shape of a Sinai billiard and found that at high energies (i.e., in the semi-classical limit), the level statistics is described by the Wigner-Dyson distribution, provided that one looks at a sufficiently narrow energy window. See Fig.??a. In the semiclassical regime, this conjecture has been firmly established by explicitly identifying contributions from periodic orbit theory to the random matrix spectral form factor [14–16].

4. These properties survive the limit $\hbar \to 1$. In fact, Eqs.(13)(15) and (16) have been shown to apply to locally interacting many-body Hamiltonians, as the ones in Eq.(34), in a myriad of numerical simulations. See Fig.??b in reference to a lattice model of spinless

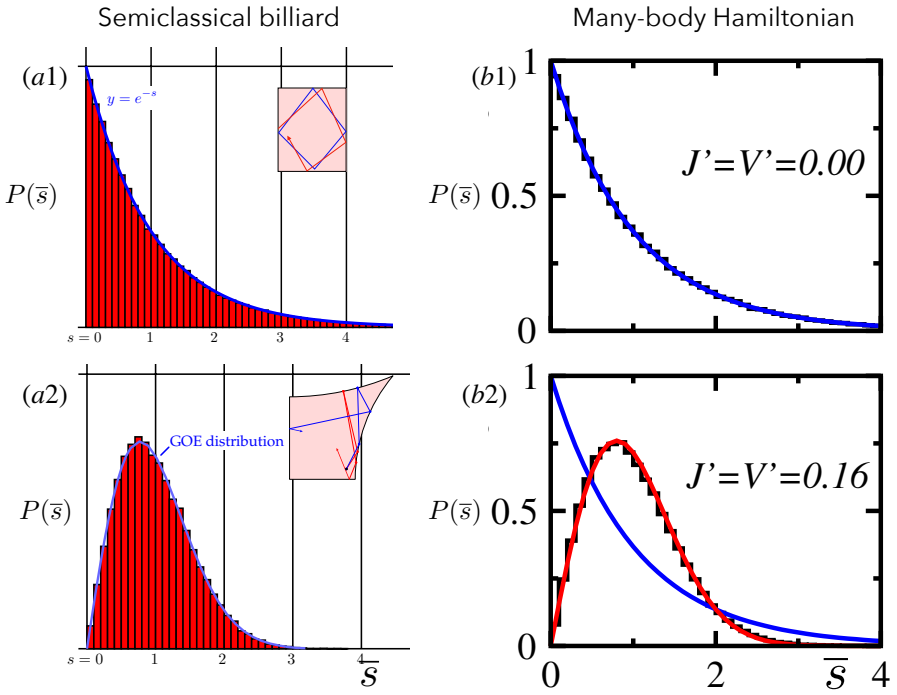

Figure 3: Examples of level spacing distribution (a) in semi-classical models (adapted from Ref. [11]) and (b) many-body lattice-system (adapted from Ref. [12]).

fermions studied in Ref. [12]. Remarkable progress has been achieved recently in the field of quantum circuits, where the linear in-time growth of the spectral form factor was proved in two different "minimal models" of chaos in quantum local circuits [17, 18].

5. The emergence of Wigner-Dyson statistics for the level spacing has been considered for a long time *the* defining property of "quantum chaos" and almost its own definition (even for systems that do not have a semiclassical limit at all). More recently, it has become apparent that the level spacing ratio (15) alone is not an exhaustive probe for the dynamic behavior of many-body systems. We will see examples later.

6. One of the few counter-examples for the conjectures described above is given by *arithmetical chaos* [19, 20]. Classical chaotic motion is defined as free motion on hyperbolic billiards generated by the so-called arithmetic groups. However, the level spacing is found to obey Poisson level spacing.

1.3 The Berry conjecture

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1.4 A toy model for ETH

To gain insights on the statistical properties of the ETH matrix elements, let us now pause with energy eigenstates and consider *random vectors* $|i\rangle$ of "truly" random matrices of dimension $D \times D$ [8]. Such vectors are characterized by components

$$\langle i\,|\alpha\rangle \equiv U_{i\alpha} \tag{19}$$

[8] Here with truly we refer to the fact that there is an ensemble we are can average on, as opposed to the ETH case.

which are uniformly distributed over the unitary group $U(D)$ (or the orthogonal or symplectic).

To truly appreciate the significance of the statistical properties of eigenvectors $|i\rangle$ of random matrices, one must analyze the matrix elements of general Hermitian operators A on a random basis. Hence, we will now study the statistical properties of

$$A_{ij} = \langle i|A|j\rangle \ , \tag{20}$$

where $|i\rangle$ is "random eigenvectors", which can also be seen as the eigenvectors of a rotationally invariant random matrix. The operator A can always be diagonalized as $A = \sum_\alpha A_\alpha |\alpha\rangle \langle\alpha|$, therefore Eq.(20) reads

$$A_{ij} = \sum_\alpha A_\alpha \langle i|\alpha\rangle \langle\alpha| j\rangle = \sum_\alpha A_\alpha U_{i\alpha} U^\dagger_{\alpha j} \ , \tag{21}$$

where we have used $U_{i\alpha} \equiv \langle i|\alpha\rangle$ in Eq.(199). This shows that the eigenvectors' statistical properties are reflected in operators' statistical properties or matrix elements on a random (rotationally invariant) basis.

We restrict ourselves to the *unitary case*, whose exact expressions are given by the Weingarten calculus discussed in the Appendix. Thus, using Es.(200a), we can compute the average

$$\overline{A_{ij}} = \sum_\alpha A_\alpha \overline{U_{i\alpha} U^\dagger_{\alpha j}} = \frac{1}{D} \sum_\alpha A_\alpha \delta_{ij} = \langle A\rangle \delta_{ij} \ , \tag{22}$$

where we have defined the normalized trace $\langle A\rangle = \mathrm{Tr}\,(A)/D$. The average of products of two matrix elements reads

$$\overline{|A_{ij}|^2} = \overline{A_{ij} A^\dagger_{ji}} = \sum_{\alpha\beta} A_\alpha A_\beta \overline{U_{i\alpha} U^\dagger_{\alpha j} U_{j\beta} U^\dagger_{\beta i}} \ . \tag{23}$$

Using Eq.(200b) for

$$\overline{U_{i\alpha} U^\dagger_{\alpha j} U_{j\beta} U^\dagger_{\beta i}} = \frac{1}{D^2-1} \left(\delta_{ij} + \delta_{\alpha\beta} - \frac{1}{D} - \frac{\delta_{\alpha\beta}\delta_{ij}}{D} \right) , \tag{24}$$

one has for $i = j$:

$$\overline{A^2_{ii}} - \overline{A_{ii}}^2 = \frac{\langle A^2\rangle - \langle A\rangle^2}{D} + \mathcal{O}(D^{-2}) \tag{25}$$

while for $i \neq j$:

$$\overline{|A_{ij}|^2} = \frac{\langle A^2\rangle - \langle A\rangle^2}{D} + \mathcal{O}(D^{-2}) \ , \tag{26}$$

thus the fluctuations of the matrix elements depend on the second cumulant

$$\kappa_2(A) = \langle A^2\rangle - \langle A\rangle^2 \ ,$$

where one can also define the first cumulant $\kappa_1(A) = \langle A\rangle = \overline{A_{ii}}$ which instead appears in the diagonal averages. The final steps of the derivations of Eqs.(25)-(26) are left as Exercise 4.9. Note that the "non-gaussian term" on the right-end side of Eq.(200b) is essential to get the $\langle A\rangle^2$ contribution. This is, for instance, absent in Eqs.(18-19) of Ref. [21], derived using only the Gaussian approximation. Combining these expressions, we find that at the leading order in $1/D$, the first two moments of the matrix elements of generic operators can be re-written as

$$\boxed{A_{ij} = \langle A\rangle \delta_{ij} + \frac{R_{ij}}{\sqrt{D}} \sqrt{\kappa_2(A)}} \ . \tag{27}$$

Comments:

- Here we have introduced a new random matrix R_{ij} (complex for the UE), which has zero mean and variance one, i.e., $\overline{R_{ij}} = 0$, $\overline{|R_{ij}|^2} = 1$.
- Note that Eqs.(25)-(26) express the known fact that for a unitary rotationally invariant matrix, $\mathrm{Var}(A_{ii}) = \mathrm{Var}(A_{i\neq j})$, see e.g. Ref. [22]
- The same calculation can be repeated for the COE ensemble, leading to exactly the same result as in Eq.(27), where now R_{ij} is a real symmetric matrix, such that $\overline{R_{ij}} = 0$, $\overline{R_{ii}^2} = 2$ and for $i \neq j$: $\overline{R_{ij}^2} = 1$, in compliance with the fact that $\mathrm{Var}(A_{ii}) = 2\mathrm{Var}(A_{i\neq j})$ as discussed for instance in Ref. [23] .
- We remark that Eq.(27) is devised only to reproduce the first and second moment of expectation values of matrices at the leading order in $1/D$:

$$\overline{A_{ii}} = \kappa_1(A) + \mathcal{O}(D^{-1/2}) \tag{28a}$$

$$\overline{A_{ii}A_{ii}} = \overline{A_{ii}}^2 + \mathcal{O}(D^{-1}) \tag{28b}$$

$$\overline{A_{ij}A_{ji}} = \frac{\kappa_2(A)}{D} + \mathcal{O}(D^{-2}) \quad i \neq j \ . \tag{28c}$$

  and it does not say anything about the averages of higher-order products. In particular, it does *not* say that the matrix elements are uncorrelated.
- Actually, one can show that the previous relation generalizes to expectation values of higher order products, at the leading order in $1/D$, in the following way [24]:

$$\overline{A_{i_1i_2}A_{i_2i_3}\ldots A_{i_ni_1}} = \frac{\kappa_n(A)}{D^{n-1}} \qquad \text{dinstinct indices} \tag{29}$$

$$\overline{A_{i_1i_2}\ldots A_{i_{k-1}i_1}A_{i_1i_{k+1}}\ldots A_{i_ni_1}} = \overline{A_{i_1i_2}\ldots A_{i_{k-1}i_1}A_{i_1i_{k+1}}\ldots A_{i_ni_1}} \qquad \text{repeated indices} \tag{30}$$

  where $\kappa_n(A)$ are higher order connected correlation functions, known as *free cumulants*, defined in Free Probability theory that we will introduce at length in the next section. At the first few orders, they read

$$\kappa_1(A) = \langle A\rangle \tag{31a}$$

$$\kappa_2(A) = \langle A^2\rangle - \langle A\rangle^2 \tag{31b}$$

$$\kappa_3(A) = \langle A^3\rangle - 3\langle A^2\rangle\langle A\rangle + 2\langle A\rangle^3 \ , \tag{31c}$$

$$\kappa_4(A) = \langle A^4\rangle - 2\langle A^2\rangle^2 - 4\langle A\rangle\langle A^3\rangle + 10\langle A^2\rangle\langle A\rangle^2 - 5\langle A\rangle^4 \ . \tag{31d}$$

1.5 Standard ETH

Let us now turn to the properties of chaotic eigenvectors $|E_i\rangle$. ETH is formulated as an ansatz for the matrix elements of observables on the basis of the eigenstates. For a generic physical observable $\hat{A}$, it reads [2]

$$A_{ij} \equiv \langle E_i|\hat{A}|E_j\rangle = \mathcal{A}(e^+)\,\delta_{ij} + e^{-S(E^+)/2} f_{\hat{A}}(e^+,\omega) R_{ij} \ , \tag{32}$$

where

- $E^+ = (E_i + E_j)/2$ is the average energy of the two eigenstates, which is extensive, i.e. $E^+ = Ne^+$ with e^+ the energy density;
- $\omega = E_i - E_j$ is the energy difference;

- $\mathcal{A}(e^+)$ is the micro-canonical expectation value and it is assumed to be a smooth function of e^+;
- $S(e^+)$ is the thermodynamic entropy, defined as the logarithm of the density of states, i.e. $e^{S(E)} = \sum_i \delta(E_i - E)$. This is also an extensive function and sometimes we will denote it $S(E) = Ns(E/N) = Ns(e)$ in terms of the entropy density $s(e)$;
- $f_{\hat{A}}(e^+, \omega) = f_{\hat{A}}(e^+, -\omega)$ is a real smooth function of its two arguments of order one; for Hamiltonian systems, this function decays at large frequencies indicating that eigenstates with very different energies shall have small matrix elements when probed with local observables;
- The numbers R_{ij} are erratically fluctuating variables – *pseudorandom matrices* –, that can be seen as random real or complex numbers with zero mean and unit variance ($\overline{R_{ij}^2} = 1$ or $\overline{|R_{ij}|^2} = 1$ respectively) [9].

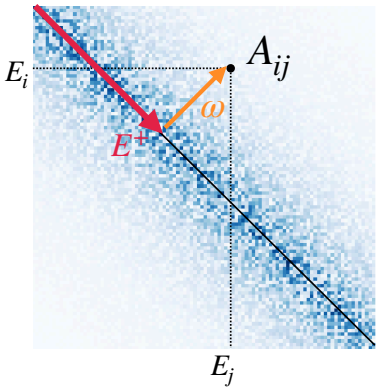


Figure 4: Pictorial representation of the matrix elements A_{ij} in the energy eigenbasis as described by ETH (32).

Comments:

1. The validity of ETH is restricted to states which have a finite energy density, away from the edges of the spectrum. This excludes the ground state and low-lying excited states or states with the highest energies, where the spectrum is more sparse. ETH is said to hold in a *strong* (weak) sense if all (almost all) the eigenstates at the centre of the spectrum obey Eq.(32). We will discuss this below.

2. ETH is expected to hold for a class of **physical observables**, which are usually referred to as "local" or sum of local ones, as demonstrated numerically by an extensive quantity of numerical experiments in spin chains [10]. We can think about the definition of physical observables in the following way. The Hamiltonian which defines the problem is a function of some basis operators which define the local structure of the problem, for instance in a spin chain one has $\hat{H} = \hat{H}(\{\hat{\sigma}_i^{x,y,z}\})$. Hence the class of physical observables shall be defined from operators which are also analytic functions of the basis operators, i.e. $\hat{A} = \hat{A}(\{\hat{\sigma}_i^{x,y,z}\})$. It is clear then that ETH will fail for very non-local and non-analytic

[9] The fictitious statistical average $\overline{\bullet}$ can be intended as a average over small energy windows or as an average over some small perturbations applied to the Hamiltonian [3].

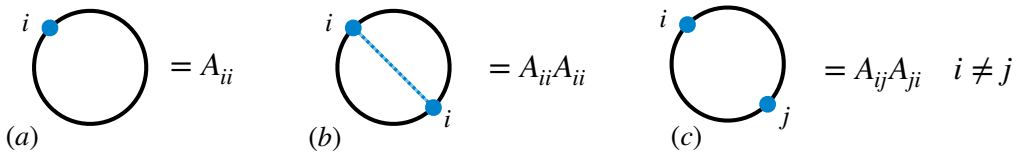
[10] For example in Ref. [31], Garrison and Grover conjecture that Eq.(32) holds for all operators within a subsystem A when the volume V_A of subsystem A is such that $V_A/V \to 0$ when the total volume $V \to \infty$. Furthermore, they discuss the case in which the support of the operator scales like a fraction $f = V_A/V$ of the total value and argue that Eq.(32) holds up to $f = 1/2$.

function operators such as eigenstate projectors $|E_i\rangle\langle E_i|$.

3. **Rewriting ETH.** At this level, no assumption has been made on the distribution of the R_{ij} (besides its mean and variance). Summaryzing, ETH states that *physical observables A in the energy eigenbasis look like pseudorandom matrices with smooth statistical properties*:

$$\overline{A_{ii}} = \mathcal{A}(E_i) \tag{33a}$$
$$\overline{A_{ii}A_{ii}} = [\overline{A_{ii}}]^2 \tag{33b}$$
$$\overline{A_{ij}A_{ji}} = |f(e^+,\omega)|^2 e^{-S(E^+)} \quad \text{for } i \neq j\ , \tag{33c}$$

where the corrections to these estimates are exponentially suppressed to respect to the leading term expressed here. To represent these conditions is useful to introduce some simple diagrams:



Here the matrix elements A_{ij} are represented by edges which connect two energy indices (dots) i and j. The contractions between two or more indices are represented by lines that connect the vertices. The blue dots indicate that the indices are all different.

Example. It is useful to fix the ideas with an example that we will use to illustrate the main features of the Eigenstate Thermalization Hypothesis. The one-dimensional Ising spin-1/2 chain in a tilted field is described by the Hamiltonian

$$H = \sum_{i=1}^{L} w\hat{\sigma}_i^x + \sum_{i=1}^{L} h\hat{\sigma}_i^z + \sum_{i=1}^{L} J\hat{\sigma}_i^z\hat{\sigma}_{i+1}^z\ , \tag{34}$$

where $\hat{\sigma}_i^\alpha$ are Pauli matrices on site i in the direction $\alpha = x, y, z$. We measure the energies in units of J and set $w = \sqrt{5}/2$, $h = (\sqrt{5}+5)/8$. With periodic boundary conditions, this system is characterized by translational and inversion symmetry. We consider the subsector at fixed $k = 0$ momentum and the even inversion subsector. Being one-dimensional, here $N = L$.

Following the comment above, examples of physical observables in this case are

$$\hat{\sigma}_{L/2}^\alpha\ , \hat{\sigma}_1^\beta\hat{\sigma}_2^\alpha \quad \text{few site observables}$$
$$\frac{1}{\sqrt{L}}\sum_{r=1}^{L}\hat{\sigma}_1^\alpha \quad \text{collective observables}\ .$$

1.6 From matrix elements to observables

We now focus on the implication of ETH on observables at equilibrium, with reference to equilibrium distributions in statistical mechanics. This will describe correlations and dynamics

after the equilibriation process descrived in the previous section has already occurred. We will hence consider Gibbsian expectation values defined as

$$\langle \bullet \rangle_\beta \equiv \mathrm{Tr}\left(\frac{e^{-\beta \hat{H}} \bullet}{Z} \right) , \tag{35}$$

where the inverse temperature β is intended as implicit function of the energy density $e_\beta = E_\beta/N = \langle \hat{H} \rangle_\beta /N$.

1.6.1 Thermalization from ETH

The ETH ansatz (32) is enough to deduce thermalization in isolated quantum systems. The infinite time-average of a local operator $\hat{A}$, by virtue of Eq.(7), is given by

$$\begin{aligned} [A]_\infty &= \sum_i |c_i|^2 A_{ii} = \sum_i |c_i|^2 \mathcal{A}(E_i) + O(e^{-S/2}) \\ &\simeq \mathcal{A}(E) + \frac{1}{2}\left(\frac{\partial^2 \mathcal{A}}{\partial E^2} \right) \Delta_E^2 = \langle E|\hat{A}|E\rangle + O(1/N) , \end{aligned} \tag{36}$$

where Δ_E^2 is the energy variance of the initial state [cf. Eq.(3)]. On the right end side of the first line, we have substituted the ETH ansatz (32) and considered the extensivity of the entropy $S(E,N) = Ns(E/N)$. On the second line, we first used the fact that generic initial states have a narrow distribution around an average energy E with small energy variance [11] [cf. Eq.(3)]. Secondly, one notices that the correction to the microcanonical result is subleading, because of Eq.(3) again.

ETH not only describes the relaxation to the microcanonical prediction, but it also explains why instantaneous observables remain close to it at most later times. In fact, one can compute the time fluctuations of the expectation values of the observable $\hat{A}$ as

$$\sigma^2_{\hat{A}} \equiv [A^2]_\infty - [A]^2_\infty = \sum_{mn, m\neq n} |c_i|^2 |c_m|^2 \, |A_{nm}|^2 \leq \max |A_{nm}|^2 \propto e^{-S(\bar{E})} , \tag{38}$$

where one first inserts the energy eigenbasis in the definition (6), then one computes the infinite-time average and lastly one estimates the maximum with the ETH ansatz in Eq.(32). Thus, the time fluctuations of the expectation values of the observable are exponentially small in the system size. This implies the existence of the limit in Eq.(39), without the need for infinite time-averaging.

As we will comment below discussing equilibrium observables, in the presence of equivalence of ensembles[12], local observables thermalize to the thermal expectation value at temperature $1/\beta_E$ (2), i.e.[13] ,

$$\lim_{t\to\infty} \lim_{N\to\infty} \langle \hat{A}(t) \rangle = \mathrm{Tr}\left(\hat{A} \frac{e^{-\beta_E \hat{H}}}{Z} \right) . \tag{39}$$

[11] In this case, one can perform a Taylor expansion around that energy

$$\mathcal{A}(E_i) = \mathcal{A}(E) + (E_i - E)\,\mathcal{A}'(E) + \frac{1}{2}(E_i - E)^2 \,\mathcal{A}''(E) + \dots \tag{37}$$

where $\mathcal{A}'(E)$, $\mathcal{A}''(E)$ are the first and second partial derivatives of the microcanonical function $\mathcal{A}(x)$ evaluated at $x = E$. Substituting back, at $o(\Delta_E^3)$ we obtain Eq.(36).

[12] Equivalence of ensemble holds when observables take practically the same value for the greatest part of the microscopic states accessible in each ensemble, namely the relevant fluctuations are negligible in the thermodynamic limit. Mathematically this is associated with the extensivity, additivity, and concavity of the thermodynamic functions, such as the energy, the entropy, or the free energy [**?**]. This is well established for standard statistical mechanics of short-range interacting Hamiltonians, see e.g. Ref. [**?**]. Notable exemptions include long-range interacting systems [**?**].

[13] Notice the order of the limits in Eq.(39): The thermodynamic limit $N \to \infty$ has to be taken first because, for finite-size systems, recurrences occur at exponentially large times in N. Alternatively, one resorts to the infinite time-average in Eq.(6).

1.6.2 Expectation values

We now compute $\langle \hat{A} \rangle_\beta$ using Eq.(32). By expanding in the energy eigenbasis, and passing to the continuum ($\sum_i \to \int dE e^{S(E)}$), one obtains

$$\langle \hat{A} \rangle_\beta = \frac{1}{Z} \int dE e^{S(E)-\beta E} \mathcal{A}(E) + O(e^{-S/2}) \, . \tag{40}$$

The energy integral is then performed via the saddle point technique, by noticing that both the arguments appearing in the exponent are extensive. This fixes the energy and the temperature according to the standard thermodynamic prescription $\beta = \partial S / \partial E$. The result with the proper normalization yields

$$\langle \hat{A} \rangle_\beta = \mathcal{A}(e_\beta) + O(1/N) \simeq \langle E_\beta | \hat{A} | E_\beta \rangle \, . \tag{41}$$

Putting together the integral in Eq.(40) with Eq.(36), one obtains, at the leading order in N,

$$[A]_\infty \simeq \langle \hat{A} \rangle^{\beta_e} \simeq \mathcal{A}(e) \simeq \langle E | \hat{A} | E \rangle \, . \tag{42}$$

Namely, the equilibrium values of observables after a quenched dynamics correspond to their thermal expectation values which can in turn and be calculated on single eigenstates corresponding to the average energy. This is the essence of the eigenstate thermalization hypothesis.

Example. The example of local observables is shown in Fig.6 where we display the diagonal expectation value of the collective observable along z, i.e. $\hat{A} = \frac{1}{\sqrt{L}} \sum_r \sigma_r^z$. The plot shows that by increasing the system size the fluctuations of the diagonal elements A_{nn} become smaller and the average becomes a smooth function of the energy density.

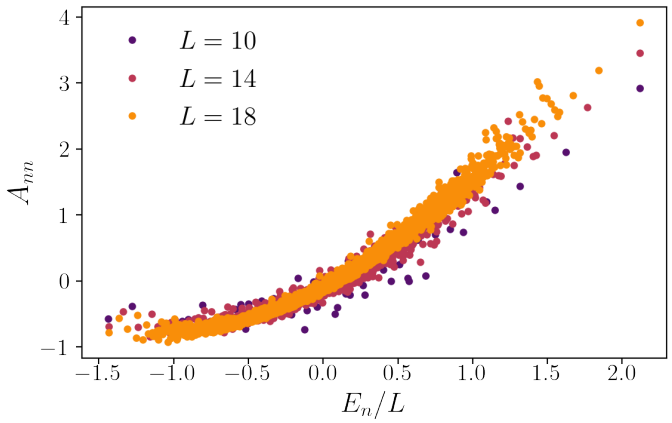


Figure 5: Diagonal expectation values of the collective magnetization $\hat{A} = \frac{1}{\sqrt{L}} \sum_r \sigma_r^z$ in the eigenstates of the Ising model (34) for increasing system size $L = 10, 14, 18$.

1.6.3 Dynamical correlation functions

We now want to see how ETH enters the description of *two-point* correlation functions at equilibrium such as

$$\langle \hat{A}(t_1) \hat{A}(t_2) \rangle_\beta \tag{43}$$

These correlation functions describe fundamental properties of classical and quantum dynamics: they encode the linear response to external perturbation (via the Kubo formula), they allow to define the fluctuation-dissipation theorem and describe the transport properties [32].

Since these depend only on the time differences $t = t_1 - t_2$, we may write everything in terms of independent time differences t, or set $t_2 = 0$, without loss of generality. In particular, we are interested in the following connected correlation function

$$\kappa_2(t) \equiv \langle \hat{A}(t) \hat{A} \rangle_\beta - [\langle \hat{A} \rangle_\beta]^2 \, . \tag{44}$$

In what follows, we will see that the ETH has a precise meaning for the two-point connected correlator. We focus here the case of *local* or *intensive* operators, see the comments below for a broader discussion. The first ETH implication is that the two-point function is given, at the leading order in N, only by sums over distinct indices, namely:

$$\kappa_2(t) =_{\text{ETH}} k_2(t) + \mathcal{O}(N^{-1}) \qquad \text{with} \quad k_2(t) \equiv \sum_{i\neq j} \frac{e^{-\beta E_i}}{Z} |A_{ij}|^2 e^{i(E_i-E_j)t/\hbar} \,. \tag{45a}$$

The consequence is that this quantity can be computed using the off-diagonal ETH in Eq.(32), leading to

$$\kappa_2(t) =_{\text{ETH}}= \text{FT}\left[e^{-\beta\hbar\omega/2}|f(E_\beta,\omega)|^2\right] + \mathcal{O}(N^{-1}) \,, \tag{45b}$$

where $\text{FT}[f(\omega)] = \int d\omega e^{i\omega t} f(\omega)$ is the Fourier transform and $E_\beta = \langle \hat{H} \rangle_\beta$ is the thermal energy. Below, you can find the step-by-step derivations of the ETH result, which may also be left as an exercise, see Exercise 3 . The reader can also directly skip to the comments below.

Derivation of Eq.(45a) By expanding the first term of the definition in Eq.(57b) in the energy eigenbasis we have

$$\kappa_2(t) = \sum_{i\neq j} \frac{e^{-\beta E_i}}{Z} |A_{ij}|^2 e^{i(E_i-E_j)t/\hbar} + \sum_i \frac{e^{-\beta E_i}}{Z} A_{ii}^2 - [\langle \hat{A} \rangle_\beta]^2 \,, \tag{46}$$

where we have rewritten the double sum $\sum_{ij}$ distinguishing dinstinct and repeated indices. Let us now focus on the second term of the expression:

$$\begin{aligned} \sum_i \frac{e^{-\beta E_i}}{Z} A_{ii}^2 &= \sum_i \frac{e^{-\beta E_i}}{Z} \mathcal{A}(E_i)^2 + O(e^{-S/2}) \\ &\simeq \frac{1}{Z} \int dE e^{S(E)-\beta E} \mathcal{A}(E)^2 \\ &\simeq \mathcal{A}(e_\beta)^2 \simeq [\langle \hat{A} \rangle_\beta]^2 + \mathcal{O}(N^{-1}) \,, \end{aligned} \tag{47}$$

where from in the first line we have substituted the ETH factorization of Eq.(33b), from the first to the second we have substituted summations with integrals and from the second to the third we have solved the integral by saddle point as in Eq.(40). Finally, we notice that the result corresponds to the square of the one-point function $\langle \hat{A} \rangle_\beta$. This relation tells us a very important consequence of ETH: *the factorization of observables with repeated indices!* Subsituting Eq.(47) into Eq.(46), the term with repeated indices cancels at the leading order the disconnected component, and we can re-write the connected correlation as as in Eq.(**??**) namely

$$\kappa_2(t) = k_2(t) + \mathcal{O}(N^{-1}) \,, \tag{48}$$

where we have defined the *ETH connected correlator* as the simple sum over different indices

$$k_2(t) \equiv \sum_{i\neq j} \frac{e^{-\beta E_i}}{Z} |A_{ij}|^2 e^{i(E_i-E_j)t/\hbar} \,. \tag{49}$$

Derivation of Eq.(45b) This derivation can be found in [21]. We start from the result above and use ETH, namely:

$$\begin{aligned} k_2(t) &= \sum_{i\neq j} \frac{e^{-\beta E_i}}{Z} |A_{ij}|^2 e^{i(E_i-E_j)t/\hbar} \\ &= \sum_{i\neq j} \frac{e^{-\beta E_i}}{Z} |f(E_{ij}^+, \omega_{ij})|^2 e^{-S(E_{ij}^+)} e^{i(E_i-E_j)t/\hbar} \\ &= \int dE_1 dE_2 e^{S(E_1)+S(E_2)-S(E_{12}^+)} \frac{e^{-\beta E_1}}{Z} |f(E_{12}^+, \omega_{ij})|^2 e^{i(E_1-E_2)t/\hbar} \\ &= \int d\omega dE e^{S(E+\omega/2)+S(E-\omega/2)-S(E)} \frac{e^{-\beta E-\beta\omega/2}}{Z} |f(E,\omega)|^2 e^{i\omega t/\hbar} , \end{aligned} \tag{50}$$

where from the first to the second line we substituted the ETH ansatz (33c), from the second to the third we have substituted summations with integrals, i.e. $\sum_i \to \int e^{S(E_1)} dE_1$, finally from the third to the last line we have performed a change of variables in the integration

$$E = (E_1+E_2)/2 \quad \omega = E_1 - E_2 \quad \longleftrightarrow \quad E_{1,2} = E \pm \omega/2 \quad \text{(Wigner variables)} .$$

We can now expand the entropies around the average energy as follows

$$\begin{aligned} S(E \pm \omega/2) &= S(E) \pm S' \frac{\omega}{2} + \frac{1}{2} S'' \frac{\omega^2}{4} + \mathcal{O}(\omega^3) \\ &= S(E) \pm \beta \frac{\omega}{2} + \frac{1}{8} \frac{\beta^2}{C} \omega^2 \end{aligned} \tag{51}$$

where we used the thermodynamic definition of inverse temperature $\beta = S' = \frac{\partial S}{\partial E}$ and of extensive heat capacity $C = Nc$ defined from $\beta^2/C = S'' = \frac{\partial^2 S}{\partial E^2}$. Thus, by summing these two contributions and subtracting by $S(E)$ in the expression above, we have

$$k_2(t) = \int dE \frac{e^{S(E)-\beta E}}{Z} \int d\omega e^{-\beta\omega/2} e^{\beta^2\omega^2/8Nc} |f(E,\omega)|^2 e^{i\omega t/\hbar} . \tag{52}$$

We note that

$$e^{\beta^2\omega^2/8Nc} \simeq 1 + \mathcal{O}(\omega^2/N) , \quad N \gg 1$$

this term becomes different from one only at very large frequencies $\omega \propto \sqrt{N}$. However, we are multiplying against $|f(E,\omega)|^2$, which is expected to rapidly decay at large frequencies and hence we can substitute $e^{\beta^2\omega^2/8Nc} \simeq 1$.

Finally, we can solve the integral over energy again by saddle point as in Eq.(40), which selects the thermal energy leading to:

$$k_2(t) = \int d\omega e^{-\beta\omega/2} |f(e_\beta, \omega)|^2 e^{i\omega t/\hbar} . \tag{53}$$

Comments:

1. We have shown that the off-diagonal ETH ansatz determines two-point dynamical correlations [cf. Eq.(53)]. By taking its Fourier transform ($\tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t) dt$) we thus have

$$\tilde{\kappa}_2(\omega) = e^{\beta\hbar\omega/2} |f(e_\beta, \omega)|^2 , \tag{54}$$

which justifies why we said that the frequency dependence of $f(e_\beta,\omega)$ encodes the physical properties of the operator. Note that this function depends on frequency also at infinite temperature $\beta = 0$ and it is the peculiar feature of ETH which distinguishes it from random matrix theory.

2. The ETH ansatz accounts also for the Fluctuation-Dissipation theorem (FDT) via the exponential factor $e^{-\beta\omega/2}$ in Eq.(54). In fact by fluctuation $C(t)$ (and the dissipation $iR''(t)$) as the real (and imaginary) parts of Eq.(57b) we have [21,33]

$$\tilde{C}(\omega) = \frac{\tilde{\kappa}_2(\omega)+\tilde{\kappa}_2(-\omega)}{2} = \cosh(\beta\hbar\omega/2)|f(e_\beta,\omega)|^2 \,, \tag{55a}$$

$$\hbar\tilde{R}''(\omega) = \frac{\tilde{\kappa}_2(\omega)-\tilde{\kappa}_2(-\omega)}{2} = \sinh(\beta\hbar\omega/2)|f(e_\beta,\omega)|^2 \,. \tag{55b}$$

which acquires the ore standard expression as

$$\hbar\tilde{R}''(\omega) = \tanh(\beta\hbar\omega/2)\,\tilde{C}(\omega) \,. \tag{56}$$

The fact that ETH accounts for the FDT (one of the cornerstones of equilibrium statistical mechanics) is an important a posteriori check.

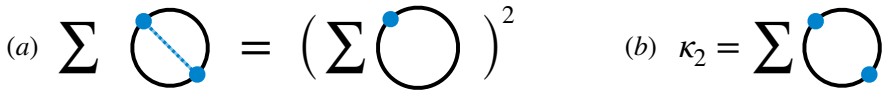
3. Let us reiterate the implications of the ETH scalings (33) on two-point functions. First of all, ETH implies the factorization over repeated indices Eq.(47):

$$\sum_i \frac{e^{-\beta E_i}}{Z} A_{ii}^2 \simeq_{\rm ETH} \mathcal{A}(e_\beta)^2 \tag{57a}$$

It is this factorization that leads to the fact that connected correlation functions are given by summations over distinct indices in Eq.(50), i.e.

$$\kappa_2(t) =_{\rm ETH} \sum_{i\neq j} \frac{e^{-\beta E_i}}{Z} |A_{ij}|^2 e^{i(E_i-E_j)t/\hbar} \tag{57b}$$

These two equations shall be regarded as an alternative definition of ETH directly on the mesoscopic observables, rather than on the individual matrix elements. It can be summarized pictorially as follows



4. Up to now, we have considered the case of local-intensive observables a_r, supported on a few adjacent sites r. All the arguments above are also valid for *collective or extensive* observables, such as sums of local ones $A_{\rm coll} = \sum_r a_r$. In this case, however, one shall proceed with some care, and the analysis shall be carried in the *microcanonical ensemble* with a small vanishing width. The issue stems from the fact that collective observables are characterized by subleading fluctuations in the system size compared to the first moment, i.e., $\kappa_2 \sim \langle\hat{A}\rangle^2/N$. Therefore, in the canonical ensemble in the expression

$$\kappa_2(t) = k_2(t) + \mathcal{O}(N^{-1}) \,,$$

the corrections $\mathcal{O}(N^{-1})$ arising from the saddle-point in the factorization, e.g., Eq.(47), becomes of the same order as the leading term, and it cannot be neglected anymore.

Thus the analysis shall be carried out in the microcanonical ensemble, where the energy width can be send to zero independently than N. We refer the reader to Ref. [34] for a detailed discussion of this issue. Note that this problem does not arise when $\overline{A_{ii}} = c$ is independent of energy density, as in the case of Floquet systems.

Example. The example of local observables is shown in Fig.6 where we display the off-diagonal matrix element of the collective observable along z, i.e. $\hat{A} = \frac{1}{\sqrt{L}} \sum_r \sigma_r^z$. The plot shows that the fluctuating off-diagonal elements in blue, selected at infinite temperature, i.e. around average energy $E_\beta = 0$.

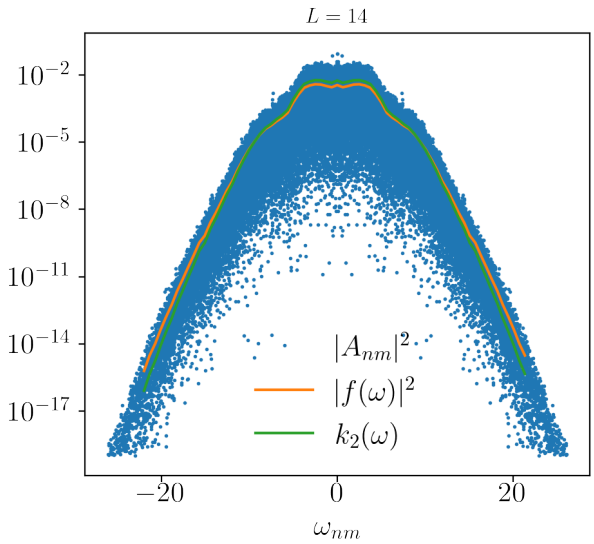
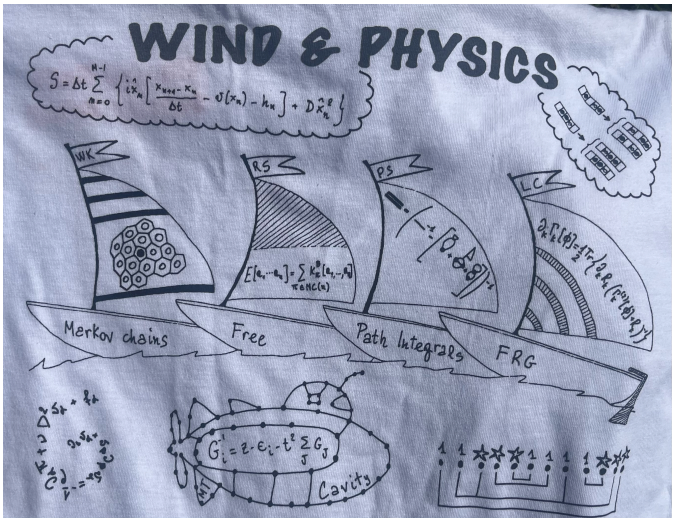


Figure 6: Off-diagonal expectation value of the collective magnetization $\hat{A} = \frac{1}{\sqrt{L}} \sum_r \sigma_r^z$ in the eigenstates of the Ising model (34) for system size $L = 14$. The individual matrix elements (blue dots) are compared with their average over small frequency windows $|f(E = 0, \omega)|^2$ (orange) and with the average of $\tilde{\kappa}_2(\omega)$.

2 Introduction to Free Probability



Concepts and Methods of Statistical Physics, The Beg Rohu Summer School

Selected bibliography:

1. (Physics-friendly introduction to Free Probability): R. Speicher, *Lecture Notes on "Free Probability Theory"*, ArXiv:1908.08125v1

Free probability – or "non-commutative probability" – can be thought of as the generalization of classical probability to non-commutative random variables, where the concept of *free independence* or "freeness" extends the one of "independence". Let us discuss some basic properties; for all the details, we refer to Refs. [35–37].

Here, we first provide a self-contained and pedagogical introduction to the definition of free cumulants, starting from the combinatorial approaches to classical cumulants. Then, we present the definition of free independence.

2.1 Classical Cumulants

Let us denote the standard cumulants of commuting random variables with "classical cumulants". Consider x a random variable with probability $p(x)$ and average $\mathbb{E}(\bullet) = \int \bullet p(x)dx$. Classical cumulants $c_n(x)$ are defined as connected correlation functions: a suitable combination of moments $m_n = \mathbb{E}(x^n)$ of the same or lower order. For instance, the first four orders read

$$c_1(x) = \mathbb{E}(x) \tag{58a}$$
$$c_2(x) = \mathbb{E}(x^2) - \mathbb{E}(x)^2 \tag{58b}$$
$$c_3(x) = \mathbb{E}(x^3) - 3\mathbb{E}(x^2)\mathbb{E}(x) + 2\mathbb{E}(x)^3 \tag{58c}$$
$$c_4(x) = \mathbb{E}(x^4) - 3\mathbb{E}(x^2)^2 - 4\mathbb{E}(x)\mathbb{E}(x^3) + 12\mathbb{E}(x^2)\mathbb{E}(x)^2 - 6\mathbb{E}(x)^4 \,. \tag{58d}$$

Notably, the specific coefficients appearing in this expression can be obtained in a combinatorial way, based on the concept of partitions. A partition π of a set $\{1, \dots n\}$ is a decomposition in blocks that do not overlap and whose union is the whole set. The set of all partitions of $\{1, 2, \dots n\}$ is denoted $P(n)$. The example of $P(4)$ for $\{1, 2, 3, 4\}$ is shown in Fig.7, where with

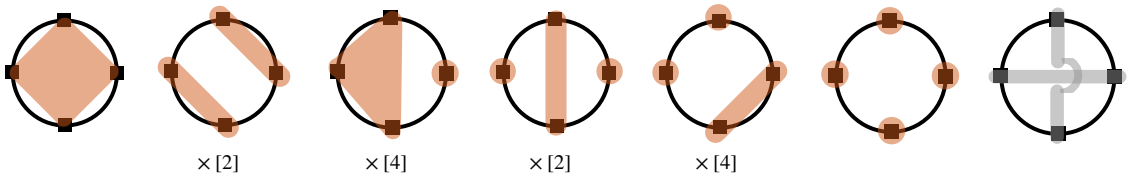


Figure 7: Set of all partitions for $n = 4$. With the colour orange, we represent the non-crossing partitions, while the crossing one is in grey. With $\times[m]$, we denote the m cyclic permutations of that partition, which determines the coefficients appearing in the moment/cumulant formulas in Eq.(58) and Eq.(66).

$\times[m]$ we denote that there are m cyclic permutations. The number of the partitions of a set with n elements is called the Bell number B_n defined recursively as $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$ with $B_1 = 1$ and $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, $B_5 = 52$, etc. The classical cumulants (58) can be defined implicitly by the moments/classical cumulants formula from the sum over all possible partitions

$$\mathbb{E}(x^n) = \sum_{\pi \in P(n)} c_\pi(x) \qquad \text{with} \quad c_\pi(x) = \prod_{b \in \pi} c_{|b|}(x)\,, \tag{59}$$

where on the right-hand side $|b|$ denotes the number of elements of the block b of the partition π. The result for the first four orders reads

$$\mathbb{E}(x) = c_1(x) \tag{60a}$$
$$\mathbb{E}(x^2) = c_2(x) + c_1^2(x) \tag{60b}$$
$$\mathbb{E}(x^3) = c_3(x) + 3c_2(x)c_1(x) + c_1^3(x) \tag{60c}$$
$$\mathbb{E}(x^4) = c_4(x) + 3c_2^2(x) + 6c_2(x)c_1^2(x) + 4c_3(x)c_1(x) + c_1^4(x)\,. \tag{60d}$$

Note that the coefficients correspond exactly to the multiplicities of each diagram. By inverting these relations one immediately finds the classical cumulants in Eq.(58).

Here we only reported the definition for a single random variable, but the same can be easily extended to families of random variables $(x_1, x_2, \dots)$ from

$$\mathbb{E}(x_1 x_2 \dots x_n) = \sum_{\pi \in P(n)} c_\pi(x_1 x_2 \dots x_n) \quad \text{with} \quad c_\pi(x_1 x_2 \dots x_n) = \prod_{b \in \pi} c_{|b|}(x_{b(1)} x_{b(2)} \dots x_{b(n)})\,, \tag{61}$$

where $b = (b(1), b(2), \dots, b(n))$ denotes the element of the block of the partition.

Summarizing, classical cumulants are connected correlation functions, whose coefficients can be computed from the combinatorial counting of partitions. A crucial property of Gaussian distributions is that cumulants of order greater than two vanish. Hence classical cumulants can be thought of as *the* connected correlations such that $c_{n>2} = 0$ for Gaussian random variables.

2.2 Free cumulants

We are now in the position to define free cumulants, which extends the previous definition to *non-commuting* variables. For definiteness, let us start by considering a $D \times D$ random matrix A and the so-called "expectation value"

$$\langle \bullet \rangle = \lim_{D \to \infty} \frac{1}{D} \mathbb{E}_{\mathcal{E}} \left[\mathrm{Tr}(\bullet) \right]\,, \tag{62}$$

which is well defined in the large D limit and normalized, i.e. $\langle \mathbb{1} \rangle = 1$. Here $\mathcal{E}$ represents a generic random matrix ensemble [14]. Note the slight abuse of notation since we also denote

[14] In the literature of Free Probability, the expectation value is usually noted by $\phi(\bullet)$ [36]

$\langle \bullet \rangle = \mathrm{Tr}\,(\bullet)\,/D$ when the argument does not involve any random matrix.

The definition of *free* cumulants is based on the combinatorics of *non-crossing partitions*, which are partitions that do not cross. The set of non-crossing partitions of $\{1,2,\dots n\}$ is denoted by $NC(n)$ and enumerated by Catalan numbers $C_n = (1+1/n)\binom{2n}{n}$ with $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, etc. Hence the number of crossing and non-crossing partitions differs from $n = 4$ on, as shown in Fig.7. Free cumulants $\kappa_n(A)$ are hence defined implicitly by

$$\langle A^n \rangle = \sum_{\pi \in NC(n)} \kappa_\pi(A) \quad \text{with} \quad \kappa_\pi(A) = \prod_{b\in\pi} \kappa_{|b|}(A)\,, \tag{63}$$

where we recall that $|b|$ is the size of each block in the partition π. This expression for the first few orders reads

$$\langle A \rangle = \kappa_1(A) \tag{64a}$$
$$\langle A^2 \rangle = \kappa_2(A) + \kappa_1(A)^2 \tag{64b}$$
$$\langle A^3 \rangle = \kappa_3(A) + 3\kappa_2(A)\kappa_1(A) + \kappa_1(A)^3 \tag{64c}$$
$$\langle A^4 \rangle = \kappa_4(A) + 2\kappa_2(A)^2 + 6\kappa_2(A)\kappa_1(A)^2 + 4\kappa_3(A)\kappa_1(A) + \kappa_1(A)^4\,, \tag{64d}$$

which, by inverting for κ_n leads to

$$\kappa_1(A) = \langle A \rangle\,, \tag{65a}$$
$$\kappa_2(A) = \langle A^2 \rangle - \langle A \rangle^2\,, \tag{65b}$$
$$\kappa_3(A) = \langle A^3 \rangle - 3\langle A^2 \rangle \langle A \rangle + 2\langle A \rangle^3\,, \tag{65c}$$
$$\kappa_4(A) = \langle A^4 \rangle - 2\langle A^2 \rangle^2 - 4\langle A \rangle \langle A^3 \rangle + 10\langle A^2 \rangle \langle A \rangle^2 - 5\langle A \rangle^4\,. \tag{65d}$$

The first difference between classical and free cumulants appears in the fourth order, as one notices by comparing the factor $2 \times \langle A^2 \rangle^2$ in Eq.(64) instead of the $3 \times \mathbb{E}(x^2)^2$ in Eq.(58).

For Gaussian random *matrices*, the free cumulants of order greater than two vanish. It is now clear that free cumulants are the direct generalization of classical cumulants to non-commuting objects and that they can be thought of as *the* connected correlations such that $\kappa_{n>2} = 0$ for Gaussian random matrices.

The definition of free cumulants can be immediately extended to different random matrices $(A_1, A_2, \dots)$ as

$$\boxed{\langle A_1 A_2 \dots A_n \rangle = \sum_{\pi \in NC(n)} \kappa_\pi(A_1 A_2 \dots A_n) \quad \text{with} \quad \kappa_\pi(A_1 A_2 \dots A_n) = \prod_{b\in\pi} \kappa_{|b|}(A_{b(1)} A_{b(2)} \dots A_{b(n)})}\,, \tag{66}$$

where $b = (b(1), b(2), \dots, b(n))$ denotes the element of the block of the partition and $|b|$ its length. As an example, consider the following partition π for $n = 8$:

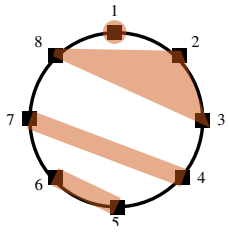


Figure 8: A non-crossing partition π of $n = 8$.

Here the the partition is $\pi = \{\{1\},\{2,3,8\},\{4,7\},\{5,6\}\}$ and the corresponding contribution reads:

$$\kappa_\pi(A_1..A_8) = \kappa_1(A_1)\kappa_3(A_2A_3A_8)\kappa_2(A_4A_7)\kappa_2(A_5A_6)\ . \tag{67}$$

By inverting the implicit definition in Eq.(66), the first few free cumulants read

$$\kappa_1(A_1) = \langle A_1\rangle\ , \tag{68a}$$
$$\kappa_2(A_1A_2) = \langle A_1A_2\rangle - \langle A_1\rangle\langle A_2\rangle\ , \tag{68b}$$
$$\kappa_3(A_1A_2A_3) = \langle A_1A_2A_3\rangle - \langle A_1A_2\rangle\langle A_3\rangle - \langle A_1A_3\rangle\langle A_3\rangle - \langle A_2A_3\rangle\langle A_1\rangle + 2\langle A_1\rangle\langle A_2\rangle\langle A_3\rangle\ . \tag{68c}$$

The inversion of Eq. (66) can be made systematic using combinatorial tools [35], leading to the so-called Möbius inversion formula:

$$\kappa_\pi(A_1,\ldots,A_n) = \sum_{\sigma\in NC(n),\sigma\le\pi} \langle A_1,\ldots,A_n\rangle_\sigma\, \mu(\sigma,\pi)\ , \tag{69}$$

where $\langle\bullet\rangle_\sigma$ is the product of moments, one for each term of the partition σ, viz.

$$\langle A_1,\ldots,A_n\rangle_\sigma := \prod_{\beta\in\sigma}\left\langle A_{j_1}\ldots A_{j_\beta}\right\rangle \tag{70}$$

and $\mu(\sigma,\pi)$ is the so-called **Möbius function** and $\sigma\le\pi$ indicates that the sum is restricted to the partitions where each block of σ is contained in one of the blocks of π.

2.3 A little on generating functions

...

2.4 Free independence or freeness of non-commuting variables

To present the definition of freeness, we focus our analysis to the case of $D\times D$ random matrices

$$A \quad \text{and} \quad B$$

which could be drawn from different ensembles. We comment on a technical assumption: the ensembles of A and B must be defined as a function of D; this can be done naturally in the context of many-body physics, where local (or few-body) operators can be naturally extended to act on a larger Hilbert space. In free probability, it is further required that the moments $\langle A^k\rangle$ exist for all k (and similarly for B). Again, this assumption is naturally satisfied for local (or few-body) operators embedded in a many-body system. Finally, an operator with zero expectation value is said to be *centered*.

Definition of freeness In terms of the expectation value above, we can define the notion of freeness for A and B. These are said to be *asymptotically free* if [38,39]

$$\left\langle \left(A^{n_1} - \langle A^{n_1}\rangle\right)\left(B^{m_1} - \langle B^{m_1}\rangle\right)\cdots\left(A^{n_k} - \langle A^{n_k}\rangle\right)\left(B^{m_k} - \langle B^{m_k}\rangle\right)\right\rangle = 0 \tag{71}$$

for all $n_1,\ldots m_k \ge 1$. Here, asymptotically refers to the fact that it has to be valid in the limit $D\to\infty$, which in free probability is absorbed in the definition of the expectation value in Eq.(62). The product of the terms above is said to be *alternating*, since consecutive powers of neither A nor B appear. The definition of freeness can then be enunciated as: "A and B are free *if the alternating product of centered elements is centered*". The definition can be extended to

different sequences of matrices $\{A_1, A_2, \dots\}$ and $\{B_1, B_2, \dots\}$. Let us remark that the definition of freeness always depends on the expectation value $\langle \bullet \rangle$, cf. Eq.(62).

This definition is not particularly transparent, but it has to be understood as a rule for computing mixed moments of non-commuting variables from knowledge of the moments of individual variables. For instance, if A and B are two free random matrices, we can simplify the expectation value of the product

$$\langle ABAB \rangle = \langle A^2 \rangle \langle B \rangle^2 + \langle B^2 \rangle \langle A \rangle^2 - \langle B \rangle^2 \langle A \rangle^2 \,, \tag{72}$$

as can be seen by expanding the products and setting to zero all alternating products, e.g. using that

$$\Big\langle (A - \langle A \rangle)(B - \langle B \rangle)(A - \langle A \rangle)(B - \langle B \rangle) \Big\rangle = 0. \tag{73}$$

Definition using free cumulants The definition of freeness simplifies using the concept of *free cumulants* introduced above in Section 2.2. A salient property of free cumulants is that they characterize freeness by the vanishing of the *mixed* [15] free cumulants, as shown e.g. in Ref. [37]. In the case of the two random matrices, it can be shown that the definition of freeness in Eq.(71) is equivalent to the following: B and A are said to be *free* (asymptotically, for large size) if their free cumulants vanish as follows:

$$\boxed{\kappa_{2n}(A, B, \dots B) = 0 \quad \forall n \,.} \tag{74}$$

Actually, the definition of freeness involves arbitrary mixed sequences of A and B.

In the case of different matrices, one has that two families $\{A_1, A_2, \dots\}$ and $\{B_1, B_2, \dots\}$ are **asymptotically free** if all mixed cumulants are zero:

$$\kappa_n(a_1, a_2, \dots, a_n) = 0 \quad \forall n \,, \tag{75}$$

where $a_1, \dots, a_n$ are letters of $\{A_1, A_2 \dots\}$ and $\{B_1, B_2, \dots\}$, containing at least a pair (A_i, B_j) for some i, j. in particular for any n and for any set of indices $i_1, \dots, i_n$, and $j_1, \dots, j_n$

$$\begin{gathered} \text{if } \{A_1, A_2, \dots\} \text{ and } \{B_1, B_2, \dots\} \text{ are asymptotically free,} \\ \text{then } \kappa_{2n}(A_{i_1}, B_{j_1}, A_{i_2}, B_{j_2}, \dots A_{i_n}, B_{j_n}) = 0 \quad \forall n \,. \end{gathered} \tag{76}$$

This works as a rule for the computation of mixed moments (as in Eq.(72)) in terms of moments in which $A_1, \dots, A_n$ and $B_1, \dots, B_n$ appear separately.

Even more nicely, one can find an explicit expression that relates the two. Concretely, if $A_1, \dots A_n$ and $B_1, \dots B_n$ are asymptotically free, then the mixed moments are given by [37]

$$\langle A_1 B_1 A_2 B_2 \dots A_n B_n \rangle = \sum_{\pi \in NC(n)} \kappa_\pi(A_1, \dots, A_n) \, \langle B_1, \dots, B_n \rangle_{\pi^*} \,, \tag{77}$$

where π^* is the dual of the partition π (also known as Kraweras complement), which is defined below. Graphically (see Fig. 9 for $n = 4$), the blocks composing π^* are the maximal blocks (polygons) with vertices on "B" that do not cross the blocks of π. As a simple check, one can verify that for the simple case of $\langle ABAB \rangle$, the general result (77) immediately yields Eq.(72).

While the right-hand side of Eq. (77) is not manifestly symmetric under $A \leftrightarrow B$, one can also exchange the role of A and B. In this case one obtains

$$\langle A_1 B_1 A_2 B_2 \dots A_n B_n \rangle = \sum_{\pi \in NC(n)} \langle A_1, \dots, A_n \rangle_{K^{-1}(\pi)} \, \kappa_\pi(B_1, \dots, B_n) \,, \tag{78}$$

[15] Mixed means that the random variables appearing in the free-cumulant do not all come from the same subalgebra.

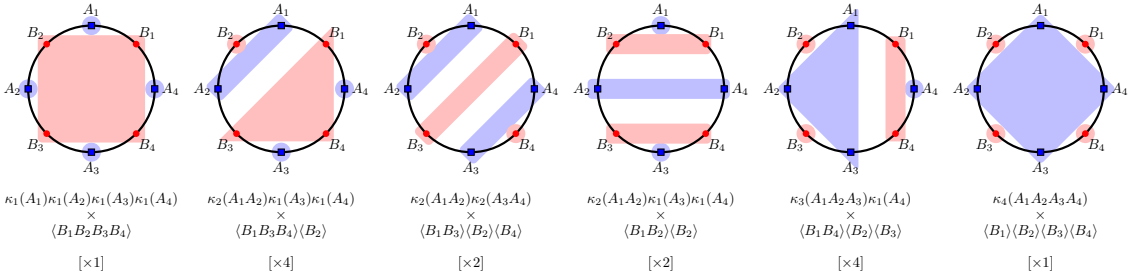


Figure 9: Graphical illustration of some of the terms appearing in Eq. (77) for $n = 4$. The shaded blue areas denote non-crossing partitions of the set $\{A_1, A_2, A_3, A_4\}$. To each non-crossing partition π, one can associate a dual non-crossing partition π^* on the set $\{B_1, B_2, B_3, B_4\}$, denoted here as shaded red areas. The contributions of these terms to the sum in Eq. (77) is the product of two terms reported below each partition. With $[\times n]$ we indicate that there are n arrangements of that non-crossing diagram.

where K^{-1} is the inverse of the duality transformation $\bullet^*$, i.e. $K^{-1}(\pi^*) = \pi$[16].

Thus the definition of freeness in Eq.(71) is greatly simplified when expressed in terms of cumulants as in Eq.(76). First of all, one gets rid of the condition of "centerness" for the variables and relaxes the condition of "alternating" to "mixed". Secondly, there are several properties of free variables which become particularly easy. For instance, the free cumulants of the sum of two-free variables are give by the sum of the free cumulants, e.g.

$$\kappa_n(A+B) = \kappa_n(A) + \kappa_n(B) \ , \tag{79}$$

where, for ease of notation, we have denoted $\kappa_n(X, \ldots, X)$ with X appearing n times as $\kappa_n(X)$. These nice properties translate over the generating functions, such as the R-transform or the S-transform, see e.g. Ref. [37]. We will, however, not deal with these topics in the present manuscript.

Examples of free random matrices While the notion of freeness might seem exotic, many well-known distributions of random matrices produce free matrices asymptotically in the $D \to \infty$ limit. For example, independent Gaussian random matrices [38] are free w.r.t. each other, and Wigner matrices are free w.r.t. deterministic ones [37] — in both cases the statement holds asymptotically in the $D \to \infty$ limit. Another instance of free random matrices that will be relevant for us is the following. Given two non-random (deterministic) matrices A and B (such that again $\langle A^n \rangle$ is finite for all n and similarly for B) and given $U \sim$ Haar, then [38]

$$U^\dagger A U \quad \text{and} \quad B \quad \text{are asymptotically free as} \quad D \to \infty. \tag{80}$$

This theorem says that unitarily invariant random matrix models are asymptotically free from deterministic matrices. As a consequence, all the relations of Eqs.(76)-(77) hold. The eigenvalue distribution is not changed by the random rotation that, however, changes the relationship between the eigenvectors of A and B which we would have called "generic" or "typical" and we can now refer to as "free'.

2.5 Free Probability in Physics

Free Probability deals with non-commuting variables (operators or large matrices). Hence, it is natural to expect that it will have a lot of applications in quantum systems. Here is a

[16] Formally K^{-1} is distinct from the duality transformation *. In fact, applying the duality twice, one would generate a cyclic permutation, corresponding to a clockwise rotation by two units in the notation of Fig. 9.

non-comprehensive list of different fields in Physics where Free Probability has appeared:

- **Planar Field Theory**. It is important to know that Free Probability actually originated in Planar Field Theory. The first time the generating function of free cumulants appeared in the perturbative treatment of Ref. [40], later on fully established by Cvitanovic in Refs. [41, 42]. See also Ref. [43]
- **Quantum Information**: see the review work [44];
- **Tensor Networks**: see Refs. [45–48];
- **Noisy and disordered systems**: in condensed matter systems see e.g. Refs. [49–51] and for stochastic models of transports see Refs. [52–54];
- **Gravity:** in quantum black holes [55, 56] and in the double scale limit of the SYK [57, 58];
- **Thermalization and chaos**: in the study of thermalization of systems evolving with a Wigner matrix [59] and in connection to the Eigenstate Thermalization Hypothesis [34, 60, 61], k-designs [62] and eigenstate correlations [63];
- if you know about other works involving free probability in the quantum realm, please let me know!

3 Full ETH and Free Probability

> "Quantum chaos is a nickname for the investigation of quantum systems which do not permit exact solutions. [...] One tries to understand statistical properties of quantum quantities by organizing them in suitable ensembles."
>
> — E. BOGOMOLNY, Quantum and Arithmetical Chaos

Based on:

1. (Full ETH): L. Foini and J. Kurchan, *Eigenstate thermalization hypothesis and out of time order correlators*, Phys. Rev. E 99, 042139 (2019);
2. (Full ETH and Free Probability): S. Pappalardi, L. Foini and J. Kurchan, *Eigenstate thermalization hypothesis and free probability*, Phys. Rev. Lett. 129, 170603 (2022);
3. (... in lattice models): S. Pappalardi, F. Fritzsch and T. Prosen, *General Eigenstate Thermalization via Free Cumulants in Quantum Lattice Systems*, arXiv:2303.00713 (2023).

We here introduce the full version of the ETH Eigenstate Thermalization Hypothesis [22], the Free Probability approach to it as discussed in Ref. [60], and finally contrast the result with what we would obtain with rotationally invariant random matrices [61].

3.1 Beyond two-point functions: correlations and the full ETH

The standard ETH "à la Srednicki" has proved to be excellent in describing thermalization, equilibrium observables and two-times dynamical correlation functions. But what about observables that depend on k times such as

$$\langle \hat{A}(t_1)\hat{A}(t_2)\dots\hat{A}(t_k)\rangle_\beta \quad ? \tag{81}$$

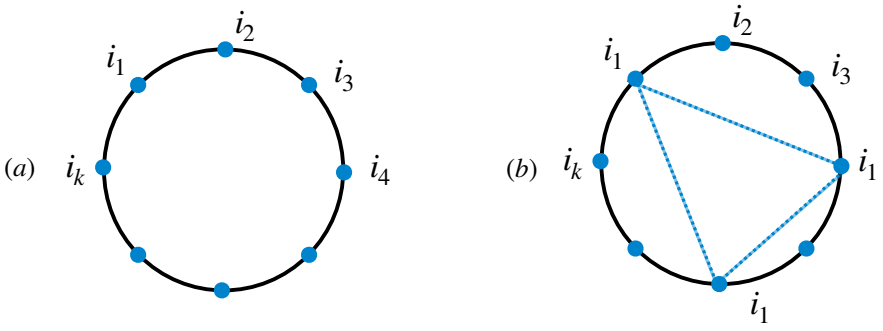
These are important every time one wants to go beyond linear response, to understand non-markovian effects or to characterize quantum chaos, via out-of-time order correlators such as $\langle \hat{A}(t)\hat{A}\hat{A}(t)\hat{A}\rangle_\beta$. This object contains correlations between *four* matrix elements and as such, it can not be described by the standard ETH (33). This led Foini and Kurchan to introduce the full version of the ETH [22], as an ansatz for the k-point correlations: the average product with distinct indices $i_1, i_2, \dots i_k$ reads

$$\overline{A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_k i_1}} = e^{(1-k)S(e_+)} F^{(k)}_{e^+}(\omega_{i_1 i_2}, \dots, \omega_{i_{k-1} i_k}) \tag{82a}$$

while, with repeated indices, it shall factorize in the large N limit as

$$\overline{A_{i_1 i_2} \dots A_{i_{p-1} i_1} A_{i_1 i_{p+1}} \dots A_{i_k i_1}} = \overline{A_{i_1 i_2} \dots A_{i_{p-1} i_1}}\ \overline{A_{i_1 i_{p+1}} \dots A_{i_k i_1}}\ . \tag{82b}$$

where $e_+ = (E_{i_1} + \dots + E_{i_k})/k$, $\omega_{i_k i_{k+1}} = E_k - E_{k+1}$ are the energy differences and S is the thermodynamic entropy. The smooth functions $F^{(k)}$ define the operator, generalizing Eq.(33), which is retrieved as $F^{(1)}_{e^+} = \mathcal{A}(e^+)$ and $F^{(2)}_{e^+}(\omega) = |f(e^+, \omega)|^2$. The full ETH can be represented pictorially as follows



where diagram (a) represents the new correlations as in Eq.(82a), while in diagram (b) the index i_1 is repeated three times leading to the factorization in three blocks $\overline{A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_1} A_{i_1 i_5} A_{i_5 i_1} A_{i_1 i_7} A_{i_7 i_k} A_{i_k i_1}}$.

Comments:

1. The correlations in Eq.(82a) are exponentially suppressed and may seem small. However, they contribute to correlation functions since they are re-summed by exponentially many states $\sum_{i_1,\ldots,i_k}$;

2. the functions $F^{(k)}$ are empty boxes that define the operator. Exactly as in the case of the two-point function, all the physical information is encoded within them and ETH at this level does not say anything about their structure.

3. Understanding how to put to work this ansatz with observables is now the focus of current research and it can be understood using the framework of Free Probability [60, 61].

To understand how this ansatz applies to multi-time correlation functions as

$$\langle A(t_1)A(t_2)\ldots A(t_n)\rangle_\beta \ , \tag{83}$$

one should sum over all indices of $A_{i_1 i_2} A_{i_2 i_3} \ldots A_{i_n i_1}$, as

$$\langle A(t_1)A(t_2)\ldots A(t_n)\rangle_\beta = \frac{1}{Z}\sum_{i_1,\ldots i_n} e^{-\beta E_i} [A(t_1)]_{i_1 i_2} \ldots [A(t_n)]_{i_n i_1} \ . \tag{84}$$

Thus, to determine the contribution of the different matrix elements, one must consider all the possible contractions. Due to the proliferation of possible choices, this a priori seems a hopeless job. We will see how Free Probability enters to help us.

First of all, let us understand the different contributions to the sum *diagrammatically*. Let us consider the example of four-point functions that we illustrate pictorially in Fig.10. Products $A_{ij}A_{jk}A_{km}A_{mi}$ are represented on a loop with four vertices i, j, k, m, depicting energy eigenstates. The contractions between two or more indices are represented by lines that connect the vertices. The blue dots indicate that the indices are all different. For instance, the first diagram represents $A_{ij}A_{jk}A_{km}A_{mi}$, the second $A_{ij}A_{ji}A_{im}A_{mi}$, the third $A_{ij}A_{jj}A_{jm}A_{mi}$ with all distinct indices, and so on. One recognizes that there are two types of diagrams: (1) *non-crossing* ones – in which the polygons created by the indices do not cross (the simple loops with all different indices are in this class) – and (2) *crossing ones*, in which the lines cross. The full ETH ansatz in Eqs.(82a) implies two properties:

fETH 1. crossing diagrams are suppressed with the inverse of the density of states as

$$\frac{1}{Z}\sum_{i\neq j} e^{-\beta E_{ii}} e^{i\omega_{ij}(t_1-t_2+t_3)} |A_{ij}|^4 \simeq e^{-S(E)} \sim D^{-1} \,, \tag{85}$$

which means that they can be neglected to compute higher-order correlation functions.

fETH 2. all non-crossing diagrams yield a finite contribution with *factorization of non-crossing diagrams into products of irreducible simple loops*:

$$\frac{1}{Z}\sum_{i\neq j\neq m} e^{-\beta E_{ii}} e^{i\omega_{ij}(t_1-t_2)+i\omega_{im}t_3} |A_{ij}|^2 |A_{im}|^2 \simeq \left(\frac{1}{Z}\sum_{i\neq j} e^{-\beta E_{ii}} e^{-i\omega_{ij}(t_1-t_2)} |A_{ij}|^2\right)\left(\frac{1}{Z}\sum_{i\neq m} e^{-\beta E_{ii}} e^{-i\omega_{im}t_3} |A_{im}|^2\right) . \tag{86}$$

which means that the diagrams (b-f) in Fig.10 can be cut along the blue line.

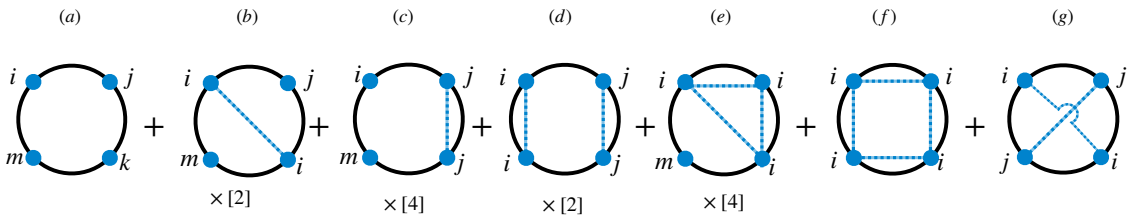


Figure 10: Bookeeping of ETH matrix elements for $n = 4$. Matrix elements A_{ij} lie on the vertex connecting two dots representing the energy index. Blue dots represent different indices, and the edges connecting two or more dots represent a contraction among them.

3.2 Full ETH and Free Probability

First of all, revisiting the definition (66), one can *define thermal free cumulants* k_n^β by

$$\langle A(t_1)A(t_2)\ldots A(t_n)\rangle_\beta = \sum_{\pi\in NC(n)} \kappa_\pi^\beta (A(t_1)A(t_2)\ldots A(t_n)) \,, \tag{87}$$

where $\langle\bullet\rangle_\beta = \frac{1}{Z}\mathrm{Tr}\left(e^{-\beta H}\bullet\right)$ with $Z = \mathrm{Tr}\left(e^{-\beta H}\right)$ plays the role of the expectation value $\phi()$. Here k_π^β are products of thermal free cumulants one for each block of π, i.e.

$$\kappa_\pi^\beta (A(t_1)A(t_2)\ldots A(t_n)) = \prod_{b\in\pi} \kappa_{|b|}^\beta (\prod_{j\in b} A(t_j)) \,, \tag{88}$$

where $|b|$ is the size of the block b in the partition π See the example in Fig.8, where now t_j is playing the role of the index j. Exactly as in Eq.(66), this is just an implicit definition of cumulants in terms of moments, which can be defined in principle also for integrable or non-ergodic systems. We will now discuss how this definition simplifies the discussion of the full ETH, which, in turn, implies a particularly simple form for the thermal-free cumulants.

The correspondence with Free Probability allows one to generalize the result at every n. First of all, all the contributions to multi-time correlations have to be found in non-crossing partitions. Specifically, the non-crossing ETH diagrams ((a-f) in Fig.10) can be read as the "dual" of non-crossing partitions π in which every element of the set is not associated with an observable [(a-f) in Fig.10]. Secondly, the ETH factorization implies a particularly simple form

for the k_q^β defined in Eq.(87). Namely, the thermal free cumulants of ETH-obeying systems are given only by summations with distinct indices

$$k_n^{\rm ETH}(A(t_1)A(t_2)\dots A(t_n)) = \frac{1}{Z}\sum_{i_1\neq i_2\neq\dots\neq i_n} e^{-\beta E_i} A_{i_1 i_2}A_{i_2 i_3}\dots A_{i_n i_1} e^{it_1\omega_{i_1 i_2}+\dots it_q\omega_{i_n i_1}} \tag{89}$$

$$= {\rm FT}\left[F^{(n)}_{e_\beta}(\vec{\omega})e^{-\beta\vec{\omega}\cdot\vec{\ell}_n}\right], \tag{90}$$

where in the second line ${\rm FT}[\bullet]=\int d\vec{\omega}e^{i\vec{\omega}\cdot\vec{t}}\bullet$ is the Fourier transform and $e_\beta=\langle H\rangle_\beta/N$ is the thermal energy density. The thermal weight with $\vec{\ell}_n=\left(\frac{n-1}{n},\dots,\frac{1}{n},0\right)$ corresponds to a generalization of the fluctuation-dissipation theorem. *This result shows that all the correlations of the full ETH* (82a) *are encoded precisely in the thermal free cumulants.* On four-point functions, the validity of the ETH ansatz implies:

$$\begin{aligned}\langle A(t_1)A(t_2)A(t_3)A(t_4)\rangle_\beta =& k_4^{\rm ETH}(t_1,t_2,t_3,t_4)+k_2^{\rm ETH}(t_1,t_2)k_2^{\rm ETH}(t_3,t_4)+k_2^{\rm ETH}(t_1,t_4)k_2^{\rm ETH}(t_2,t_3)\\ &+k_1^{\rm ETH}\Big[k_3^{\rm ETH}(t_1,t_2,t_3)+k_3^{\rm ETH}(t_1,t_3,t_4)+k_3^{\rm ETH}(t_1,t_2,t_4)+k_3^{\rm ETH}(t_2,t_3,t_4)\\ &\quad+k_1^{\rm ETH}k_2^{\rm ETH}(t_1,t_3)+k_1^{\rm ETH}k_2^{\rm ETH}(t_2,t_4)+(k_1^{\rm ETH})^3\\ &\quad+k_1^{\rm ETH}k_2^{\rm ETH}(t_1,t_2)+k_1^{\rm ETH}k_2^{\rm ETH}(t_1,t_4)+k_1^{\rm ETH}k_2^{\rm ETH}(t_2t_3)+k_1^{\rm ETH}k_2^{\rm ETH}(t_3,t_4)\Big].\end{aligned} \tag{91}$$

For $k_1=0$, this expression reduces to Eq.(4) of the main text, where we make explicit use of time-translational invariance, i.e. $k_q(t_1,t_2,...t_q)=k_q(t_1-t_q,t_2-t_q,...,t_{q-1}-t_q)$.

3.3 Free cumulants in rotationally invariant systems

Rotationally invariant models are characterized by probability distribution of the matrix elements $P(A)\equiv P(A_{ij})$ that is invariant under a change of basis:

$$P(A)=P(U^\dagger AU), \tag{92}$$

where U may be an orthogonal, unitary, or symplectic matrix ($UU^\dagger=U^\dagger U=\mathbb{1}$). Let us denote $\overline{\bullet}$ averages over these ensembles. A class which enjoys this property is given by $P(A)\propto\exp[-\frac{D}{2}{\rm Tr}V(A)]$ where $V(A)$ is a generic polynomial - the potential ($V(A)=A^2/2$ in the case of the Gaussian ensemble). These matrices have the property that their moments only depend on the distributions of the eigenvalues a_i, i.e. $\langle A^m\rangle=\frac{1}{D}{\rm Tr}(A^m)=\frac{1}{D}\sum_i a_i^m$.

For rotationally invariant systems, Ref. [24] proved that the free cumulants are given by averages over simple loops (diagrams with different indices)

$$\overline{A_{i_1 i_2}A_{i_2 i_3}\dots A_{i_n i_1}} = D^{1-n}\,k_n(A) \quad \text{with} \quad i_1\neq\dots\neq i_n \tag{93}$$

at the leading order in D. Here the average is taken according to $P(A)$ in Eq. (92) and $k_n(A)$ is defined in Eq. (63). The equality holds for each product of matrix elements, without the summation [in contrast with Eq. (5) of the main text]. The overall constant normalization D^{n-1} stands for the fact that the average over each element is the same. For $n=2$ this equation reads

$$\overline{A_{ij}A_{ji}} = D^{-1}\,k_2(A)=D^{-1}\left(\langle A^2\rangle-\langle A\rangle^2\right), \tag{94}$$

where $k_2(A)$ is given in Eq. (65b). This result corresponds to the ETH ansatz for $n=2$ for CUE matrices.

We now compute free cumulants in the frequency domain

$$\tilde{k}_n(\omega_1,\omega_2,\dots,\omega_{n-1})=\frac{1}{D}\sum_{i_1\neq i_2\neq\dots\neq i_n}\overline{A_{i_1 i_2}A_{i_2 i_3}\dots A_{i_n i_1}\delta(\omega_1-(E_{i_1}-E_{i_2}))\dots\delta(\omega_{q-1}-(E_{i_{n-1}}-E_{i_n}))}. \tag{95}$$

We assume a decoupling between the average over $A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_n i_1}$ and the average over the delta functions (absence of statistical correlation between matrix elements and the spectrum) [17]. Substituting Eq. (93) into Eq. (95) we thus conclude

$$\tilde{k}_n(\omega_1, \omega_2, \dots, \omega_{n-1}) = k_n(A) \frac{R_n(\omega_1, \omega_2, \dots \omega_{q-1})}{D^n} \,, \tag{96}$$

where

$$R_n(\omega_1, \omega_2, \dots \omega_{n-1}) = \sum_{i_1 \neq i_2 \neq \dots \neq i_n} \overline{\delta(\omega_1 - (E_{i_1} - E_{i_2})) \delta(\omega_2 - (E_{i_2} - E_{i_3})) \dots \delta(\omega_{n-1} - (E_{i_{n-1}} - E_{i_n}))} \tag{97}$$

is the *n—point spectral correlator*, which encodes all n—point correlations between the energy eigenvalues. This generalizes to many n the connected two-point spectral correlation $R_2(\omega) = \sum_{i \neq j} \delta(\omega - \omega_{ij})$, whose Fourier transform yields the spectral form factor.

For $n = 2$, Eq.(93) gives $\overline{A_{ij} A_{ji}} = D^{-1} k_2(A)$ and this result reads

$$\tilde{k}_2(\omega) = k_2(A) \frac{R_2(\omega)}{D^2} \,, \tag{98}$$

and it implies that the second cumulant in frequency ($\tilde{k}_2(\omega) = |f(\omega)|^2$ and $f(\omega)$ in the standard ETH notations) is only a function of the spectral correlations $\frac{R_2(\omega)}{D^2}$ and a constant function $k_2(A)$ which only depends on the eigenvalues of the operator. This corresponds to the standard ETH ansatz for CUE matrices, see e.g. Ref. [64]

To summarize, in basis-rotationally-invariant systems, the free cumulant in the frequency domain is given by the product of the free cumulant of the matrix (constant in frequency) and the spectral correlations, which are constant almost everywhere, i.e. when all frequencies ω_{ij} are much larger than mean level spacing [65].

4 Freeness in chaotic dynamics

4.1 Long-time freeness for chaotic Hamiltonians

4.2 Calculation of the Page-Curve via Free Probability techniques

[17] This is ensured by a sort of self-averaging of Eq. (93) which is valid at the leading order in D, thus the fluctuations which would contribute to the cross term in the average with the deltas are subleading.

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A Hints of Random Matrix Theory

Selected bibliography:

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The full many-body problem of quantum dynamics is encoded in the spectrum of the many-body Hamiltonian:

$$\hat{H}|E_i\rangle = E_i\,|E_i\rangle \ , \quad i = 1,\dots D = \dim \mathcal{H}. \tag{99}$$

In the absence of small parameters such as $\hbar$, we will employ a new technique: Random Matrix Theory (RMT), which addresses the following question: if we have a large random matrix whose elements are random variables with a given probability law, what can we say about the property of its eigenvalues and its eigenvectors?
Even if it is different from what we encountered in classical dynamics, the statistical approach has the same spirit as what was discussed on the emergence of Statistical Mechanics. There, we renounced the knowledge of the exact dynamical state, and we required only that the system may be in any of the many possible states compatible with the symmetries. With exactly the same philosophy, we shall consider an ensemble of Hamiltonians, whose properties are determined by the global symmetries.

With this aim, we will here introduce some basic facts of random matrix theory, which will then come in handy for physical Hamiltonians.

Historical notes The study of random matrix theory was initiated in mathematical statistics in the 30s, but it started to be intensively studied in the 50s in the context of nuclear physics, with the invaluable work of Wigner and Dyson. There, the goal was to describe the behavior of neutron resonances (peaks) obtained experimentally with neutron scattering.

Wigner proposes that the local statistical behavior of levels in a simple sequence is identical to the eigenvalues of a random matrix. The corresponding symmetry has to be imposed but, besides that, the elements can be taken to be distributed at random with a Gaussian distribution.

This line of thought proved to be extremely successful. The use of random matrix theory has now spread to several branches of knowledge: from the excitation spectra of metals to Riemann Zeta functions to... chaotic quantum systems!

A.1 Gaussian ensembles ++

A.1.1 The ensembles

Consider a $D \times D$ random matrix H, such that each of its individual elements is *independently* distributed with a Gaussian probability distribution $P(H_{ij}) \propto e^{-H_{ij}^2/2}$. At this level, there are no symmetries $H_{ij} \neq H_{ji}$, and therefore the eigenvalues are generically complex. Its joint probability density function is factorized as

$$\rho(H_{11}, H_{12}, \dots H_{DD}) = \prod_{ij}^{D} \frac{e^{-H_{ij}^2/2}}{\sqrt{2\pi}} \ . \tag{100}$$

Box 1: Reminder on random variables

We call $\rho(x)$ the probability density function of the random variable X if $\int_a^b \rho(x)dx$ is the probability that X takes values in the interval (a,b). It is normalized $\int sx\rho(x) = 1$ and it allows to define the moments of the distribution $\langle X^n\rangle = \int x^n\rho(x)dx/$ The cumulative distribution function $F(x) = \int_{-\infty}^{x} dy\rho(y)$ is the probability that X is smaller or equal than x. If we have D random variables, then the joint probability distribution function is $\rho(x_1,\dots x_D)$ and the *marginal* probability distribution function of x alone is

$$\rho(x) = \int dx_2\dots dx_D\rho(x_1,\dots x_D)\,. \tag{101}$$

If a set of random variables is a function of another one $x_i = x_i(\mathbf{y})$, the relation of the probability functions between the two sets is

$$\rho(x_1,\dots x_D)dx_1\dots dx_n = \rho(x_1(\mathbf{y}),\dots x_D(\mathbf{y}))|\det J(\vec{x}\to\mathbf{y})|dy_1\dots dy_D\,, \tag{102}$$

where $J(x\to\mathbf{y}) = \frac{\partial x_i}{\partial y_j}$ is the determinant of the Jacobian of the transformation.

We start by considering the three different main symmetry classes of RMT, which have real eigenvalues. Steee also Fig.12.

Real symmetric matrices: the Gaussian Orthogonal Ensemble (GOE) We consider real symmetric matrices

$$H_s = \frac{H+H^s}{2} \quad s.t. \quad H^s_{ij} = \frac{H_{ij}+H_{ji}}{2} \tag{103}$$

This matrix now has real eigenvalues. Now we have $D + D(D-1)/2$ independent variables. Show as exercise 4.1 that

$$\rho(H^s_{11},H^s_{12},\dots H^s_{DD}) = \prod_i^D \frac{e^{-(H^s_{ii})^2/2}}{\sqrt{2\pi}} \prod_{j>i} \frac{e^{-(H^s_{ij})^2}}{\sqrt{\pi}}\,. \tag{104}$$

This ensemble is called the Gaussian Orthogonal Ensemble (GOE). The name orthogonal refers to the fact that real symmetric matrices are diagonalized by orthogonal transformations $OO^T = O^TO = \mathbb{1}$. We note an important property of this ensemble, namely that the variance of the off-diagonal is half the variance of the diagonal, i.e.

$$\mathrm{Var}(H^s_{ij}) = \frac{1}{2}\mathrm{Var}(H_{ii})\,. \tag{105}$$

If we now sample N times these matrices and do the histogram, we find that it has a variance

$$\pm\sqrt{2D} = \pm\sqrt{2\beta D} \qquad \beta = 1\,. \tag{106}$$

where we defined the constant $\beta = 1$ for this ensemble. In what follows, we will simply drop the index s when discussing the GOE.

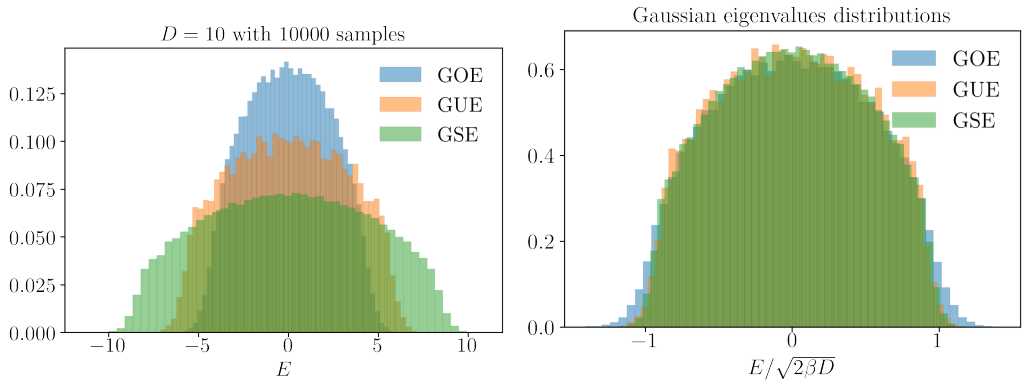


Figure 11: (Left) Histograms of GOE, GUE, and GSE for $D = 10$ and 10000 samples. (Right) Rescaled by $\sqrt{2\beta D}$ with $\beta = 1, 2, 4$ respectively.

Hermitian complex matrices: Gaussian Unitary ensemble (GUE) We consider complex hermitian matrices

$$H_H = \frac{H + H^\dagger}{2} \quad s.t. \quad H^H_{ij} = \frac{H_{ij} + H^*_{ji}}{2} \tag{107}$$

Also, matrix now has real eigenvalues. Now we have D^2 independent variables: the D diagonals $H^J_{ii} = H_{ii}$ and $D(D-1)$ the off-diagonals $\mathrm{Re}(H_{ij})$ and $\mathrm{Im}(H_{ij})$ for $i > j$. As done above, one can show that

$$\rho(H^H_{11}, \mathrm{Re}(H^H_{12}), \mathrm{Im}(H^H_{12}), \dots H^H_{DD}) = \prod_i^D \frac{e^{-(H^H_{ii})^2/2}}{\sqrt{2\pi}} \prod_{j>i} \frac{e^{-(\mathrm{Re}H^H_{ij})^2/2}}{\sqrt{2\pi}} \frac{e^{-(\mathrm{Im}H^H_{ij})^2/2}}{\sqrt{2\pi}} \,. \tag{108}$$

This ensemble is called the Gaussian Unitary Ensemble (GUE). The name unitary refers to the fact that real symmetric matrices are diagonalized by unitary transformations $UU^\dagger = U^\dagger U = \mathbb{1}$. In this case, the variance of the off-diagonal is the same as the variance of the diagonal, i.e.

$$\mathrm{Var}(\mathrm{Re}H^H_{ij}) = \mathrm{Var}(\mathrm{Im}H^H_{ij}) = \mathrm{Var}(H^J_{ij}) \,. \tag{109}$$

If we now sample T times these matrices and we do the histogram, we find that it has a variance

$$\pm\sqrt{4D} = \pm\sqrt{2\beta D} \qquad \beta = 2 \,. \tag{110}$$

where we defined the constant $\beta = 2$ for this ensemble. As before, we will drop the index H in what follows.

Self-Dual quaternionic matrices: Gaussian Symplectic ensemble (GSE) These are $2D \times 2D$ random matrices, whose entries are quaternionic numbers [18] sampled with a Gaussian distribution and are self-dual [19]. They can be constructed from $2D \times 2D$ complex matrix X as

$$H = \frac{X + X^\dagger - \mathbb{J}(X + X^\dagger)\mathbb{J}}{4} \tag{111}$$

These matrices are diagonalized by symplectic transformations, as such, they define the Gaussian symplectic ensemble. If we now sample T times these matrices and we do the histogram, we find that it has a variance

$$\pm\sqrt{8D} = \pm\sqrt{2\beta D} \qquad \beta = 4 \,. \tag{112}$$

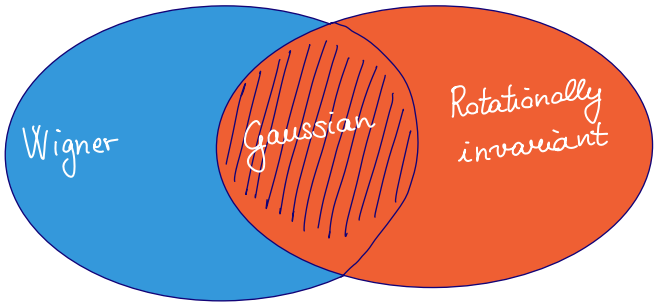
[18] Quaternionic numbers are $q = x_1 + ix_2 + jx_3 + kx_4$ such that $i^2 = j^2 = k^2 = ijk = -1$.

[19] Self-dual quaternion matrices are that such the matrix is equal to its dual, i.e. $Q^D = Q$. The dual of a quaternion is defined as the conjugation (not the complex conjugation) of all its elements: $[Q^D]_{ij} = \overline{Q_{ji}}$. A quaternion is self-dual if its complex matrix representation A obeys $A = -\mathbb{J}A^T\mathbb{Z}$. See e.g. Ref. [66].

where we defined the constant $\beta = 2$ for this ensemble.

These three different classes (see also Fig.12 are associated to the symmetry classes of given Hamiltonians, see Section A.1.6 below.

A.1.2 Classification of random matrices



There is a simple classification of random matrices with real eigenvalues:

1. *Matrices with independent entries, also called Wigner matrices.* In this case, the entries are independent random variables:

$$\rho(H) \propto \prod_{ij} f(H_{ii}) g(H_{ij}) \,.$$

Matrices in this class include adjacency matrices of random graphs [67], power-law bounded matrices [68], etc.

2. *Rotationally invariant ensambles, also called Matrix Models*. These ensembles contain matrices characterized by the so-called rotational invariance: this is the property that two matrices are related by the similarity transformation

$$H' = UHU^{-1} \,, \tag{113}$$

occur with the same probability, namely:

$$\rho(H_{11}, \dots H_{DD}) dH_{11} \dots dH_{DD} = \rho(H'_{11}, \dots H'_{DD}) dH'_{11} \dots dH'_{DD} \tag{114}$$

Here U is an orthogonal/unitary/symplectic matrix if H is real symmetric/complex hermitian/quaternionic self-dual respectively. This invariance needs two conditions: (a) that

$$dH_{11} \dots dH_{DD} = dH'_{11} \dots dH'_{DD}.$$

which is always true because the flat measure is invariant under conjugation and (b) that

$$\rho(H) = \rho(UHU^{-1}) \quad \Longleftrightarrow \tag{115}$$
$$\rho(H) \propto \exp\left(-\frac{1}{2}\mathrm{Tr}\left(H^2\right) + V(H)\right) \quad \text{with} \quad V(H) = \alpha_1 \mathrm{Tr}\,(H) + \alpha_3 \mathrm{Tr}\left(H^3\right) + \dots$$

in other words, the probability distribution can be written only as a function of the traces of powers of the Hamiltonian. This implication follows from the cyclic property of the trace.

This means that we can rotate our matrices at will, but this will not change the distribution of the matrix: in other words, the eigenvectors are irrelevant! As we will see, for this class of systems there is a complete factorization between eigenvalues properties and eigenvectors properties.

The Gaussian ensembles discussed so far lie at the intersection between these two. In fact, you can show as an exercise that:

$$\rho(H^s_{11},\dots H^s_{DD}) \propto \exp\left(-\frac{1}{2}\mathrm{Tr}\left(H^2\right)\right) , \tag{116}$$

which therefore corresponds to the above equation without $V(H)$, which is in jargon is referred to as the "random matrix potential".

A.1.3 Eigenvalues distributions: The Wigner surmise!

One of the most remarkable facts is the great universality that random matrices, and in particular the Gaussian ensembles, have in predicting the eigenvalues distribution of a large class of different systems. We now have seen how to sample different Hamiltonians with some Gaussian distribution probabilities $\rho(H)$, which leads to a collection of real eigenvalues

$$\Lambda = (\lambda_1,\dots,\lambda_D) \qquad \lambda_{i+1} \geq \lambda_i \, . \tag{117}$$

We now ask the following questions:

1. What is their distribution $\rho(\lambda)$?
2. What is the distribution of the level spacing $s_i = \lambda_{i+1} - \lambda_i$?

To gain some intuition, let us start with a simple and instructive exercise. We consider from a 2×2 GOE matrix:

$$H = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_1 \end{pmatrix} , \quad x_1, x_2 = \mathcal{N}(0,1), \quad x_2 = \mathcal{N}(0,1/2) , \tag{118}$$

where $\mathcal{N}(a,b)$ indicates the normal distribution of average a and variance b. The GUE case is left as exercise. The eigenvalues of the matrix can be found by using the formula for 2×2 matrices

$$\lambda^2 - \mathrm{Tr}\,(H)\,\lambda + \det H = 0 \tag{119}$$

which leads to

$$\lambda_{1,2} = \frac{x_1 + x_2 \pm \sqrt{(x_1-x_2)^2 + 4x_3^2}}{2} , \quad s = \sqrt{(x_1-x_2)^2+4x_3^2} \, . \tag{120}$$

Thus the level spacing distribution can be computed by

$$p(s) = \int dx_1 dx_2 dx_3 \frac{e^{-x_1^2/2}}{\sqrt{2\pi}} \frac{e^{-x_2^2/2}}{\sqrt{2\pi}} \frac{e^{-x_3^2}}{\sqrt{\pi}} \delta(s - \sqrt{(x_1-x_2)^2+4x_3^2}) \, . \tag{121}$$

We then perform the following change of variables

$$\begin{cases} x_1 - x_2 = r\cos\theta \\ 2x_3 = r\sin\theta \\ x_1 + x_2 = \Psi \end{cases} \qquad \begin{cases} x_1 = \dfrac{\psi + r\cos\theta}{2} \\ x_2 = \dfrac{\psi - r\cos\theta}{2} \\ x_3 = \dfrac{r\sin\theta}{2} , \end{cases} \tag{122}$$

and one finds

$$\det J = -r/4.$$

Plugging this into Eq.(121) one gets

$$p(s) = \frac{1}{2\pi^{3/2}} \int dr d\Psi d\theta \delta(s-r) e^{-\psi^2/4 - r^2/4} = \frac{s}{2} e^{-s^2/4} \,. \tag{123}$$

We are almost done. We usually proceed with the "*unfolding*" procedure: which amounts to re-scale the level spacing by its average $\langle s \rangle$ as

$$\bar{p}(s) = \langle s \rangle p(\langle s \rangle s) \quad s.t. \quad \langle \bar{s} \rangle = \int \bar{p}(s) s = 1 \,. \tag{124}$$

with this choice this leads to

$$\boxed{\bar{p}(s) = \frac{\pi}{2} s e^{-\frac{\pi}{4} s^2} \,,} \tag{125}$$

which is the Wigner surmise for the GOE ensemble. Even if we computed it as a simple exercise for 2×2 matrices, it remarkably applies to generic random matrices in the limit $D \to \infty$.

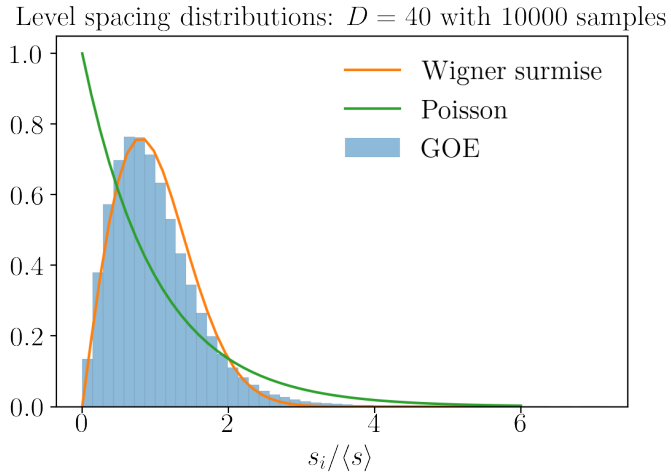


Figure 12: Level spacing distributions for $D \times D$ GOE matrices compared with the Wigner surmise (125) and the prediction for uncorrelated variables.

A.1.4 General eigenvalues distribution: Vandermorte

As an analogous exercise as the one leading to Eq.(125), compute the eigenvalue probability distribution, which gives:

$$\rho(\lambda_1, \lambda_2) = \frac{|\lambda_1 - \lambda_2|}{4\pi^{1/2}} e^{-\lambda_1^2/2} e^{-\lambda_2^2/2} \,. \tag{126}$$

More generally, it can be shown that the eigenvalues of a Gaussian random matrix are characterized by the following joint probability distribution function:

$$\rho(\lambda_1, \dots \lambda_D) = \frac{1}{Z_{\beta,D}} e^{-\frac{1}{2}\sum_{i=1}^{D} \lambda_i^2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \tag{127}$$

for the different $\beta = 1, 2, 4$ associated with the different ensembles (GOE, GUE, and GSE). In this equation

$$\prod_{i<j} |\lambda_i - \lambda_j|^\beta = \det(\lambda_i^{j-1})_{i \le j \le D} = \det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots \lambda_1^{D-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots \lambda_2^{D-1} \\ & & \dots & \\ & & \dots & \end{pmatrix} \tag{128}$$

is called the *Vandermonde determinant* (determinant due to the right-hand side, which is a rewriting valid only for $\beta = 1$) and, as we will see, it is the determinant of the Jacobian done in the transformation in the diagonalization procedure.

Let us comment on two competing effects in Eq.(127): the Gaussian factor kills the probability of having eigenvalues that are too far from the origin, acting as a sort of *confinement*. On the other hand, the Vandermonde determinant acts as a repulsion: all the eigenvalues fill each other and repel. The probability distribution does not factorize, the eigenvalues of Gaussian random matrices are correlated!

To derive Eq.127, we restrict ourselves to the GOE and we proceed by diagonalizing the matrix

$$H = O\Lambda O^T \ , \tag{129}$$

where $\Lambda = (\lambda_1, \dots \lambda_D)$ is the diagonal matrix containing the eigenvalues and O are the orthogonal matrices containing the eigenvectors. We want to perform the following change of variables:

$$H_{ij} \longrightarrow \{\lambda_i, O_{ij}\} \ . \tag{130}$$

From the rule on the distribution transformation, we need to compute:

$$\rho(H_{11}, \dots H_{DD}) dH_{11} \dots dHDD = \rho(H_{11}(\Lambda, O), \dots, H_{DD}(\Lambda, O)) |J(H \to (\Lambda, O))| dO \prod_i d\lambda_i \ , \tag{131}$$

here, dO represents the equivalent of the "angular variables", which is the volume element in the space of the orthogonal matrices. The uniform measure in the space of the orthogonal group is called the *orthogonal Haar measure*.

Now we consider more generally rotational invariant models (115) which contain the gaussian case. In this case, the distribution depends only on the eigenvalues, since

$$\rho(H) \propto \exp\left(-\frac{1}{2}\mathrm{Tr}\left(H^2\right) + \alpha_1 \mathrm{Tr}\left(H\right) + \alpha_3 \mathrm{Tr}\left(H^3\right) + \dots\right) \tag{132}$$

$$= \exp\left(-\frac{1}{2}\sum_i \lambda_i^2 + \alpha_1 \sum_i \lambda_i + \alpha_3 \sum_i \lambda_i^3 + \dots\right). \tag{133}$$

As mentioned, there is a complete decoupling between the properties of the eigenvalues and those of the eigenvectors. Basically one can rotate the matrices as one wishes and the statistical properties remain unchanged!

Therefore we only need to compute the Jacobian of the transformation, which amounts in computing $\frac{\partial H_{ij}}{d\lambda_k}$ and $\frac{\partial H_{ij}}{dO_{kl}}$. Using Eq.(129), we compute the infinitesimal transformation

$$\begin{aligned} dH &= dO\Lambda O^T + O^T d\Lambda O^T + O\Lambda dO^T \\ &= dO\Lambda O^T + O^T d\Lambda O^T - O\Lambda O^T dOO^T \\ &= O d\tilde{H} O^T \ , \end{aligned} \tag{134}$$

where from the first to the second line we have used $dO^T = -O^T dOO^T$ from $d(OO^T) = 0$ and in the second line we have defined

$$d\tilde{H} = d\Lambda + [d\Omega, \Lambda] \quad \text{with} \quad d\Omega = O^T dO \quad \text{angular variables.} \tag{135}$$

Hence we can write this matrix in the eigenbasis of λ which yields

$$d\tilde{H}_{ij} = \delta\lambda_i \delta_{ij} + (\lambda_i - \lambda_j) d\Omega_{ij} \ , \tag{136}$$

from which

$$J = \det \begin{pmatrix} \frac{d\tilde{H}_{11}}{d\lambda_1} & \frac{d\tilde{H}_{11}}{d\lambda_1} & \cdots & \cdots \\ \cdots & \frac{d\tilde{H}_{22}}{d\lambda_2} & & \\ \cdots & \cdots & & \\ & & \cdots \frac{d\tilde{H}_{DD}}{d\Omega} & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \\ 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & \cdots & \\ & \cdots & & \lambda_1 - \lambda_2 & \\ & \cdots & & 0 & \lambda_2 - \lambda_3 \end{pmatrix} = \prod_{i<j} (\lambda_i - \lambda_j) \tag{137}$$

Hence we have written:

$$\rho(H)dH = \rho(\Lambda) \prod_{i<j} (\lambda_i - \lambda_j) d\Lambda d\Omega \tag{138}$$

and by integrating the angular variables we obtain:

$$\boxed{\rho(\lambda_1, \dots \lambda_D) = Z_{D,\beta}^{-1} \quad e^{-\frac{1}{2}\sum_i \lambda_i^2 + V(\lambda)} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \ ,} \tag{139}$$

where $Z_{D,\beta}^{-1}$ is the normalization. We showed this result for the orthogonal case $\beta = 1$, but for the other symmetry classes it can be shown in the same way and it only leads to a difference in the factor β.

The above formula tells us that the probability of sampling two eigenvalues very close $s \to 0$ is zero. This means that the eigenvalues feel each other and they *repell* each other. For this reason one talks about "*level repulsion*". In other words, the energy levels are correlated. This fact is in stark contrast with the level spacing of independent random variables, discussed in Section A.1.5 below.

A.1.5 Spacing between iid variables

Let us compare the previous derivation, with the statistics of gaps between adjacent *identically independent* distributed random variables. In this case, we will not see any repulsion, rather *levels attraction*. In fact, the distribution of the *local level spacing* is given by the exponential law

$$\lim_{D\to\infty} p_D(\bar{s}) = e^{-\bar{s}} \tag{140}$$

which is the law for the spacing in a Poisson process. Here, the local level spacing appears [20]

$$\bar{s} = s\ D p_X(x) \tag{141}$$

Hence the Poisson distribution in Eq.(140) tells us that the probability of crossings (vanishing gaps $\bar{s} \to 0$) goes to one in the thermodynamic limit. For the derivation, see the Box below.

[20] This is chosen because this is $\bar{s} = O(1)$, while the typical level spacing s round a point x is $\sim 1/Dp_X(x)$, because increasing D more and more variables need to occupy the same space and the level spacing goes down.

Box 2: Derivation of Eq.(140).

Consider $\{X_1, X_2, \dots X_D\}$ from a common probability distribution $p_X(x)$ with cumulative distribution function (cfd) $F(x) = P[X \leq x]$.

- The conditional probability that, given the variable $X_j = x$ there is a differen variable $X_{k\neq j} = x+s$ is given by the probability that one variable sits around $x+s$ (this happens with probability $p_D(x+s)$) times the probability that the remaining $D-2$ variables are either to the left of x (this happens with probability $F(x)$) or to the right of $x+s$ (this happens with probability $1-F(x+s)$), i.e.

$$p_D(s|X_j = x) = p_X(x+s)[F(x)+1-F(x+s)]^{D-2} . \tag{142}$$

- The probability of the gap s for any particle with value x is hence given by

$$p_D(s|\text{any}X = x) = \sum_{j=1}^{D} p_D(s|X_j = x)p_N(X_j = x) = Dp_D(s|X_j = x)p_X(x) , \tag{143}$$

  where we have used the property that every variable has the same distribution.

- The probability of the gap, independent of where the variable is hence is given by

$$p_D(s) = \int dx p_D(s|\text{any}X = x) . \tag{144}$$

  We now change use the change of variables in Eq.(141) which gives

$$p_D(s = \frac{\bar{s}}{Dp_X(x)}|X_j = x) = p_X(x + \frac{\bar{s}}{Dp_X(x)})[F(x)+1-F(x+\frac{\bar{s}}{Dp_X(x)})]^{D-2} \tag{145}$$

$$=_{D\to\infty} p_X(x)\left[1 - F'(x)\frac{\bar{s}}{Dp_X(x)}\right]^{D-2} \simeq p_X(x)e^{-\bar{s}} , \tag{146}$$

  we now use the previous equations and find:

$$\lim_{D\to\infty} p_D(\bar{s}) = \lim_{D\to\infty} p_D(s = \frac{\bar{s}}{Np_X(x)})\frac{ds}{d\bar{s}} = D\int dx \frac{p_X(x)^2}{Dp_X(x)}e^{-\bar{s}} = e^{-\bar{s}} . \tag{147}$$

A.1.6 Symmetries

We want to show how the different classes discussed before actually correspond to the physical symmetry classes of Hamiltonian systems.

Time-reveral symmetry... or rather symmetry under "reversal of motion". A classical system is time reversal invariant if, given a solution at time t ($\mathbf{x}(t)$ and $\mathbf{p}(t)$), one can find an independent solution obtained by setting $t' = -t$ and some conditions onto $\mathbf{x}'(t)$ and $\mathbf{p}'(t)$ which connects it easily to the previous one. The simplest example is the one of a particle for which we stop the motion at $t = 0$ and then we revert its motion $\mathbf{p}(t = 0) = -\mathbf{p}(t = 0)$ and the motion is reverse because the Hamiltonian is an even function of p, i.e. $H(\mathbf{q}, \mathbf{p}) = H(\mathbf{q}, -\mathbf{p})$.

Let denote T the time reversal operator and $|\psi\rangle$ the quantum state. It is shown in the

quantum mechanics course that the time-reversal operation must be *antiunitary* [21], i.e.

$$\langle T\psi|T\phi\rangle = \langle\psi|\phi\rangle^* = \langle\phi|\psi\rangle \ . \tag{148}$$

It can be shown that this implies that the transformation T can be rewritten as

$$T = UC \ , \tag{149}$$

where $C\,|\psi\rangle = |\psi\rangle^*$ is the conjugation transformation and U is a unitary operator. A Hamiltonian characterized by time-reversal invariance is therefore characterized by

$$[H, T] = 0 \ . \tag{150}$$

Let us derive the eigenvalues of T. First of all, physically, by applying twice the time reversal, one shall obtain the same state, besides a phase, namely

$$T^2 = \alpha\mathbb{1} \quad |\alpha| = 1. \tag{151}$$

To determine α we can square the operator:

$$T^2 = UCUC = UU^*CC = UU^* = \alpha\mathbb{1} \tag{152}$$

Using the unitariety of U: $UU^\dagger = \mathbb{1} \rightarrow U^{-1} = U^\dagger$, this implies

$$U^* = \alpha U^{-1} = \alpha U^\dagger = \alpha(U^*)^T = \alpha(\alpha U^{*T})^T = \alpha^2 U^* \ , \tag{153}$$

where on the right-end side we have used the same expression twice. We conclude that

$$\alpha^2 = 1 \quad \rightarrow \boxed{T^2 = \pm 1} \ . \tag{154}$$

Therefore we can classify time-reversal symmetry depending on the eigenvalue of T^2.

$T^2 = 1$ **Real matrices** This characterizes systems with "conventional time-reversal", as for spinless particles or an even number of spins $1/2$. In this case, we have:

$$[H, T] = 0 \quad HT = TH \quad T^{-1}HT = H \ .$$

In the case of $T^2 = 1$, it can be shown (not shown here, but one can verify that it holds as an exercise) that $T^{-1} = T = C$. From this, it follows

$$H = THT = CHC = H^*C^2 = H^* \ . \tag{155}$$

In other words, the Hamiltonian is real and symmetric:

$$H^T = H^* = H.$$

The canonical transformation that leaves invariant symmetric matrices is the from the group of *orthogonal matrrices*:

$$OO^T = O^TO = \mathbb{1} \ , \tag{156}$$

they 1) leave the spectrum invariant and 2) transform real matrices in other real matrices $OHO^T = H'$. Time reversal symmetry with $T^2 = 1$ is usually associated with rotational symmetry.

[21] As opposed to unitary transformation, which does not change the phase of the overlaps $\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle$.

$T^2 = -1$ **Quaternionic real matrices** This symmetry holds for instance for spin $1/2$ of for enseble of spins with an odd number. In fact we know that the angular momentum changes sign upon time reversal:

$$TJT^{-1} = -J. \tag{157}$$

It can be shown that time-reversal symmetry with this eigenvalue can be re-written as $T = \mathbb{J}C$, where $\mathbb{J}$ is the symplectic unit. One can show that the Hamiltonian which is left invariant by this transformation is a so-called "real quaternionic matrix" which can be expressed in the form:

$$H = H_0 + iH_1 + jH_2 + kH_3 \,, \tag{158}$$

where H_0 is a real symmetric matric, while $H_{1,2,3}$ are real antisymmetric ones. Matrices of this form belong to the GSE.

Absence of symmetries In this case the Hamiltonian is an arbitrary hermitian matrix $H = H^\dagger$. The canonical transformations are given by the unitary group $UU^\dagger = U^\dagger U = \mathbb{1}$ which 1) leave the spectrum invariant and 2) transform hermitian matrices in other hermitian matrices $UHU^\dagger = H'$. Thus the appropriate class is the unitarily invariant one, i.e.

$$P(H) = P(UHU^\dagger) \,. \tag{159}$$

In summary, we have discussed the following symmetry classes:

- unitary class: absence of symmetries → GUE with $\beta = 2$;
- orthogonal class: time-reversal invariance with $T^2 = 1$ → GOE with $\beta = 1$;
- symplectic class: time-reversal invariance with $T^2 = -1$ → GSE with $\beta = 4$.

A.2 Eigenvalues averages and fluctuations

Up to now, we have discussed the eigenvalues via joint probability distributions. But what happens if we want to infer the distribution of a single eigenvalue or of its fluctuations?

To this aim, let us define the *normalized density of eigenvalues*:

$$n(\lambda) = \frac{1}{D}\sum_{i=1}^{D} \delta(\lambda - \lambda_i) \,, \tag{160}$$

which is a sum of spikes, similar to when we did the histogram, and we expect it will become a smooth function of λ for $D \to \infty$.

A.2.1 Average level density

Given the joint distribution of the eigenvalues $\rho(\lambda_1, \ldots, \lambda_D)$, we can define its marginal

$$\bar{\rho}(\lambda) = \int d\lambda_2 \ldots d\lambda_D \rho(\lambda, \lambda_2 \ldots, \lambda_D) = \overline{n(\lambda)} \,, \tag{161}$$

where $\overline{n(\lambda)}$ represents the ensemble average (see exercise 4.6 below). This quantity is called the *ensemble-average level density*. For matrices in the Gaussian ensemble, in the limit of large D, the level density can be found exactly and it is given by

$$\bar{\rho}(\lambda) = \frac{1}{\pi}\frac{1}{\beta D}\sqrt{2\beta D - \lambda^2} = \frac{1}{\pi}\frac{1}{\sqrt{\beta D}}\sqrt{2 - \left(\frac{\lambda}{\sqrt{\beta D}}\right)^2} \,, \tag{162}$$

known as the *Wigner semi-circle law*.

Comments:

1. This expression means that in the limit $D \to \infty$ the eigenvalues of the Gaussian ensembles become centered over the interval $[-\sqrt{2\beta D}, +\sqrt{2\beta D}]$ where $\pm\sqrt{2\beta D}$ are called the *spectral edges*.

2. By performing the change of variables $x = \lambda/\sqrt{\beta D}$, the distribution in Eq.(161) becomes
$$\boxed{\bar{\rho}(x) = \frac{1}{\pi}\sqrt{2-x^2}\,.} \tag{163}$$
This corresponds to the smooth curve we saw empirically by doing the histograms in Fig.12.

3. For large but finite D, the edges of Gaussian ensembles are *soft*, which means that one can always find an eigenvalue that lies below the spectral edge. Other ensembles have *hard* edges, which means that the lowest eigenvalues are impenetrable.

4. The average level density is a non-universal quantity which depends on the specific ensemble. This is in contrast which other local quantifiers (such as the level spacings, or the level spacing ratios).

5. A general property of the level density is that it is *self-averaging* which means that, in the limit of large D, the individual realization look like the average:
$$\rho(x) \sim \bar{\rho}(x) + O(D^{-1})\,. \tag{164}$$

An interesting object to study is the *Fourier transform* of Eq.(163) in the limit $D \to \infty$. First of all, we show a general relation:

$$\mathrm{FT}\left[\bar{\rho}(x)\right]_t = \int e^{-itx}\bar{\rho}(x)dx = \int e^{-itx}\frac{1}{D}\sum_i \overline{\delta(x_i - x)}dx \tag{165}$$

$$= \frac{1}{D}\sum_i \overline{e^{-itx_i}} = \frac{\overline{\mathrm{Tr}\,(e^{-iHt})}}{D} \equiv z_1(t)\,. \tag{166}$$

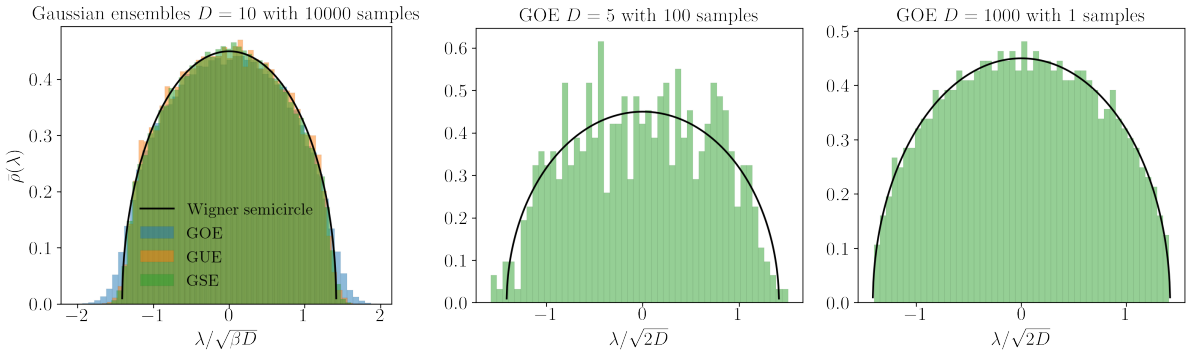


Figure 13: Distributions of the level densities for the Gaussian ensembles and their properties. (Left) The Wigner semicircle in Eq.(163) is a unique function that predicts the average density for all the ensembles. (Center) Soft edges: for small D there are eigenvalues below or above $\pm\sqrt{2\beta D}$. (Right) Self-averaging: for large D a single instance behaves like its average.

Hence the FT of the level density always can be interpreted as the average of a partition function in imaginary time. In the case of the Wigner semi-circle, this reads:

$$z_1(t) = \int e^{-itx}\bar{\rho}(x)dx = \frac{J_1(2|t|)}{t}\,, \tag{167}$$

where $J_1(2|t|)$ is the Bessel function of first type. Therefore this function starts from 1 at time $t = 0$, oscillates, and then decays at large time t with an envelope given by $z_1(t) \sim t^{-3/2}$.

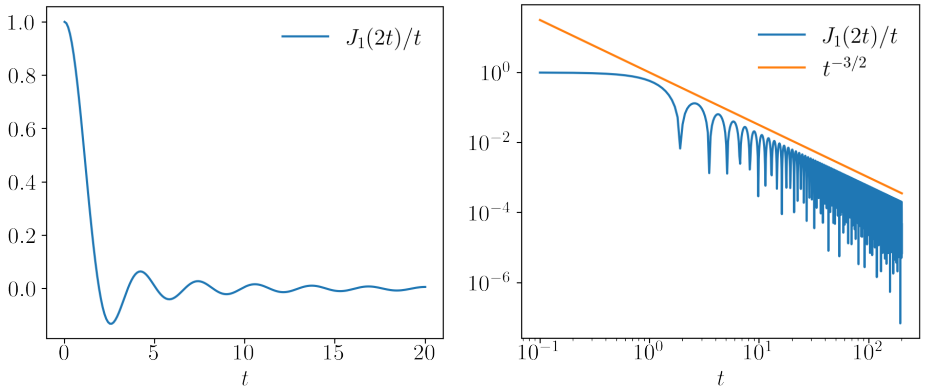


Figure 14: Fourier transform of the average level density $z_1(t)$ for the gaussian ensemble in Eq.(167) and its long-time behaviour on the left.

The semi-circle law can be proved in a series of ways, a fact that displays how varied is the extent of random matrix theory. Even if we do not reproduce here any of the derivations, let us here mention some of the approaches, because they contain several important concepts which are often used in RMT.

- *Stat. mech. approach:* Here one starts re-writing the normalization to the distribution of the eigenvalues in Eq.(139) as a partition function, i.e.

$$Z_{D,\beta} = \int d\mathbf{x} e^{-\beta N^2 \mathcal{V}[\mathbf{x}]}\,, \tag{168}$$

  of the following fictitious Hamiltonian:

$$\mathcal{V}[\mathbf{x}] = \frac{1}{2N}\sum_i x_i^2 - \frac{1}{2N^2}\sum_{i\neq j}\ln|x_i - k_j|\,, \tag{169}$$

  where we have exponentiated the Vandermonte determinant. This equation represents free particles on a line that interact via a logarithmic potential. In analogy with the Coulomb gas in 2D, this problem is known as the Coulomb gas problem. The solution to the partition function is found by saddle point and after a lengthy calculation, one finds (161). See e.g. Ref. [23] for a pedagogical derivation.

- *Combinatorics approach*. Let us introduce the resolvent also known as the Green function:

$$G(z) = \frac{1}{D}\overline{\mathrm{Tr}\left(\frac{1}{z-H}\right)} = \int dE \frac{\bar{\rho}(E)}{z-E}\,, \tag{170}$$

  where we have re-expressed the trace in terms of the eigenvalues of H and then re-written the delta functions in terms of Eq.(160). The Green function contains all the

information about the spectrum of H. In particular, we can retrieve the average density by [22]:

$$\bar{\rho}(\lambda) = \frac{1}{\pi}\mathrm{Im}G(\lambda + i0^+) \,. \tag{171}$$

In the Gaussian case, one can expand the trace and compute the combinatorics of the various products. One finds that the finite contributions come from the so-called non-crossing pairings, which can be resumed leading to the Wigner semicircle in Eq,(161).

- *Resolvent + saddle point*. One can find an algebraic solution for the $G(z)$ by finding the saddle point solution to Eq.(169), see Ref. [23].
- free probability approach!!

A.2.2 Fluctuations and the Spectral Form Factor

Besides the average level density, one may as well be interested at the marginal between n different eigenvalues, i.e. their correlations! Let us define the *n-point spectral correlator* as

$$\bar{\rho}^{(n)}(x_1, \ldots, x_n) = \int dx_{n+1} \ldots dx_D \rho(x_1, \ldots, x_D) \,. \tag{172}$$

For example, at the lowest orders, this yields the average level density:

$$\bar{\rho}^{(1)}(x_1) = \bar{\rho}(x_1) \,,$$

and the two-point correlation, which shall encode the connected correlations as

$$\bar{\rho}^{(2)}(x_1, x_2) = \bar{\rho}(x_1)\bar{\rho}(x_2) + \bar{\rho}^{(2)}(x_1, x_2)_c \,.$$

In the case of the Gaussian ensembles the spectral correlators can be determined exactly as

$$\bar{\rho}^{(n)}(x_1, \ldots, x_n) = \frac{(D-n)!}{D!} \det[K_D(x_i, x_j)]_{i,j=1,\ldots,n} \,, \tag{173}$$

where $K_D(x, y)$ is the Kernel function, which can be determined exactly and may depend on the specific ensemble [65]. For instance in the GUE one has, in the limit of $D \to \infty$,

$$\lim_{D\to\infty} K_D^{\mathrm{GUE}}(x_i, x_j) = \begin{cases} \frac{D}{\pi}\sqrt{2 - x_i^2} & i = j \\ \frac{D}{\pi}\frac{\sin(2\beta D(x_i - x_j))}{2\beta D(x_i - x_j)} & i \neq j \end{cases} \,. \tag{174}$$

In the first few orders, this implies:

$$\bar{\rho}^{(1)}(x_1) = \frac{1}{D}K(x_1, x_1) = \frac{1}{\pi}\sqrt{2 - x_1^2} \,, \tag{175}$$

$$\begin{aligned} \bar{\rho}^{(2)}(x_1, x_2) &= \frac{1}{D(D-1)}\left[K(x_1, x_1)K(x_2, x_2) - K(x_1, x_2)K(x_2, x_1)\right] \\ &= \frac{D^2}{D(D-1)}\left[\bar{\rho}(x_1)\bar{\rho}(x_2) - Y_2(x_1, x_2) \,,\right] \end{aligned} \tag{176}$$

[22] In general the resolvent is a function of the complex variable z and one has

$$G(\lambda + i\epsilon) = P\int dE \frac{\bar{\rho}(\lambda)}{\lambda + i\epsilon} + i\pi\bar{\rho}(\lambda) \quad \epsilon \to 0^+ \,.$$

where $Y_2(x_1,x_2) = \frac{1}{D^2}K(x_1,x_2)K(x_2,x_1)$ is known as the *two cluster function* and it depends on the symmetry class. In the appropriate scaling regime, one can find exact behavior for this function, which only depends on the eigenvalues differences. Setting the local level spacing as

$$r = \frac{2D\beta}{\pi}(x_1 - x_2) \tag{177}$$

one has [65]

$$\lim_{D\to\infty} Y_2^{\mathrm{GUE}}(x_1,x_2) = \frac{1}{\pi^2}\left(\frac{\sin\pi r}{\pi r}\right)^2 \tag{178a}$$

$$\lim_{D\to\infty} Y_2^{\mathrm{GOE}}(x_1,x_2) = \frac{1}{\pi^2}\left(\frac{\sin\pi r}{\pi r}\right)^2 + \int_r^\infty \frac{\sin\pi t}{\pi t}dt\left(\frac{d}{dr}\frac{\sin\pi r}{\pi r}\right) \tag{178b}$$

$$\lim_{D\to\infty} Y_2^{\mathrm{GSE}}(x_1,x_2) = \frac{1}{\pi^2}\left(\frac{\sin\pi r}{\pi r}\right)^2 - \left(\frac{d}{dr}\frac{\sin\pi r}{\pi r}\right)\int_r^\infty \frac{\sin 2\pi t}{2\pi t}dt \tag{178c}$$

Note that this function quantifies the connected correlations between two eigenvalues at distance $x_1 - x_2$. It expresses the fact that correlations exist only at very small scales, i.e. $2\beta D(x_1 - x_2) \ll 1$, for very small eigenvalue differences. Otherwise, the eigenvalues look uncorrelated.

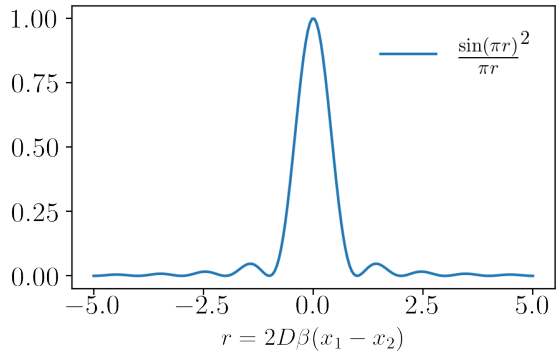


Figure 15: Spectral correlations for the GUE symmetry class in Eq.(178a).

How does this translate to the time domain? As done above in Eq.(165), we consider the time-dependent Fourier transform of these objects. In particular, let us introduce the so-called *Spectral-Form-Factor*:

$$\begin{aligned}
\mathrm{SFF}(t) &= \overline{\left|\frac{\mathrm{Tr}(e^{-iHt})}{D}\right|^2} = \frac{1}{D^2}\sum_{ij}\overline{e^{i(x_i-x_j)t}} \\
&= \frac{1}{D} + \frac{D(D-1)}{D^2}\int dx_1 dx_2 \rho^{(2)}(x_1,x_2)e^{i(x_1-x_2)t} \\
&= \left|\int dx_1 \hat{\rho}(x_1)e^{-ix_1 t}\right|^2 + \frac{1}{D} - \int dx_1 dx_2 Y_2(x_1,x_2)e^{i(x_1-x_2)t} \\
&= |z_1(t)|^2 + \frac{1}{D}(1 - z_2(t))\,,
\end{aligned} \tag{179}$$

where from the second to the third line, we plugged in the two-level spectral correlator in Eq.(176) and the disconnected result $z_1(t)$ from Eq.(165). Finally, in the last line, we have

defined the Fourier transform of the two-cluster function:

$$z_2(t) = D \int dx_1 dx_2 Y_2(x_1, x_2) e^{i(x_1 - x_2)t} \,, \tag{180}$$

whose specific expression depends on the ensemble. In the GUE case we can use Eq.(178a) and find that

$$1 - z_2^{\mathrm{GUE}}(t) = 1 - D \int dx_1 dx_2 \frac{1}{\pi^2} \left(\frac{\sin(2\beta D(x_1 - x_2))}{2\beta D(x_1 - x_2)} \right)^2 e^{i(x_1 - x_2)t} = \begin{cases} \dfrac{t}{\beta D} & \text{for} \quad t < \beta D \\ 1 & \text{for} \quad t > \beta D \end{cases} . \tag{181}$$

The linear regime is known as *the ramp*, and it encodes all the correlations of random matrices; it is believed to be one of the universal features of RMT. At $t = D$, the SFF becomes constant, this regime is known as the *plateau*, and it is the result that would hold in the case of independent random variables (see exercise 4.7).

A.3 Circular ensembles

As we have seen for the Gaussian case, the average level density distribution was not "uniform" but rather had a semi-circle shape, see Eq.(163). There have been several attempts to create a uniform distribution from the eigenvalue properties of RM. However, a uniform probability density cannot be defined on the infinite real line.

To address this issue, in 1962, Dyson introduced three new ensembles on the space of *unitary matrices* [69–71], called circular ensembles. The classification comes from the canonical transformation of their member matrices and are respectively:

- COE - circular Orthogonal matrices - for symmetric unitary matrices
- CUE - circular Unitary matrices - for unitary matrices
- CSE - circular Symplectic matrices - for self-dual unitary quaternionic matrices.

The circular ensemble is characterized by the Haar measure (uniform measure over a group) on the group of $D \times D$ orthogonal matrices $O(D)$, the unitary group $U(D)$ and the symplectic group $Sp(2D)$ respectively.

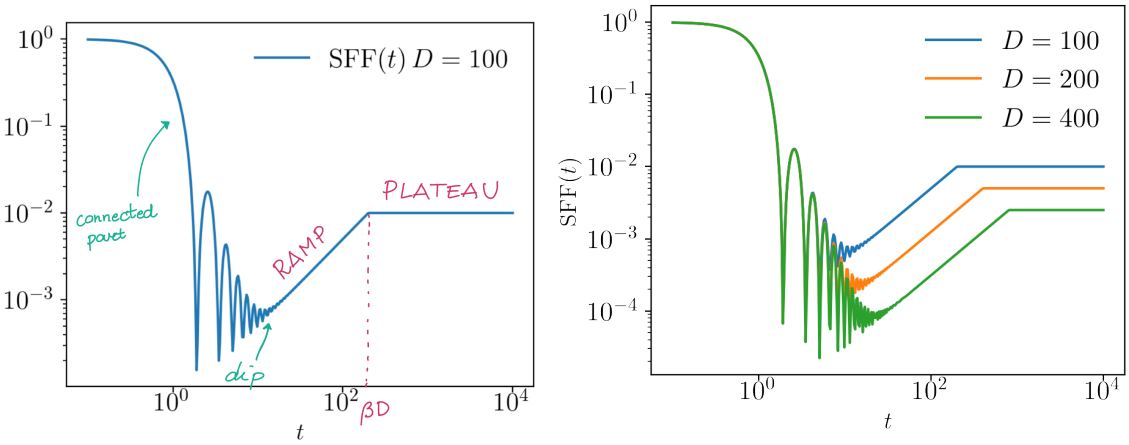


Figure 16: Spectral form factor for the GUE ensemble in Eq.(181).

Let's denote S one of its members. Since S is unitary, its eigenvalues lie on the unit circle, namely

$$S|i\rangle = e^{i\theta_i}|i\rangle \qquad 0 \leq \theta_i \leq 2\pi \ . \tag{182}$$

The joint probability of the eigenvalues is

$$\rho(\theta_1,\ldots,\theta_D) = \frac{1}{Z_{\beta,D}} \prod_{i<j} |e^{i\theta_i} - e^{i\theta_j}|^\beta = \frac{1}{Z_{\beta,D}} \prod_{i<j} |2\sin(\theta_i - \theta_j)|^\beta \ , \tag{183}$$

where $Z_{\beta,D}$ is a normalization. This probability depends only on the eigenphases difference; hence, it is homogeneous (in contrast to the Gaussian ensemble). As a consequence, the average level density is constant:

$$\bar{\rho}(\theta) = \int d\theta_2 \ldots d\theta_D \rho(\theta,\ldots,\theta_D) = \frac{1}{2\pi} \ . \tag{184}$$

The eigenphases are uniformly distributed. Hence, the circular ensemble does not necessitate any unfolding of the quasienergies.

The circular ensembles have important applications for non-integrable Floquet circuits.

B Eigenvector statistics in Random Matrix Theory

We consider $D \times D$ random matrix H. Let us restrict ourselves to *rotationally invariant random matrices*, such that the probability of the random matrix H is invariant under similarity transformation, i.e.

$$P(H) = P(UHU^{-1}) \ , \tag{185}$$

where U is an orthogonal/unitary/symplectic matrix if H is real symmetric/complex hermitian/quaternionic self-dual, respectively. This class of RMT models, also known as *matrix models* in the high-energy community, are characterized by

$$P(H) \propto \exp\left(-\frac{1}{2}\mathrm{Tr}\left(H^2\right) + V(H)\right) \quad \text{with} \quad V(H) = \alpha_1 \mathrm{Tr}\,(H) + \alpha_3 \mathrm{Tr}\left(H^3\right) + \ldots \tag{186}$$

in other words, the probability distribution can be written only as a function of the traces of powers of the Hamiltonian. This implication follows from the cyclic property of the trace. This means that one can rotate our matrices at will, but this will not change the distribution of the matrix: in other words, the eigenvectors are irrelevant! For this class of systems, there is a complete factorization between eigenvalues properties and eigenvectors properties. Notably, the Gaussian ensemble is contained in this ensemble since it corresponds to the above equation without $V(H)$, which is in jargon referred to as the "random matrix potential".

We now want to describe the properties of the eigenvectors of H, i.e.,

$$H|i\rangle = \lambda_i |i\rangle$$

where $|i\rangle$ are the vectors of size D with components $c_\alpha^{(i)}$:

$$|i\rangle = \begin{pmatrix} c_1^{(i)} \\ c_1^{(i)} \\ \cdots \\ c_D^{(i)} \end{pmatrix} = \sum_\alpha c_\alpha^{(i)} |\,|\alpha\rangle \ , \quad \text{with} \quad c_\alpha^{(i)} = \langle\alpha|\,i\rangle \ , \tag{187}$$

where we have re-expressed the eigenstate in the basis $|\alpha\rangle$ using Dirac notation.

Since in Eq.(185), one can take as U the matrix that diagonalizes the Hamiltonian, hence containing the eigenvectors, it is clear that once the symmetry class has been identified, the statistical properties of the eigenvectors shall not depend on the basis: every eigenvector can be rotated into an arbitrary vector of unit norm. Essentially, we only have two requirements:

1. normalization;
2. orthogonality.

Form the normalization requirement, one would like to conclude that the only invariant characteristic of eigenvectors is the norm. Hence the joint probability distribution of the D components is:

$$\rho_{\rm UE}(\mathbf{c}^{(i)}) = c_D \delta\left(1 - \sum_{\alpha=1}^{D} |c_\alpha^{(i)}|^2\right) \quad \forall i \tag{188a}$$

$$\rho_{\rm OE}(\mathbf{c}^{(i)}) = c'_D \delta\left(1 - \sum_{\alpha=1}^{D} (c_\alpha^{(i)})^2\right) \quad \forall i\,, \tag{188b}$$

where c_D and c'_D are normalization constants. Essentially, the probability of the components is non-zero only on the surface of a multidimensional unit sphere. Since all eigenvectors are the same, we can now forget about the subscript i.

We will discuss below the constraint of orthogonality, which is subleading in $1/D$ for large D.

B.1 Normalization and single component distribution

Let us now focus on the *distribution of a single component* $y = |c_1|^2$, by considering the following marginal:

$$\rho_{\rm UE}(y) = \int d^2c_1 \dots d^2c_D \delta(y - |c_1|^2) \rho_{\rm UE}(\mathbf{c})\,. \tag{189}$$

We will now perform the derivation in the GUE case; the GOE analysis is left as exercise 4.8.

Let us first introduce an auxiliary object:

$$\rho_{\rm UE}(y;t) = \int d^2c_1 \dots d^2c_D \delta(y - |c_1|^2)\, c_D \delta\left(t - \sum_{\alpha=1}^{D} |c_\alpha|^2\right) \tag{190}$$

such that $\rho_{\rm UE}(y) = \rho_{\rm UE}(y;1)$. We take the Laplace transform to respect to t as

$$\begin{aligned}\rho_{\rm UE}(y;s) &= \int_0^\infty \rho_{\rm GUE}(y;t) e^{-st} dt = c_D \int d^2\mathbf{c}\, \delta(y - |c_1|^2) e^{-s\sum_{\alpha=1}^{D} |c_\alpha|^2} \\ &= c_D \int d^2c_1 \delta(y - |c_1|^2) e^{-s|c_1|^2} \left(\int_0^\infty d^2c e^{-s|c|^2}\right)^{N-1}\,.\end{aligned}$$

Then, we convert the integrals into polar coordinates $d^2c = 2\pi r dr$ and re-absorb the angles in the constant and obtain

$$\rho_{\rm UE}(y;s) = \tilde{c}_D \int r dr e^{-sr^2} \delta(y - r^2) \left(\int_0^\infty \rho d\rho e^{-s\rho^2}\right)^{N-1} = \tilde{c}_D \frac{e^{-sy}}{s^{N-1}}\,. \tag{191}$$

The inverse Laplace transform [23] leads to

$$\rho_{\rm UE}(y;t) \propto (t-y)^{N-2} \theta(t-y)\,. \tag{192}$$

[23] The inverse Laplace transform is defined as $f(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma-iT}^{\gamma+iT} e^{st} f(s) ds$.

Setting $t = y$ and normalizing, we get

$$\rho_{\rm UE}(y) = (N-1)(1-u)^{N-2} \qquad 0 \le y \le 1 \,. \tag{193}$$

We now re-scale the component by its average

$$\overline{y}_{\rm UE} = \int dy \rho_{\rm UE}(y) dy = \frac{1}{D} \,, \tag{194}$$

and consider

$$\eta = \frac{y}{\overline{y}} = Dy \,. \tag{195}$$

Therefore, in the large D limit, one has

$$\rho_{\rm UE}(\eta) = \lim_{D\to\infty} \frac{1}{D} \rho_{\rm UE}(\eta/D) = e^{-\eta} \,. \tag{196}$$

One can perform similar calculations for the other symmetry classes, leading to the so-called *Porter-Thomas distribution* [72] for a single component $\eta = |c_1|^2/\overline{|c_1|^2}$ of a random vector:

$$\rho_{\rm OE}(\eta) = \frac{1}{2\pi\sqrt{\eta}} e^{-\eta} \tag{197a}$$

$$\rho_{\rm UE}(\eta) = e^{-\eta} \tag{197b}$$

$$\rho_{\rm SE}(\eta) = \eta e^{-\eta} \,. \tag{197c}$$

Comments:

1. This result shows that the single amplitude $|c_1|$ has a distribution that looks very similar to a (normalized) Gaussian. There are some differences that depend on the different symmetry classes.

2. In deriving this result, we have not used anywhere that the matrix was Gaussian: The result applies to all "random" normalized vectors of unitary matrices from the circular ensemble.

3. Since, in this case, we were interested only in one component, we did not care much about the correlations between different components, which matter! As we will see in the next section.

4. It turns out that also high energy eigenvectors of chaotic Hamiltonian follow this distribution: another instance of RMT universality, ss we will see in the next chapter on many-body chaos.

B.2 Orthogonality and Haar averages

From the discussion until now, it may seem that one may obtain these results by simply taking Gaussian entries and normalizing them. As we will see, while this prescription gives sensible insights in the large D limit, this is not accurate: we need correlations between the eigenstates. A simple way to see this is that another obvious condition for the eigenvectors statistics is the *orthogonality*. Using the definition in Eq.(187), this implies

$$\langle i\,|j\rangle = \sum_{\alpha\beta} c_\alpha^{(j)} c_\beta^{(i)*} = \delta_{ij} \,. \tag{198}$$

The matrices U are defined by the eigenvectors

$$[U]_{i\alpha} = c_\alpha^{(i)*} = \langle i|\alpha\rangle \,, \tag{199}$$

should be unitary $UU^\dagger = U^\dagger U = \mathbb{1}$. In other words, for every D, the eigenvectors are distributed uniformly on the unitary $U(D)$ or the orthogonal group $O(D)$.

The statistical properties of these Haar ensembles are known exactly $\forall D$. Let us restrict to the unitary case. The first-order average reads

$$\overline{U_{i\alpha}U^\dagger_{\alpha' i'}} = \frac{1}{D}\delta_{ii'}\delta_{\alpha\alpha'} \,, \tag{200a}$$

which shows that finite averages survive only if the indices appear at least twice. If we consider more general products, the indices i always have to be permutations of the indices i' and the same for the α. For instance, on the products of four matrix elements we have

$$\overline{U_{i\alpha}U_{j\beta}U^\dagger_{\alpha' i'}U^\dagger_{\beta' j'}} = \frac{1}{D^2-1}\left[\delta_{ii'}\delta_{\alpha\alpha'}\delta_{jj'}\delta_{\beta\beta'} + \delta_{ij'}\delta_{\alpha\beta'}\delta_{ji'}\delta_{\beta\alpha'} - \frac{1}{D}\left(\delta_{ii'}\delta_{\alpha\beta'}\delta_{jj'}\delta_{\beta\alpha'} + \delta_{ij'}\delta_{\alpha\alpha'}\delta_{ji'}\delta_{\beta\beta'}\right)\right] \tag{200b}$$

The first two leading order terms can be identified with Gaussian contribution (by doing Wick contractions). Conversely, the last two terms, which seem to go away in the thermodynamic limit, are important to have the normalization of the eigenvectors. These Eqs.(200) are the order 1 and 2 of a more general method for computing Haar averages over the unitary group, which goes under the name of *Weingarten calculus* [73–75]. The general formula for the average of products of unitary matrices reads

$$\overline{U_{i_1\alpha_1}\dots U_{i_n\alpha_n}U^\dagger_{\alpha'_1 i'_1}\dots U^\dagger_{\alpha'_n i'_n}} = \sum_{\tau\sigma\in S_n} \mathrm{Wg}(\sigma\tau^{-1})\delta_{i_1 i'_{\sigma(1)}}\dots\delta_{i_n i'_{\sigma(n)}}\delta_{\alpha_1\alpha'_{\tau(1)}}\dots\delta_{\alpha_n\alpha'_{\sigma(n)}} \tag{200c}$$

where τ,σ are permutations over the symmetric group of n elements S_n. Essentially, Eq.(200c) is telling us that only indices $\mathbf{i}'$, which are appropriate permutations of $\mathbf{i}$, survive and similarly $\boldsymbol{\alpha}'$ have to be permutations of $\boldsymbol{\alpha}$. The proportionality constant $\mathrm{Wg}(\sigma\tau^{-1})$ is a combinatorial matrix, called *Weingaten matrix* [73,74]. The latter is given by the inverse of the Gram matrix[24]

$$Q_{\sigma,\tau} = D^{\#(\sigma^{-1}\tau)} \,, \tag{201}$$

and $\#\sigma$ counts the number of cycles in the permutation σ.

[24] We assume the existence of the inverse Q^{-1}, valid for $k \le N$, which is the case under consideration.