Lecture III

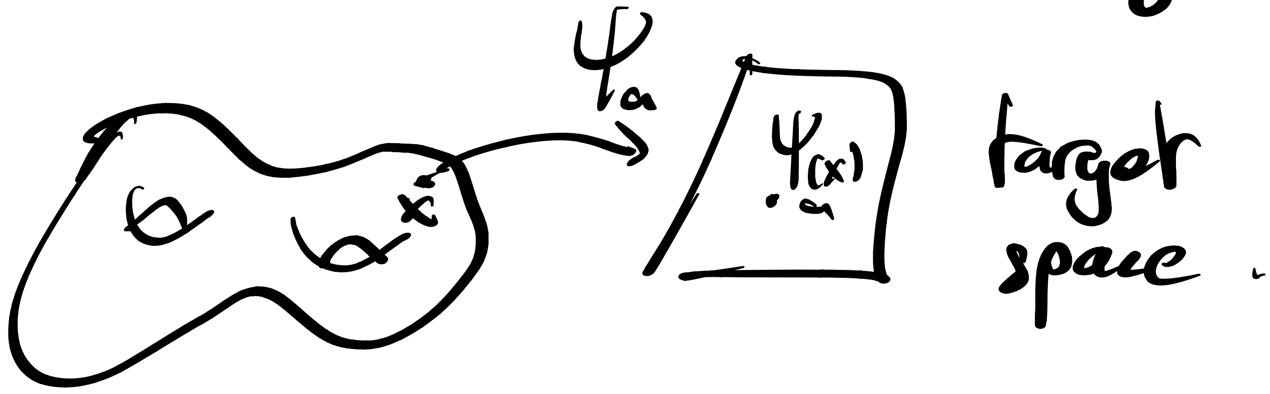
Glimpse of general 2D CFT

Stress tensor and central charge

Glimpse of general 2D CFT:
Stress tensor and central charge

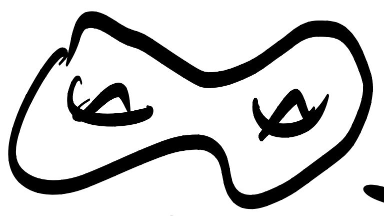
1. Motivation

What we have in mind: a surface (M, g), with some degrees of freedom, or fields, $\varphi_a(x)$, $x \in M$ and a local action $S[\varphi] = \int F(\varphi_a, \partial\varphi_a, \dots) d^2x$



if we integrate out φ_a, then we get

$$Z[\,\cdot\,] = \int [d\varphi_a] \exp(-S[\varphi_a])$$

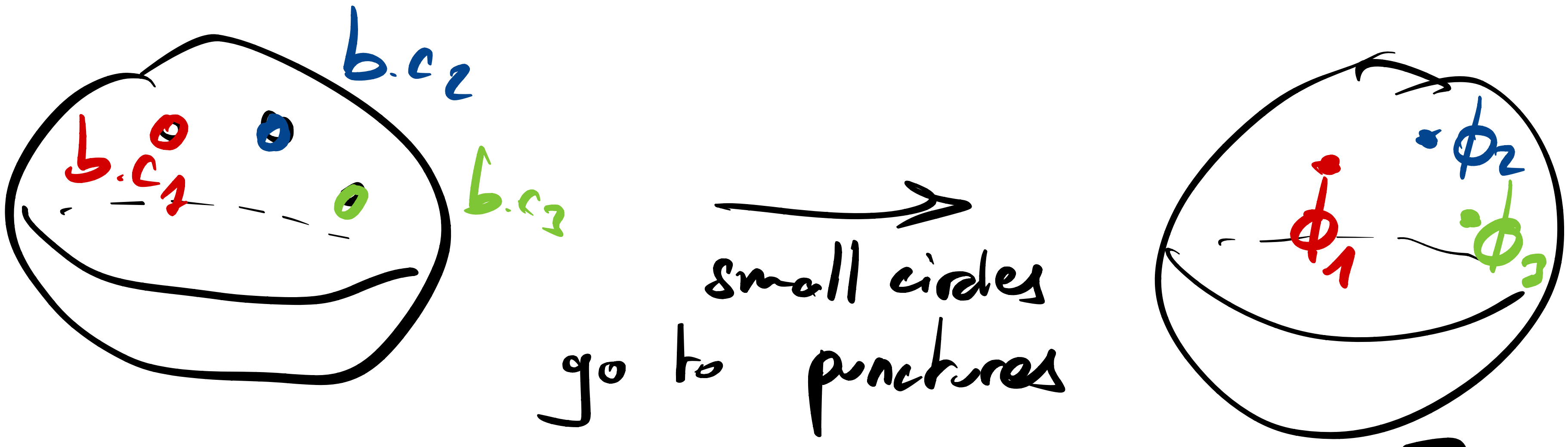


depends on (M, g)

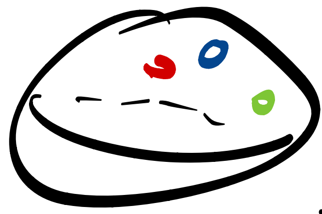
also possible: boundaries with b.c for φ

$$Z[\,\cdot\,] = \int [d\varphi_a]\, e(-S[\varphi_a])$$

with prescribed b.c.



Then $\langle \Phi_1 \Phi_2 \Phi_3 \rangle = \dfrac{Z[\;\;]}{Z[\;\;]}$.



correlation function

In summary: a CFT is a recipe to define partition functions (and correlation functions) that satisfy some consistency rules (inspired from concrete examples, e.g. lattice models)

Forget "target spaces" and explicit computations of path integrals.

We will explore basic ingredients of CFT in this lecture.

2. The stress-tensor

Local operator whose 1-point function determines variation of $Z[g]$ with metric g:

$$\boxed{-\log \frac{Z[M, g+\delta g]}{Z[M, g]} = \frac{1}{4\pi}\int_M \langle T^{ab}(x)\rangle_g \,\delta g_{ab}(x)\sqrt{g}\,d^2x}$$

relates the variation of partition function to a correlation function.

In conformal coordinates,

$$ds^2 = \lambda^2(dx^2+dy^2)$$
$$= \lambda^2 dz\, d\bar z, \qquad z = x+iy$$

the components are

$$\begin{cases} T^{zz} = T^{xx} - T^{yy} + i2T^{xy} \\ T^{\bar z\bar z} = T^{xx} - T^{yy} - i2T^{xy} \\ T^{z\bar z} = T^{xx} + T^{yy} \end{cases}$$

scale invariance implies $\boxed{T^{z\bar z} = 0}$ (assuming zero curvature)

while $\partial_a T^{ab} = 0$ (Noether's thm)

so $\partial_z T^{zz} = \partial_{\bar z} T^{\bar z\bar z} = 0$.

So, in a 2d CFT, we have a stress-tensor with components that satisfy

$$\boxed{\begin{array}{l} T(z) \ \left(= T_{zz} = \frac{1}{4} T^{\bar{z}\bar{z}}\right) \text{ holomorphic} \\ \bar{T}(\bar{z}) \ \left(= T_{\bar{z}\bar{z}} = \frac{1}{4} T^{zz}\right) \text{ anti-holomorphic} \end{array}}$$

behavior under conformal map

$$z \longmapsto w(z)$$

$$\boxed{T(z) = \left(\frac{dw}{dz}\right)^2 T(w) + \frac{c}{12}\{w, z\}}$$

This is a generalisation of what we had found for the GFF, where we had $c = 1$.

$$\boxed{c \text{ is called the "central charge".}}$$

Moreover, in the plane $\mathbb{C}$ or in $\mathbb{H}$,

$$\langle T(w) \rangle_{\mathbb{C} \text{ or } \mathbb{H}} = 0$$

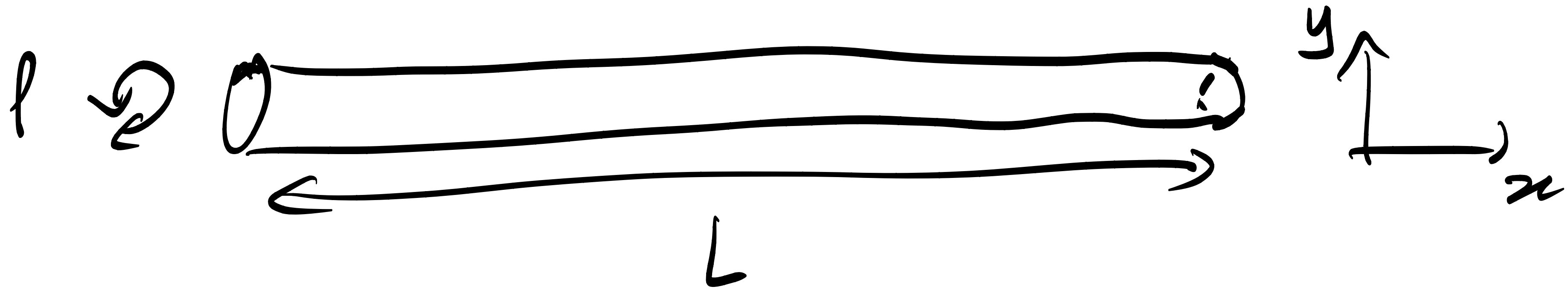
Therefore, in a domain U,

$$\boxed{\langle T(z) \rangle = \frac{c}{12}\{w, z\}}$$

Now let's explore the meaning of this in some examples.

3. The central charge in partition functions

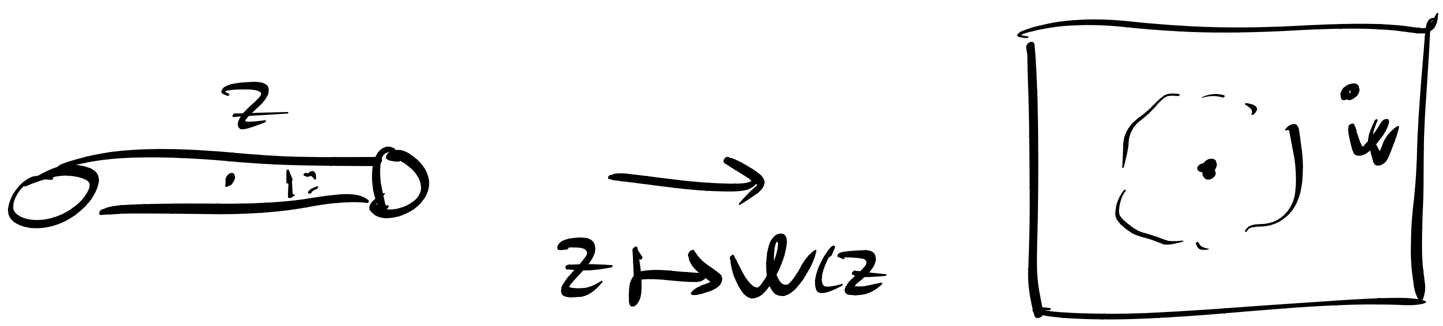
3.1 Central charge of long cylinders



claim: for $L \gg \ell$,

$$\boxed{Z \simeq \exp\left(\frac{\pi c}{6}\frac{L}{\ell}\right)}$$

This result follows from



$$w = e^{\frac{2\pi}{\ell} z}$$

The Schwarzian of this map is

$$\{w, z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2 = -\frac{1}{2}\left(\frac{2\pi}{\ell}\right)^2$$

$$\Rightarrow \boxed{\langle T(z)\rangle_{cyl} = \frac{c}{12}\{w,z\} = -\frac{\pi^2 c}{6\ell^2}}$$

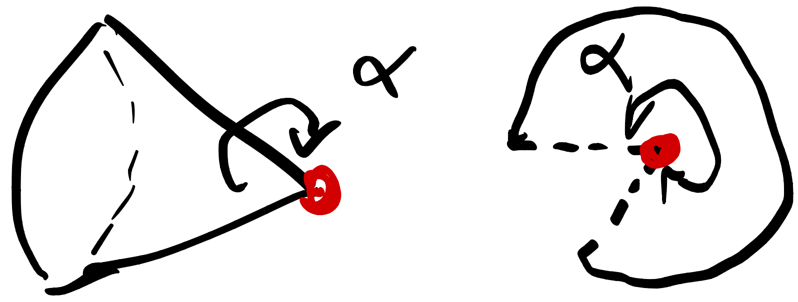
Then we can look at $ds^2 = dx^2 + \left(\frac{\ell+\delta\ell}{\ell}\right)^2 dy^2$

$$-\log\frac{Z(L,\ell+\delta\ell)}{Z(L,\ell)} = \frac{1}{4\pi}\int T^{ab}\,\delta g_{ab}\, d^2x$$

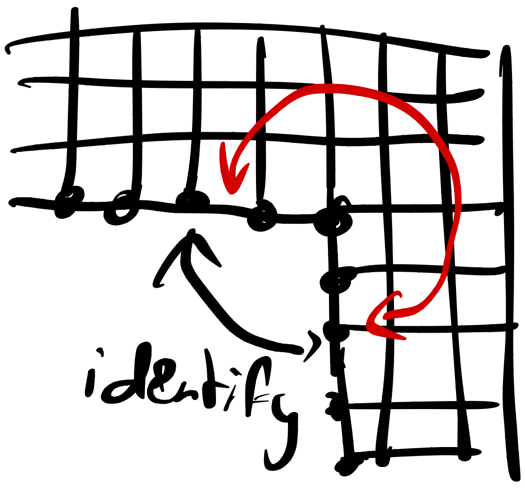
$$= \frac{1}{4\pi}\int(-T-\bar{T})\frac{2\delta\ell}{\ell}\, d^2x = \frac{\pi c}{6\ell^2}L\,\delta\ell$$

so integrating $\delta\log Z/\delta\ell$ we get the above result.

3.2 Curvature at a point: conical singularities

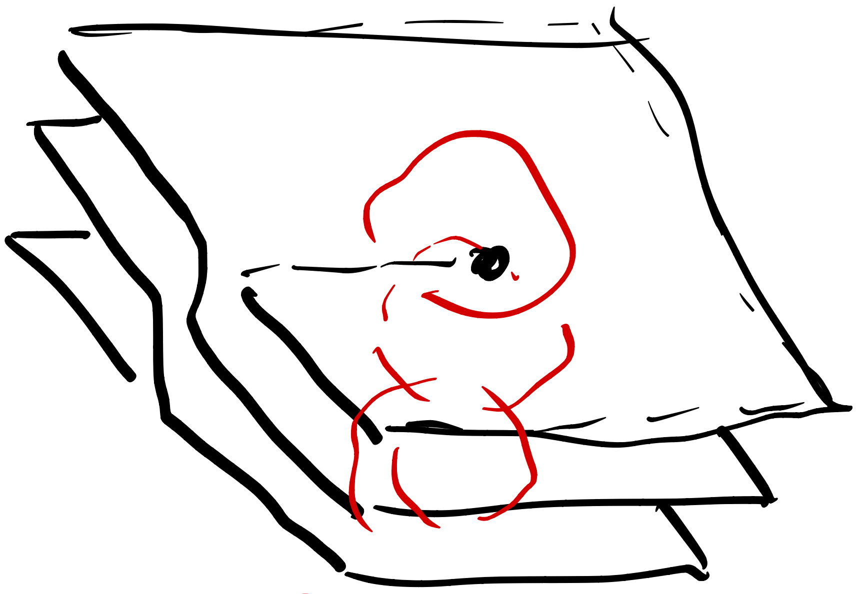


think for instance of a stat-mech model on a lattice:



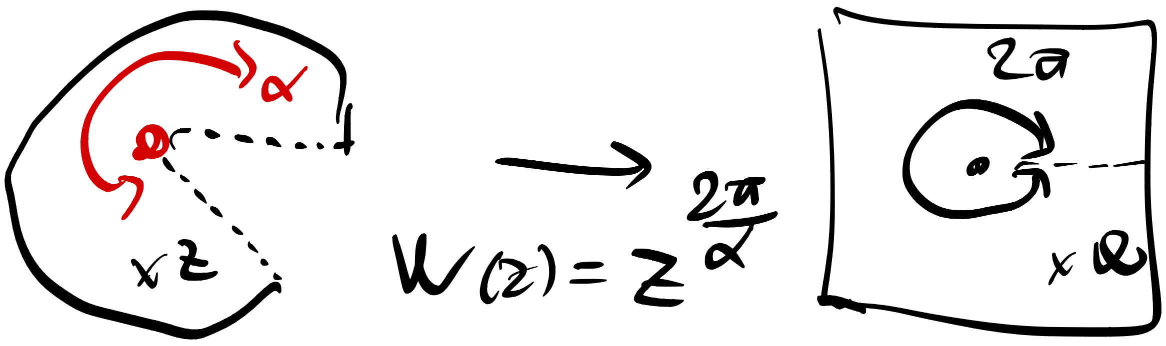
$\alpha = \frac{3\pi}{2}$

α can be larger than 2π:



$\alpha = 3 \times 2\pi$.

We can compute $\langle T(z) \rangle_\alpha$ using the conformal map:

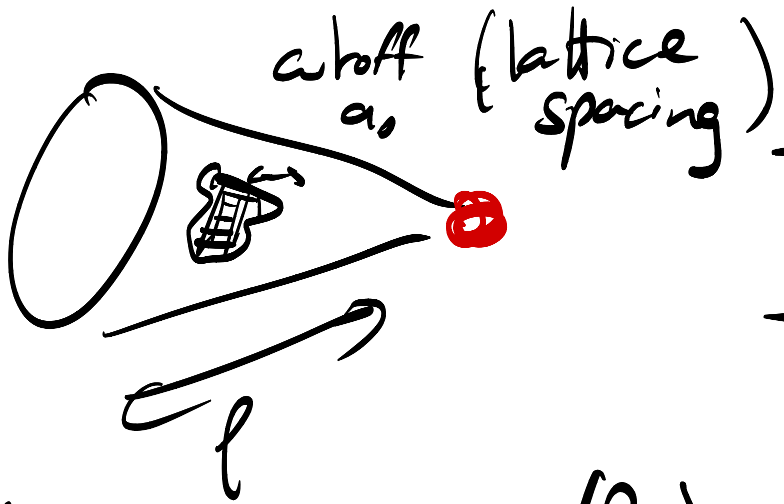


$$w(z) = z^{\frac{2\pi}{\alpha}}$$

The Schwarzian of this map gives

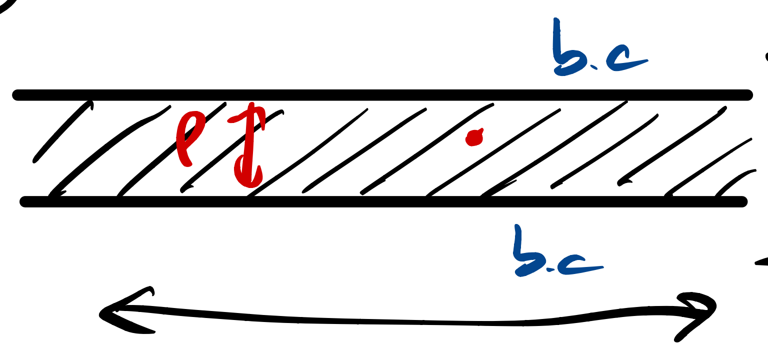
$$\langle T(z) \rangle_{\text{cone}} = \frac{c}{12} \{ w, z \} = \frac{c}{12} \left(1 - \left(\frac{2\pi}{\alpha} \right)^2 \right) \frac{1}{2z^2}$$

using similar trick as for the cylinder this leads to

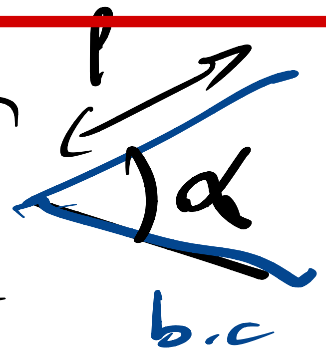
$$-\log Z\Big[$$


$$\Big] = \frac{c}{12}\left(\frac{\alpha}{2\pi} - \frac{2\pi}{\alpha}\right)\log\left(\frac{\ell}{a_0}\right).$$

3.3 Analogs with boundaries

$$Z\Big[$$


$$\Big] \simeq \exp\left(\frac{\pi c}{24}\frac{L}{\ell}\right)$$

$$Z\Big[$$


$$\Big] \simeq \frac{c}{24}\left(\frac{\alpha}{\pi} - \frac{\pi}{\alpha}\right)\log\left(\frac{\ell}{a_0}\right).$$

3.4 Applications

* Hearing the shape of a drum

a drum is a membrane that vibrates

membrane attached to frame

$\rightarrow$ vibration modes correspond to spectrum of Laplacian.

so the question (rephrased mathematically) is: can the spectrum of the Laplacian on 2 different polygonal domain be the same?

How is this related to our subject?

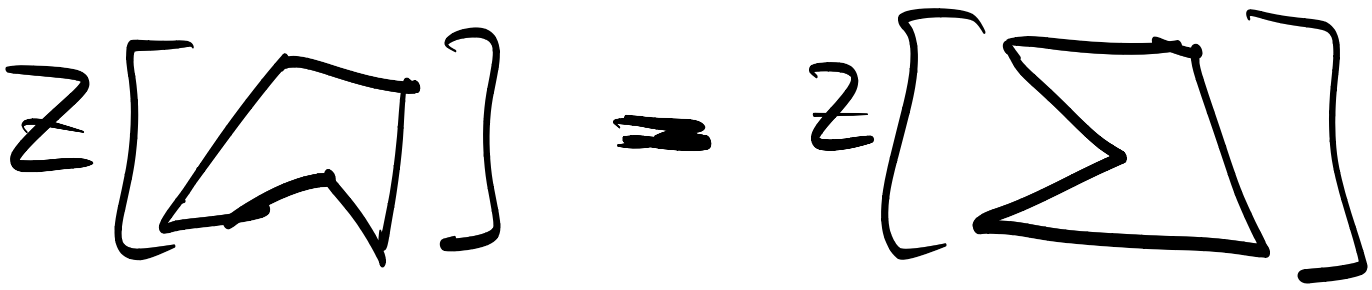
If one computes the partition of the GFF

$$Z = \int e^{\frac{1}{8\pi}\int h \Delta h \, d^2x} [dh]$$

$$= \det(-\Delta)^{-1/2} = \left(\prod_k \lambda_k\right)^{-1/2}$$

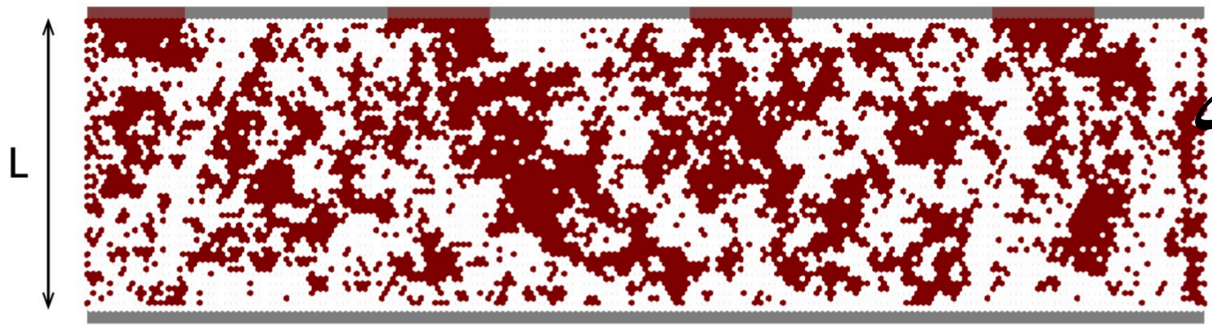
product of eigenvalues of Laplacian.

so if the 2 domains have the same spectrum, they must have the same partition function, therefore



$$\sum_i \frac{1}{24}\left(\frac{\alpha_i}{\pi} - \frac{\pi}{\alpha_i}\right) = \sum_j \frac{1}{24}\left(\frac{\beta_j}{\pi} - \frac{\pi}{\beta_j}\right)$$

* Casimir energy (attractive force between plates)



critical medium, e.g. 2 fluids at miscible transition

attractive force between plates.

$$\boxed{\text{force} = \frac{d}{d\ell}(-\log Z) = \frac{\pi c}{24}\frac{L}{\ell^2}}$$

* Heat capacity of 1D quantum critical system

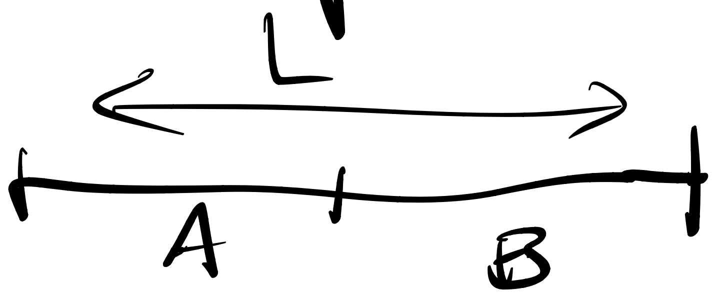
consider gapless quantum system, excitations moving at velocity $\pm v$

low temperature T. Then $\boxed{\frac{C}{L} = \frac{1}{L}\frac{dU}{dT} = \frac{\pi c}{3}\frac{T}{\hbar v}}$

this is because the internal energy is

$$U = \langle H \rangle_T$$
$$= T^2 \partial_T \log\left(\mathrm{tr}\exp\left(-\frac{H}{T}\right)\right)$$
$$= T^2 \partial_T \log Z[\text{cylinder of circumference } \hbar v/T \text{, length } L]$$
$$= T^2 \partial_T \log \exp \frac{\pi c}{6}\frac{L}{\hbar v/T}.$$

* Entanglement entropy of 1D quantum critical point

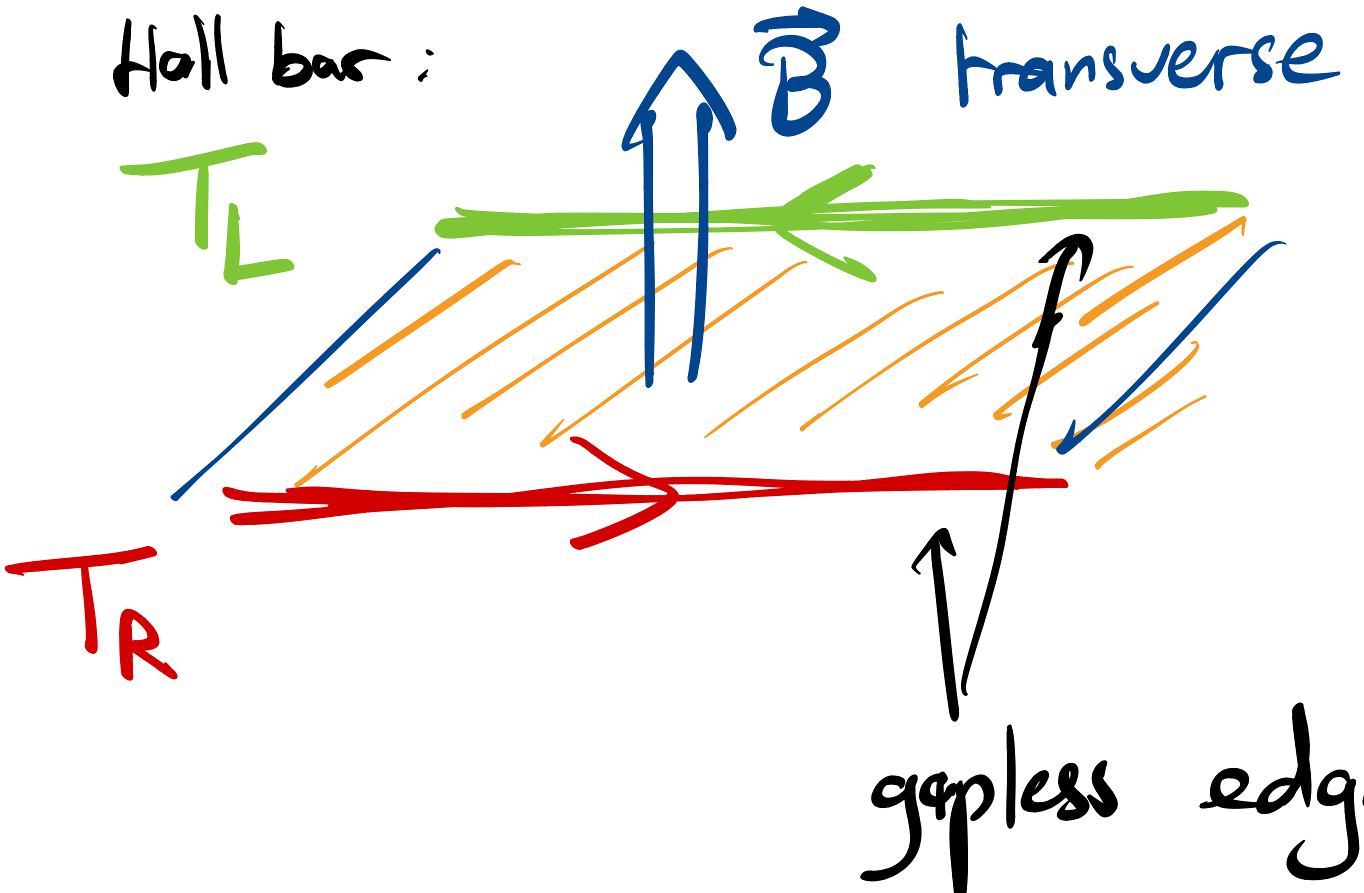


$$S_A = \frac{c}{6} \log L$$

follows from Renyi entropy $S_A^{(n)} = \frac{1}{1-n} \log L^{\frac{c}{12}(n-\frac{1}{n})}$

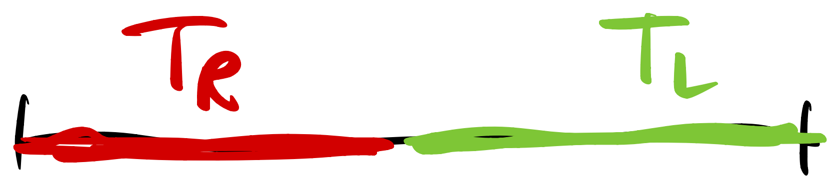
* Thermal Hall effect

Hall bar :



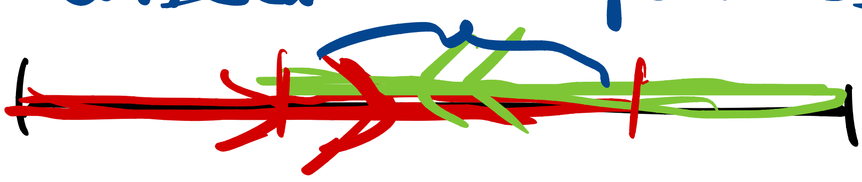
energy current : $\boxed{J_E = \frac{\pi c}{12}(T_R^2 - T_L^2)}$

* "Bipartite quench" : energy current



1D critical system) time evolution

mixture of L,R excitations at different temperatures



energy current :

$$\boxed{J_E = \frac{\pi c}{12}(T_R^2 - T_L^2)}$$

$\boxed{\text{Conclusion: central charge is a central quantity in applications.}}$

(Almost) every universal result about a physical system that can be expressed in terms of central charge only can be traced backed to:

$$T(z) = \left(\frac{dw}{dz}\right)^2 T(w) + \frac{c}{12}\{w,z\}.$$

Exercise (central charge of 1D Fermi gas)

Consider N non-interacting fermions on a ring of length ℓ. Each of them is in an eigenstate of the Schrödinger eq.

$$-\frac{\hbar^2}{2m}\partial_x^2 \psi_n = E_n \psi_n, \qquad \psi_n(x+\ell) = \psi_n(x).$$

1. Assume N is odd. What are the single-particle eigenstates that are filled?
   Express the ground state energy of the gas of N fermions in terms of the single-particle energies.

2. What is the energy of the first excited state? And its momentum? Assuming that the velocity v of gapless excitations is given by

$v = (E_{1st\ exc.} - E_0)/P_{1st\ exc.}$, show that $v = \frac{\hbar}{m}\pi n$ where $n = \frac{N}{\ell}$ is the particle density.

3. Consider the low-T limit of the partition function

$$Z = \mathrm{tr}\, \exp\left(-\frac{H}{T}\right) \underset{T\to 0}{\simeq} \exp\left(-\frac{E_0}{T}\right).$$

Using the Euler-McLaurin formula, evaluate Z for $T \to 0$, $\ell \to \infty$, $N \to \infty$, keeping the density $n = N/\ell$ fixed. Show that the result is of the form

$$Z = \exp(A\ell L) \times \exp\left(\frac{\pi c}{6}\frac{L}{\ell}\right)$$

with $L = \frac{\hbar v}{T}$.

The first term is an extensive (non-universal) contribution to the partition function, while the second term is the universal part predicted by CFT.

Question: what is the value of c?