Lecture I

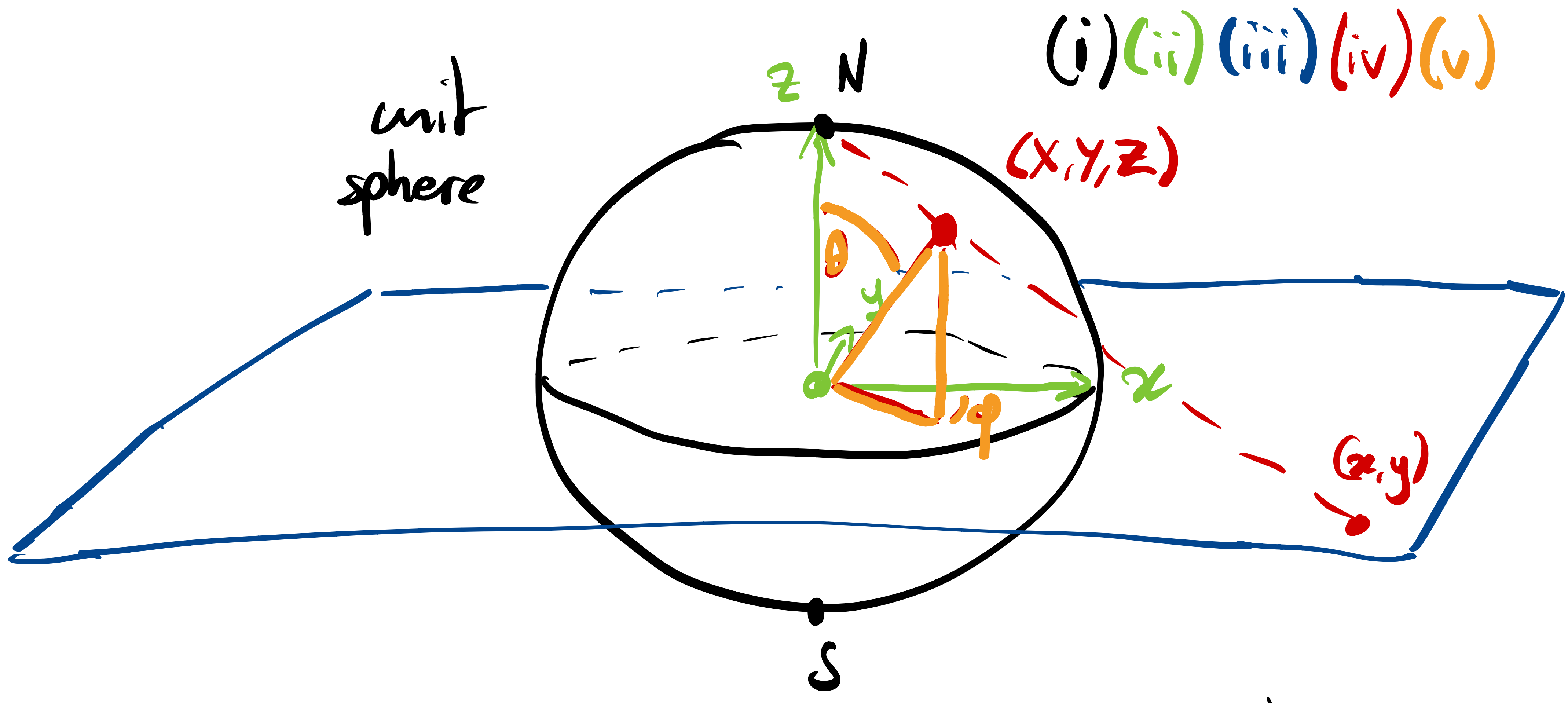
Conformal mapping

Conformal mapping

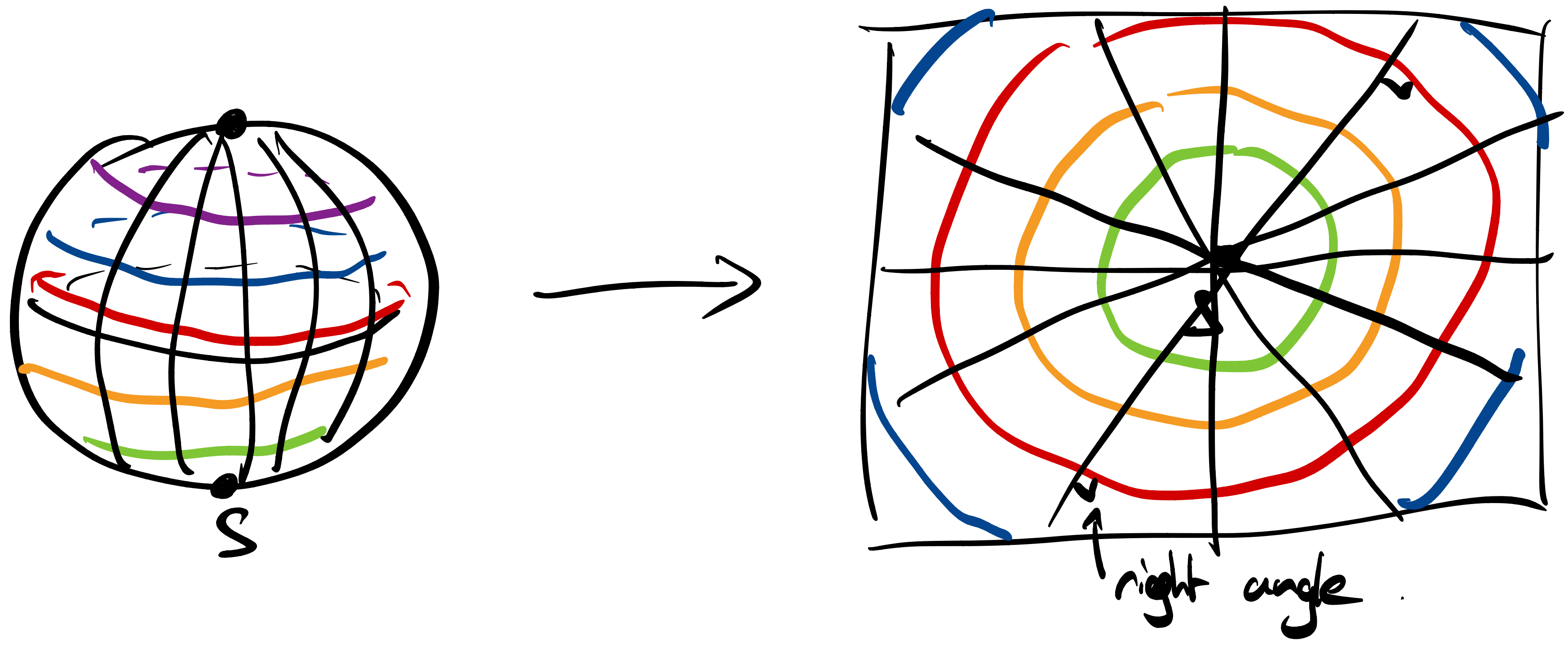
1. 2D conformal maps : a first look

Conformal map = transformation that preserves angles, not length.

example : stereographic projection.



$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1-Z}\begin{pmatrix} X \\ Y \end{pmatrix} = \operatorname{cotan}\frac{\theta}{2}\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$



line element on the sphere :

$$ds^2 = d\theta^2 + (\sin\theta)^2 d\varphi^2$$

expressed in (x,y) coordinates

$$= \lambda(x,y)^2 \underbrace{(dx^2 + dy^2)}$$

proportional to line element in the euclidean plane

$$\left(\text{with } \lambda = \frac{2}{1+x^2+y^2}\right)$$

<u>conclusion</u> :

$$g^{\text{sphere}}_{ab} = \lambda\, g^{\text{plane}}_{ab}$$

$(a,b = x,y)$.

so the map preserves angles, but not lengths $(\lambda \neq 1)$.

2. <u>Conformal map of planar domains and complex analytic functions</u>

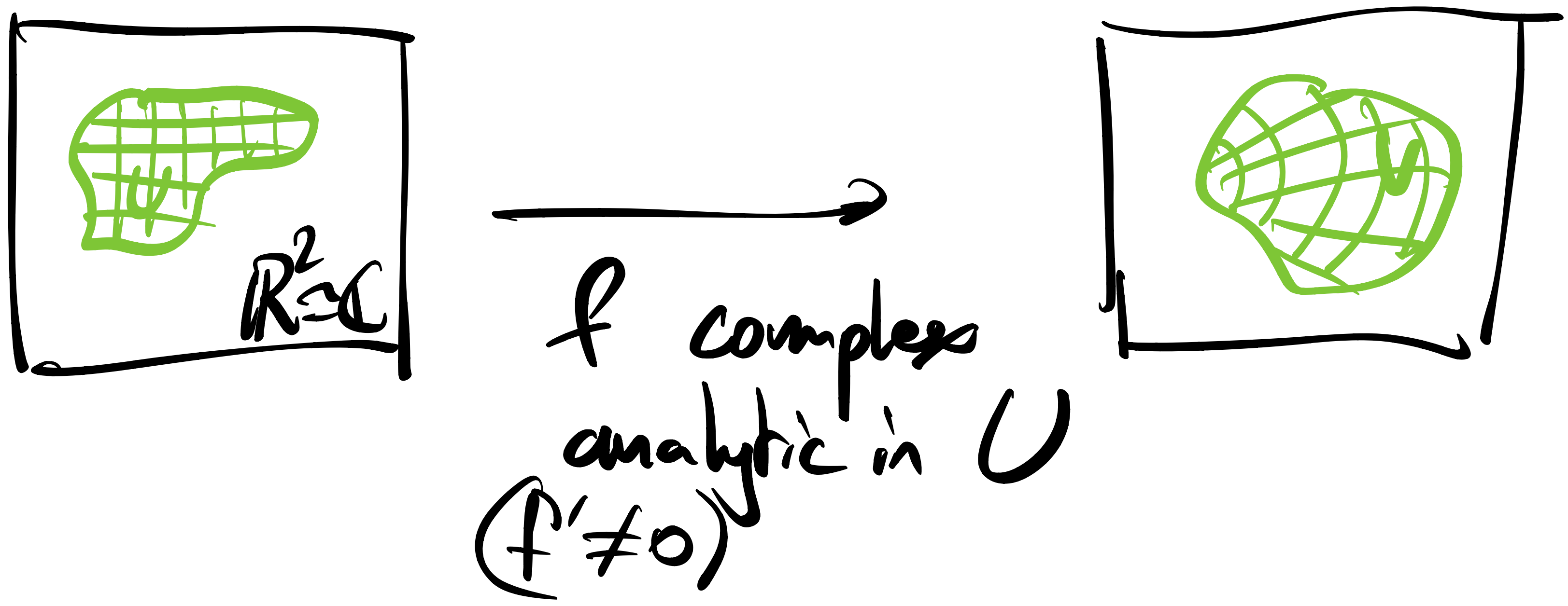
recall that, in complex coordinates,

(i) $z \longmapsto z + a$ is a translation

(ii) $z \longmapsto e^{i\theta} + a$ is rotation of angle θ

(iii) $z \longmapsto \lambda z$ $(\lambda > 0)$ is a dilation of ratio λ.

Now look at a complex-analytic (holomorphic) function on some open subset of the plane



This map is conformal.

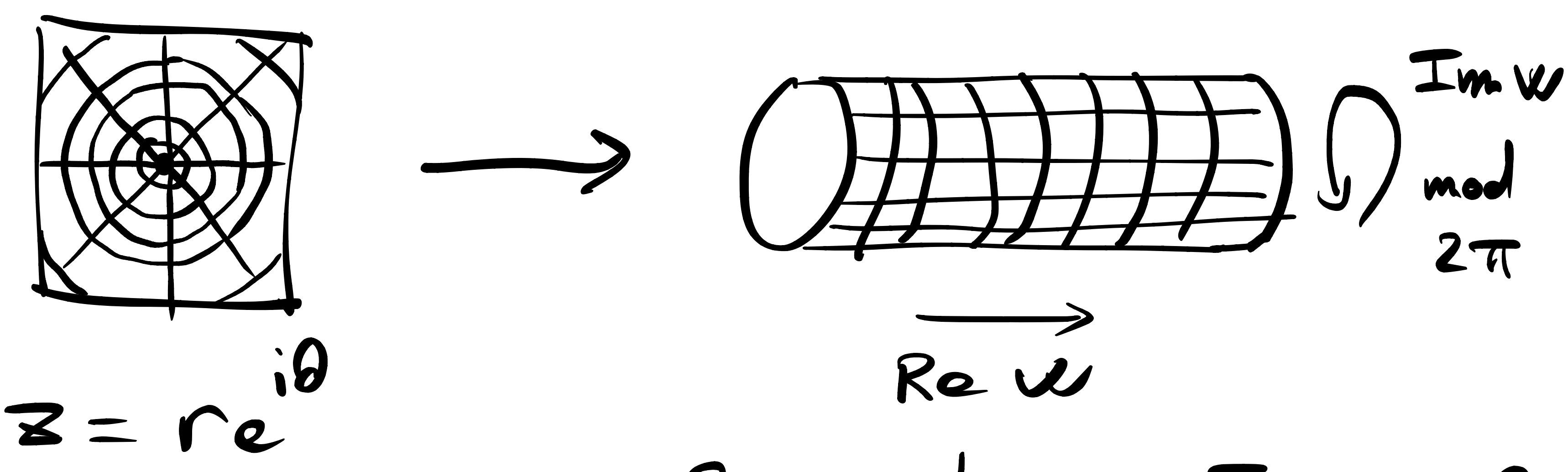
Indeed, around any point $z_0 \in U$

$$f(z) \simeq f(z_0) + (z-z_0) f'(z_0) + \dots$$

$$= \underbrace{f(z_0)}_{\text{translate}} + \underbrace{e^{i \arg f'}}_{\text{rotation}} \underbrace{|f'|}_{\text{dilation}} (z-z_0)$$

Examples of conformal maps (slides)

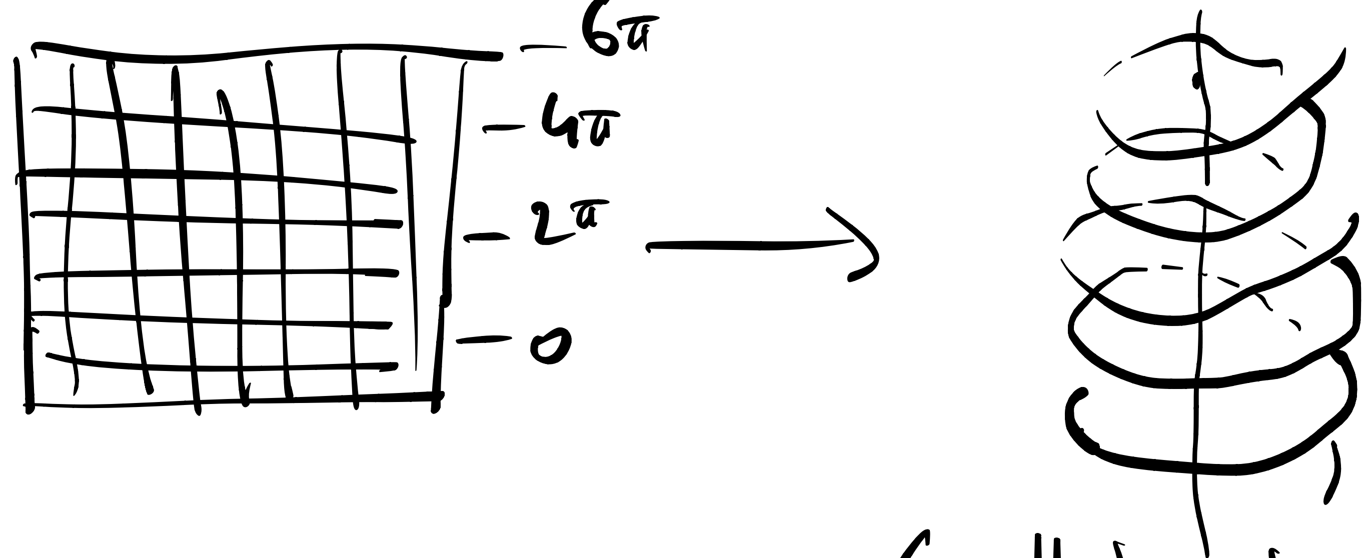
- $z \mapsto w = \ln z$



$\mathrm{Re}\, w = \ln r \qquad \mathrm{Im}\, w = \theta$

- $z \mapsto w = e^z$

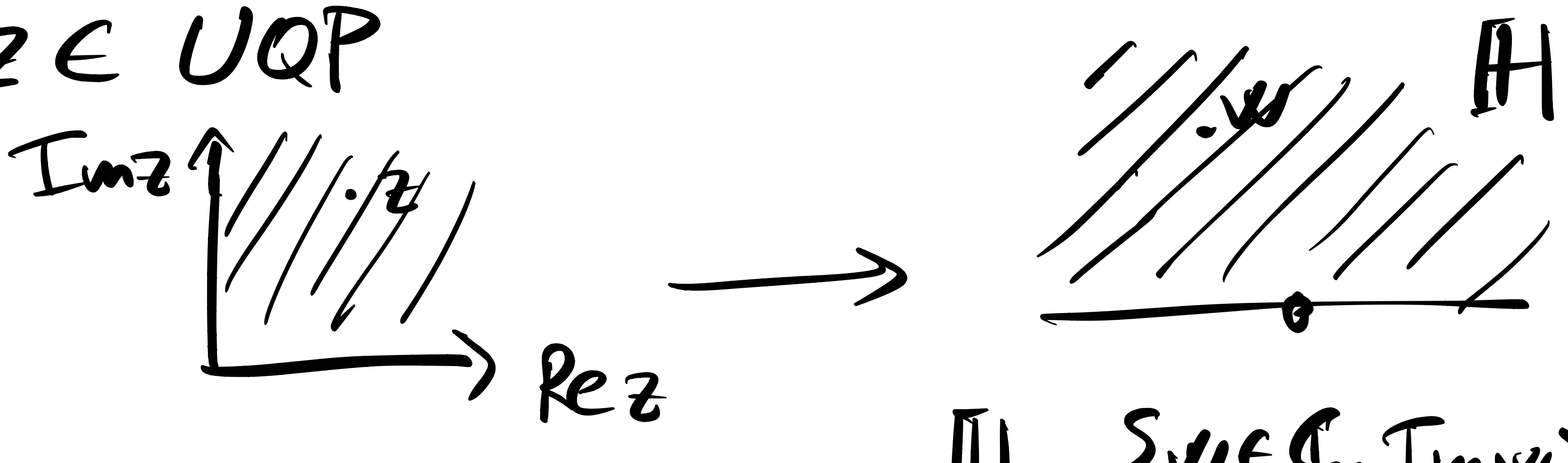
or



(multiple sheets)

- $z \in \mathbb{C} \mapsto w = z^2$

$z \in$ UQP



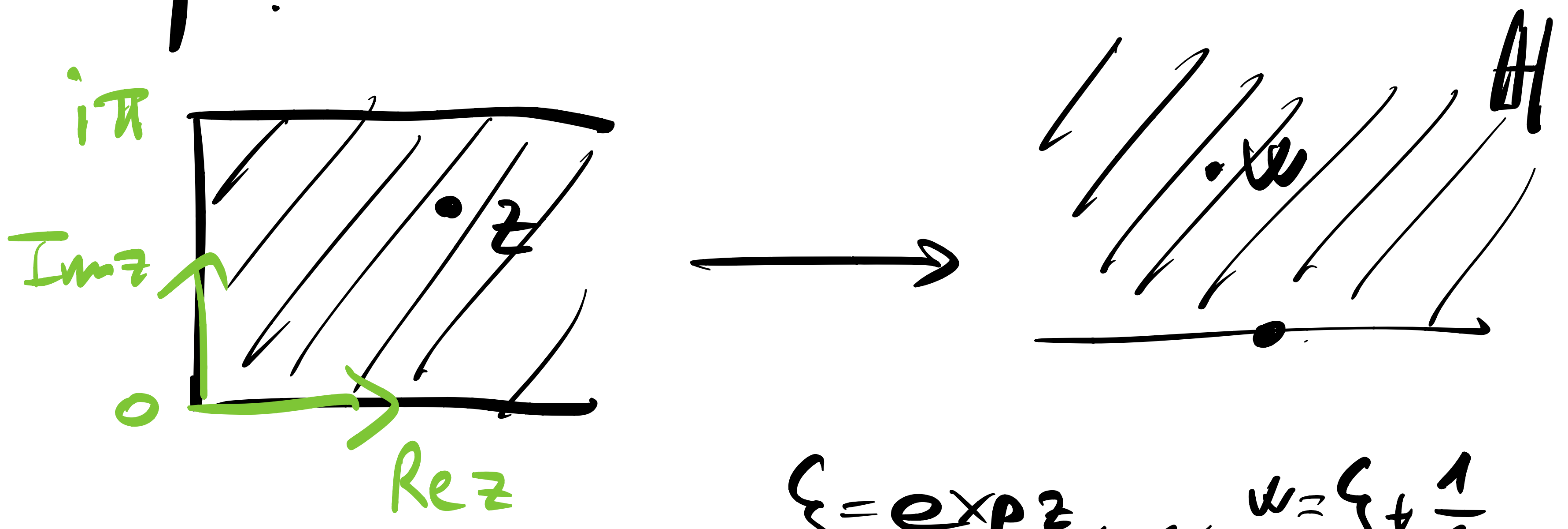
$\mathbb{H} = \{w \in \mathbb{C}, \operatorname{Im} w > 0\}$

- $z \mapsto z + \frac{1}{z}$

$\mathbb{C}$ minus disc

$\mathbb{C}$ minus slit.

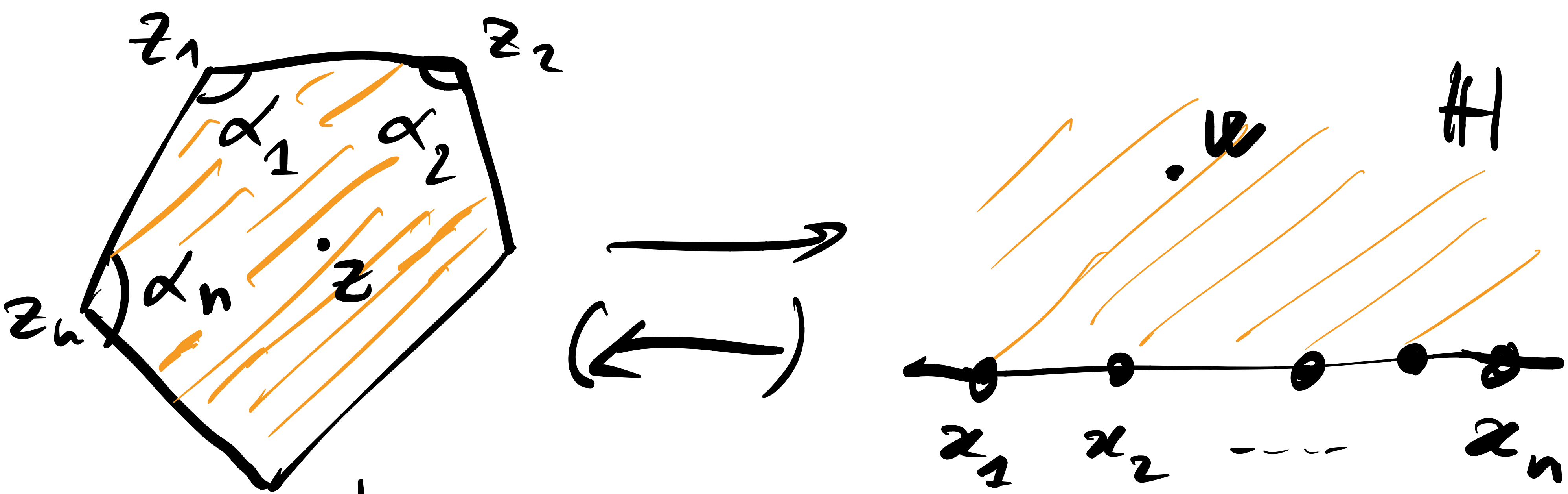
- <u>Question</u>: Can you guess the following map?



<u>Answer</u>:

so $w = \cosh z$.

- a <u>general formula</u> for mapping from <u>polygonal domain</u> to $\mathbb{H}$.



polygonal domain

near a corner: $z - z_j \propto (w - x_j)^{\alpha_j/\pi}$

differentiating: $dz \propto (w - x_j)^{\alpha_j/\pi - 1}\, dw$

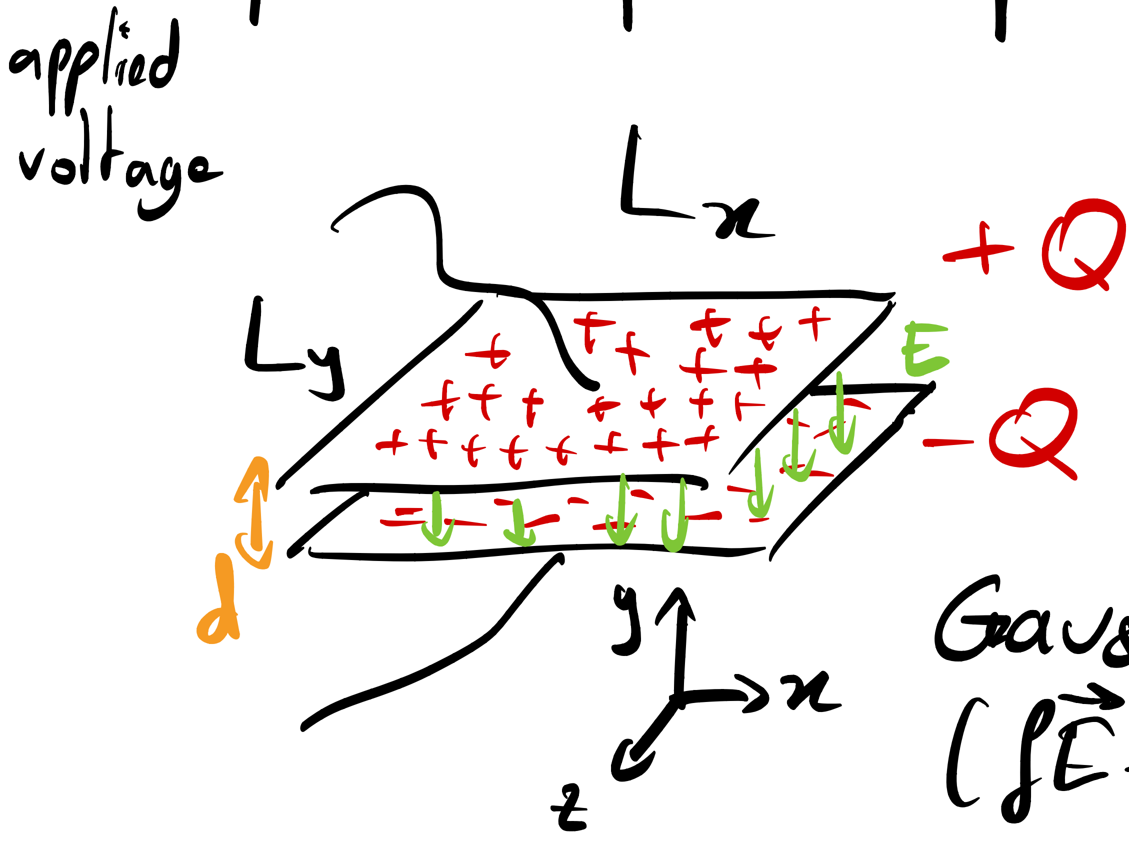
global formula: $\boxed{z = A \int^{w} \prod_{j=1}^{n} (\xi - x_j)^{\frac{\alpha_j}{\pi} - 1} d\xi}$

(Schwarz-Christoffel formula).

3. Conformal mappings in physics / engineering

3.1 Electrostatics

parallel plate capacitor (infinite)



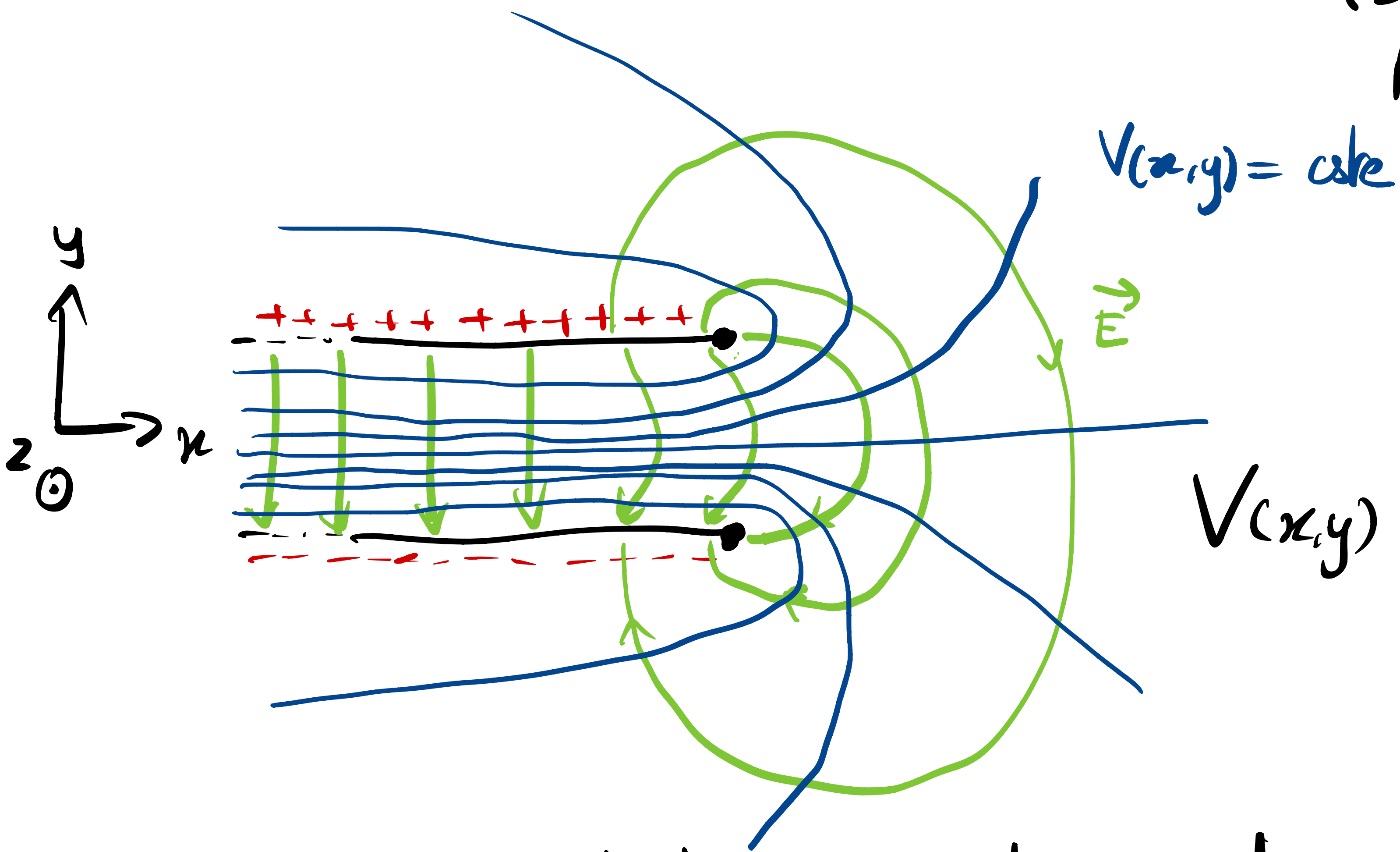
capacitance

$$C = \frac{Q}{V}$$

Gauss law $\Longrightarrow$ $C = \dfrac{\varepsilon_0 L_x L_y}{d}$

$$\left(\oint \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon_0} \right)$$

(set vacuum permittivity to 1)

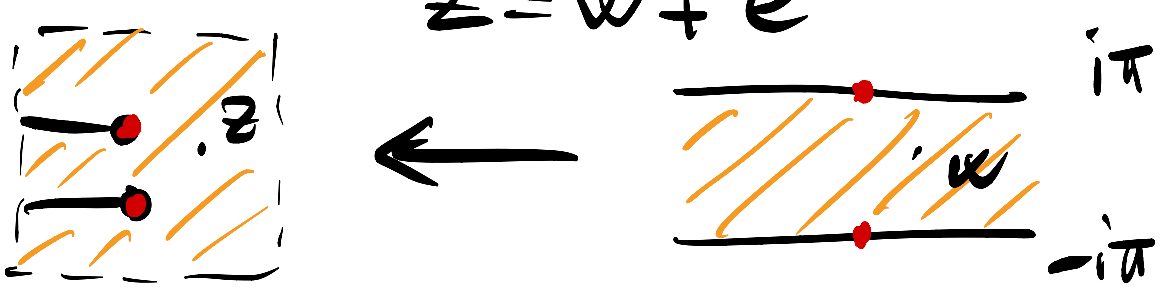


$$\Rightarrow \quad C = \frac{L_x L_y}{d} + \frac{L_y}{2\pi} \ln\left(\frac{L_x}{d}\right) + O(1)$$

(exercise).

Conformal map:

$$z = w + e^w$$



3.2 Aerodynamics / fluid dynamics

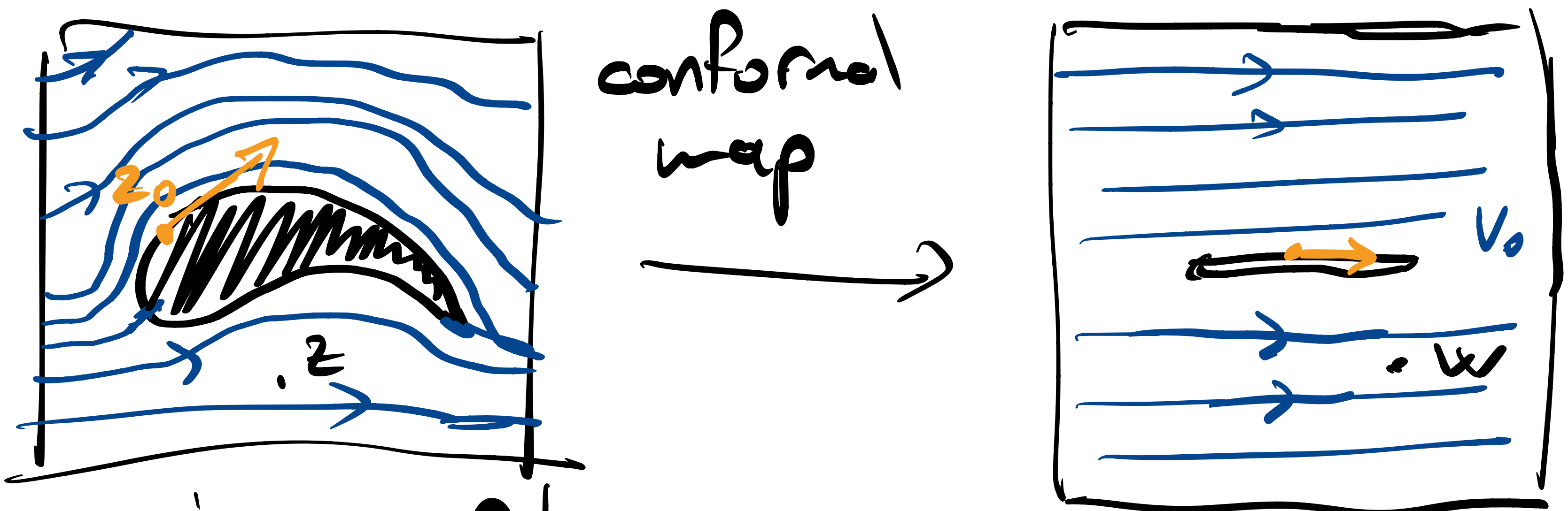
fluid flows in 2D: $\vec{U} = \begin{pmatrix} U_x \\ U_y \end{pmatrix}$

$$\begin{cases} \text{incompressible fluid: } \vec{\nabla}\cdot\vec{U} = 0 \\ \text{irrotational flow: } \vec{\nabla}\times\vec{U} = 0 . \end{cases}$$

if we set $U_z = \frac{1}{2}(U_x - iU_y)$

then $\partial_{\bar{z}} U_z = \frac{1}{4}(\partial_x + i\partial_y)(U_x - iU_y) = 0$

so $U_z(z)$ is holomorphic.



wing profile

plane minus slit where $\vec{U} = (V_0, 0)$ is a valid flow.

then $U_z(z) = \left(\frac{dw}{dz}\right)(V_0 + i0)$ is going to be a valid flow around the wing profile. (angle $\arg\left(\frac{dz}{dw}\right)$ at z_0)

4. Some important facts

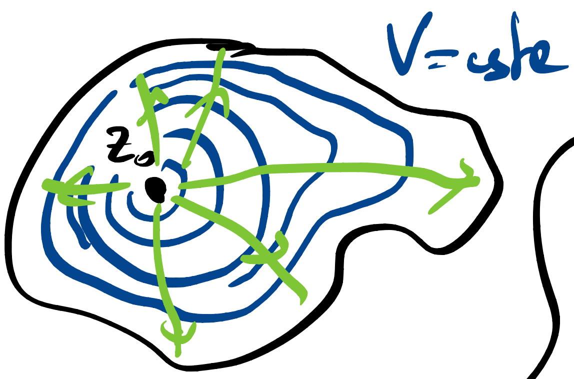
4.1. Riemann mapping theorem

Thm: Let U be an open, simply connected, subset of $\mathbb{C}$, not equal to $\mathbb{C}$ itself.

Then there exists a conformal mapping from U to the unit disc.

Sketch of proof (electrostatics):

fix $z_0 \in U$ and look at potential V created by a charge at z_0



$$\begin{cases} \Delta V = 0 & \text{for } z \neq z_0 . \\ V(z,\bar{z}) \underset{z \to z_0}{=} -\log|z-z_0|^2 & \\ V(z,\bar{z}) = 0 & \text{along } \partial U . \end{cases}$$

Harmonicity $\Rightarrow \exists \tilde{V}$ s.t. $f = V + i\tilde{V}$ is holomorphic.

Indeed, $\tilde{V}$ s.t. $\begin{pmatrix} \partial_x \tilde{V} \\ \partial_y \tilde{V} \end{pmatrix} = \begin{pmatrix} -\partial_y V \\ \partial_x V \end{pmatrix}$.

Then e^{-f} is the conformal map we are looking for sending z_0 to Δ.

Example :

$$z \longmapsto w = \frac{z-i}{z+i}$$

4.2 Special conformal transformations (Möbius)

Global conformal

Thm : all conformal maps from $\mathbb{H}$ to itself are of the form

$$z \longmapsto \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{R}, \quad ad-bc=1.$$

(argument : look at $f : \mathbb{D} \to \mathbb{D}$, use fixed point and maximum principle).

Thm bis : all conformal maps from $\mathbb{C}$ to itself are of the form

$$\boxed{z \longmapsto \frac{az+b}{cz+d}}, \quad a,b,c,d \in \mathbb{C}, \quad ad-bc=1$$

(6 free parameters vs 3 free param.)

Exercise : $z \longmapsto \xi = \dfrac{a_1 z + b_1}{c_1 z + d_1} \longmapsto w = \dfrac{a_2 \xi + b_2}{c_2 \xi + d_2}$

the composition maps $\mathbb{H}$ to $\mathbb{H}$ (or $\mathbb{C}$ to $\mathbb{C}$) so it is also a 'SCT.' Compute its parameters a,b,c,d

Conclusion of exercise: Group of SCT isomorphic to $SL(2,\mathbb{R})$ or $SL(2,\mathbb{C})$.

5.3 The cross-ratio

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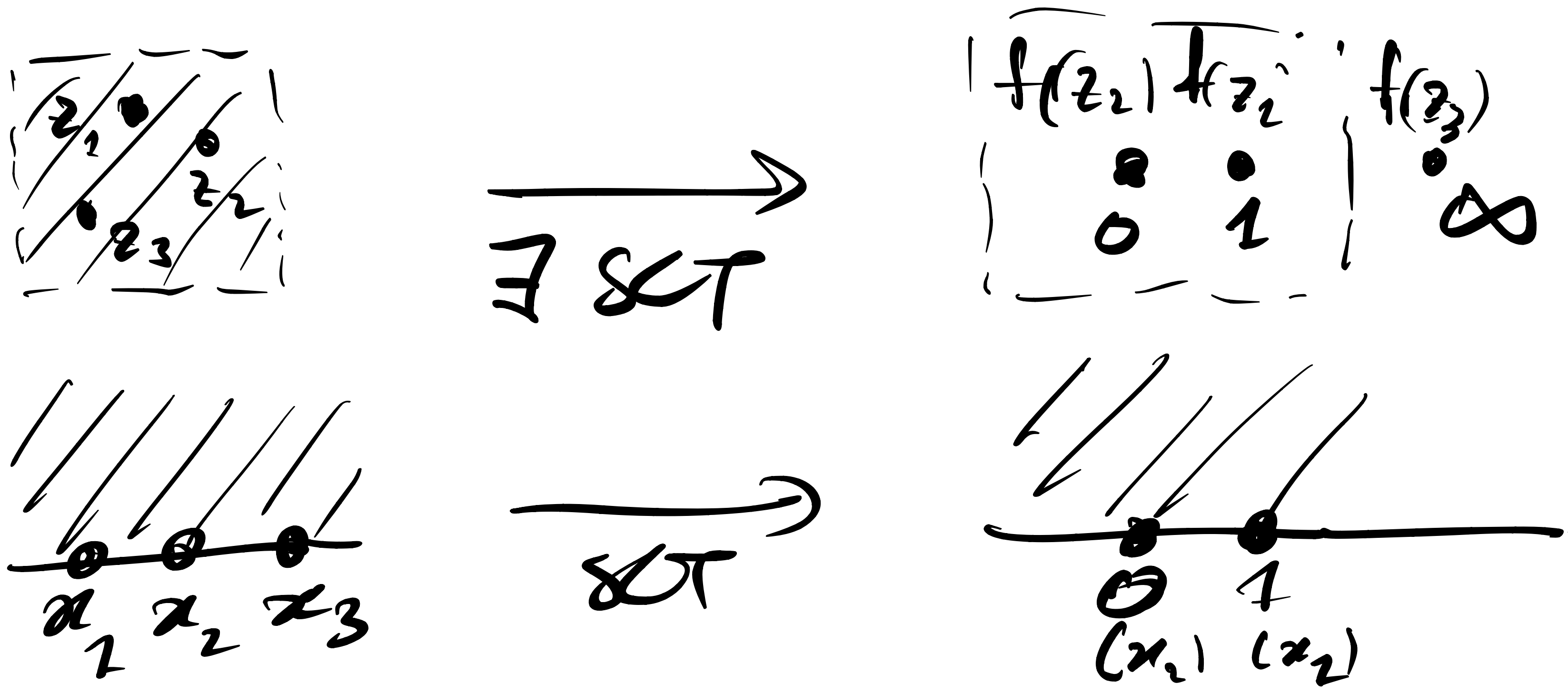


SCT can be used to fix three points

* Natural to ask: what about 4 points $z_1 z_2 z_3 z_4$?

define **cross-ratio** $\eta(z_1, z_2, z_3, z_4)$ as the position of z_4 under the SCT that sends (z_2, z_3, z_4) to $(1, 0, \infty)$:

$$\left(\text{this is } z \mapsto \frac{z-z_3}{z-z_4}\,\frac{z_2-z_4}{z_2-z_3} = f(z)\right)$$

so

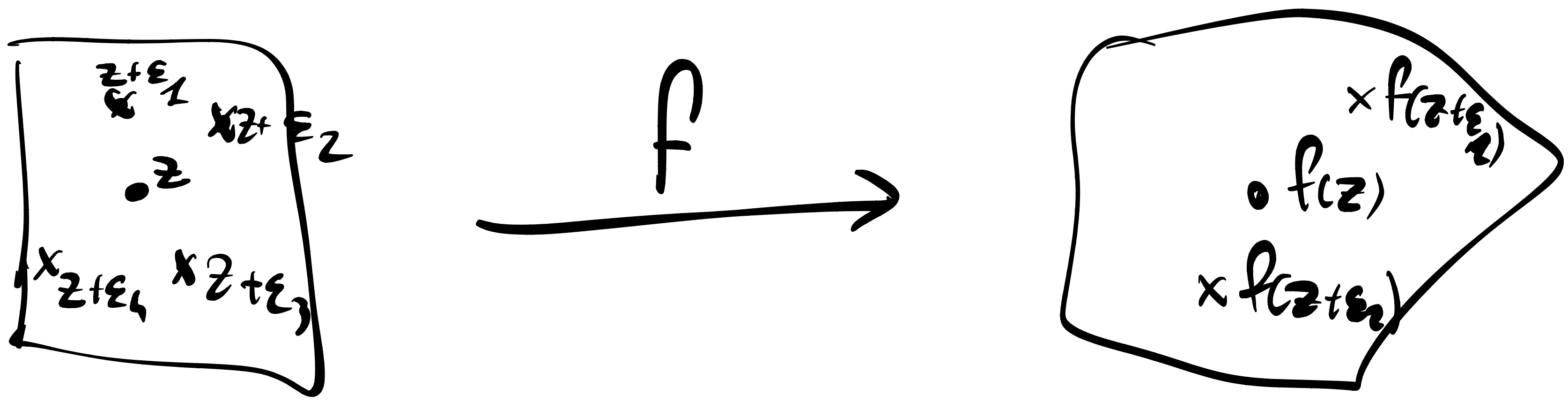
$$\boxed{\eta = f(z_4) = \frac{z_1-z_3}{z_1-z_4}\,\frac{z_2-z_4}{z_2-z_3}}.$$

By definition, η is invariant under all SCTs.

* <u>Thm</u>: cross-ratio is <u>real</u> iff z_1, z_2, z_3, z_4 are on a circle (or a line, wich is a circle with ∞ radius).

<u>Corollary</u>: SCTs map circles to circles.

* It is instructive to ask what happens to the cross-ratio under a conformal map f that is not SCT: Local distorsion:



$$\eta(f(z+\varepsilon_1), f(z+\varepsilon_2), f(z+\varepsilon_3), f(z+\varepsilon_4))$$
$$= \eta(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)\left[1 + (\varepsilon_1-\varepsilon_2)(\varepsilon_3-\varepsilon_4)\frac{1}{6}\{f,z\} + \dots\right]$$

where $$\boxed{\{f,z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2}$$

<u>Schwarzian derivative</u>.

5. Remark: conformal vs. Weyl transformations

Conformal and Weyl transformations are not the same thing, although they are related.

Conformal maps $\quad \overset{(z)}{x^a} \to \overset{(w)}{y^a}(x^a)$.

(change of coordinates).

Weyl transformation:

$$g_{ab}(x) \to (g')_{ab} = e^{2\sigma(x)} g_{ab}(x)$$

Not the same!

However: under a conformal transformation, metric changes as

$$ds^2 = dz\,d\bar{z} \qquad (z = x^1 + i x^2)$$
$$= \left(\frac{dz}{dw}\right)\left(\frac{d\bar{z}}{d\bar{w}}\right) dw\, d\bar{w}$$

so $\quad g_{ab}(y) = \lambda^2(x) g_{ab}(x)$

with $\lambda^2(x) = \dfrac{dz}{dw}\dfrac{d\bar{z}}{d\bar{w}}$.

So we may remove the $\lambda^2(x)$ by a Weyl transformation.

note : in isothermal coordinates

$$ds^2 = \lambda^2 \, dz \, d\bar{z}$$

Gauss curvature is

$$\boxed{K = -\frac{1}{\lambda^2} \Delta \log \lambda}$$

example : sphere $\lambda = \dfrac{2r}{1+z\bar{z}}$ radius of sphere

$$K = -\frac{(1+z\bar{z})^2}{4r^2} \, 4 \, \partial_z \partial_{\bar{z}} \left(-\log(1+z\bar{z})\right)$$

$$= \frac{1}{r^2}$$

so a conformal map given by complex analytic function never modifies curvature. But Weyl transformation does modify it.