

Thermal axion production in the Early Universe

Giovanni Villadoro



2211.03799 with A. Notari and F. Rompineve

Axion thermal ~~Thermal axion~~ production in the Early Universe

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The (Minimal) QCD Axion

$$\mathcal{L}_{SM}^{\theta=0} + \frac{1}{2}(\partial_\mu a)^2 + \frac{a}{f_a} \frac{\alpha_s}{8\pi} G\tilde{G} + \dots$$

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- contribute to part (or all) of Ω_{dm}

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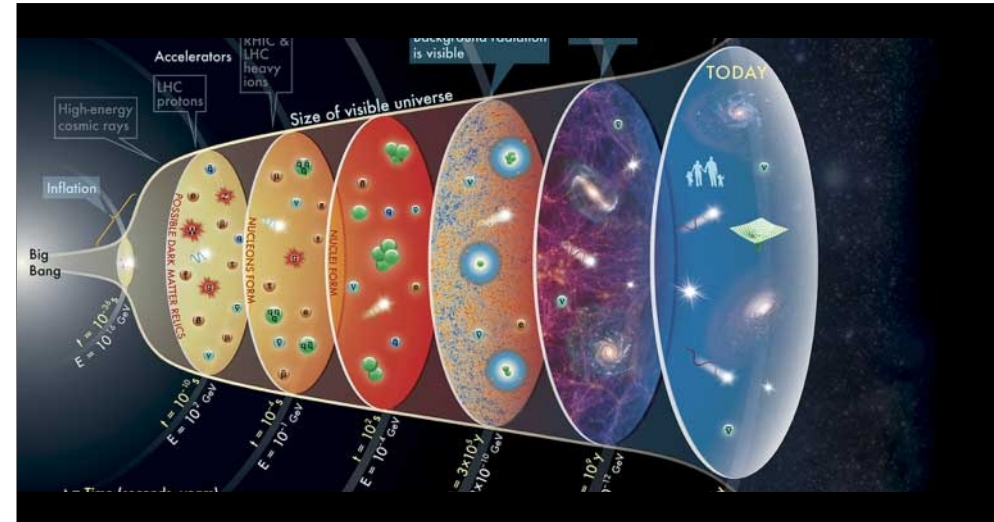
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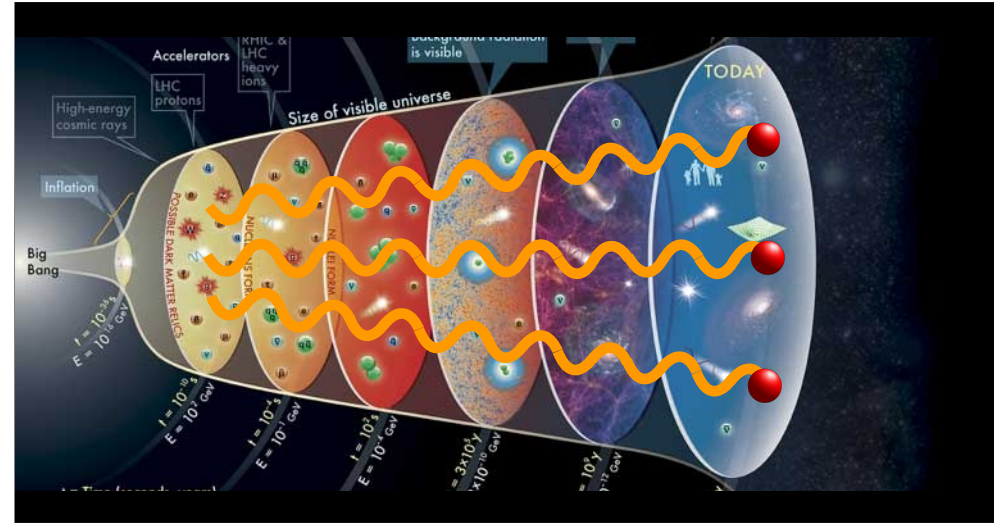
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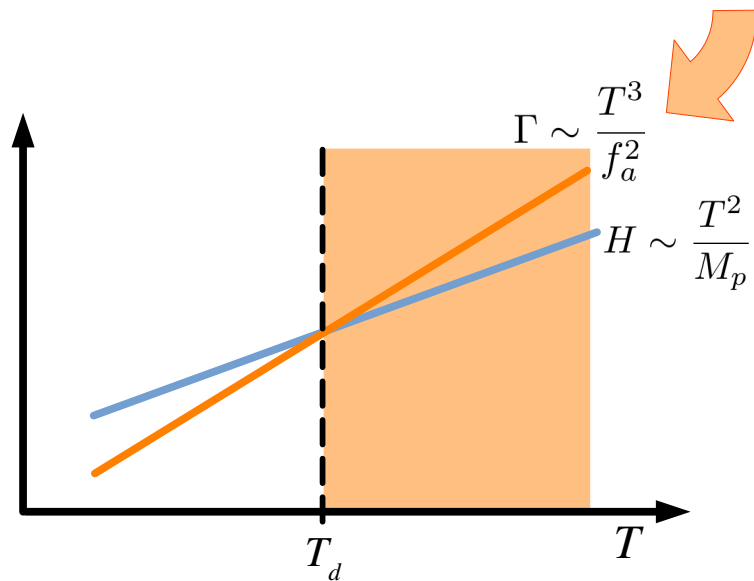
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$$\Gamma \sim \frac{T^3}{f_a^2}$$



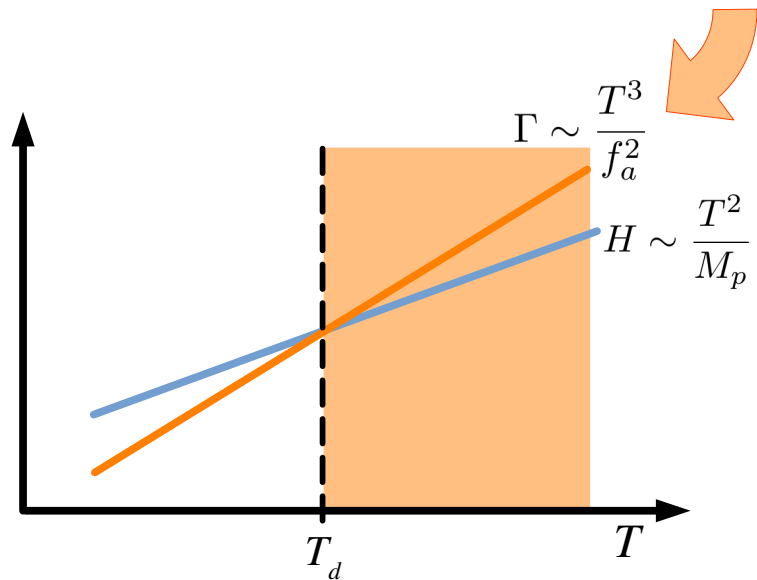
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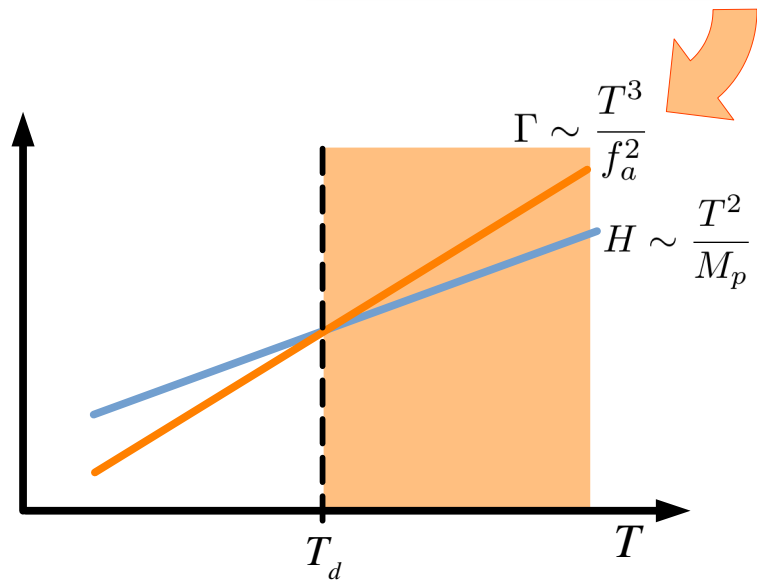
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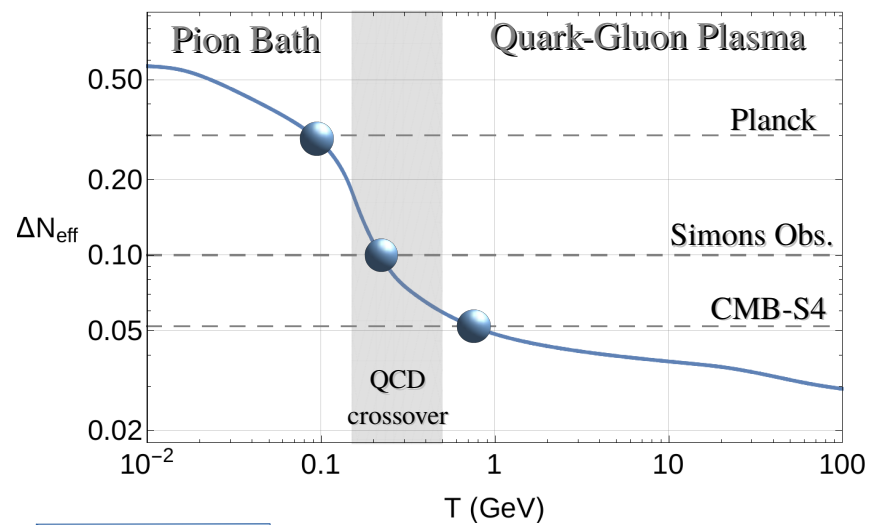
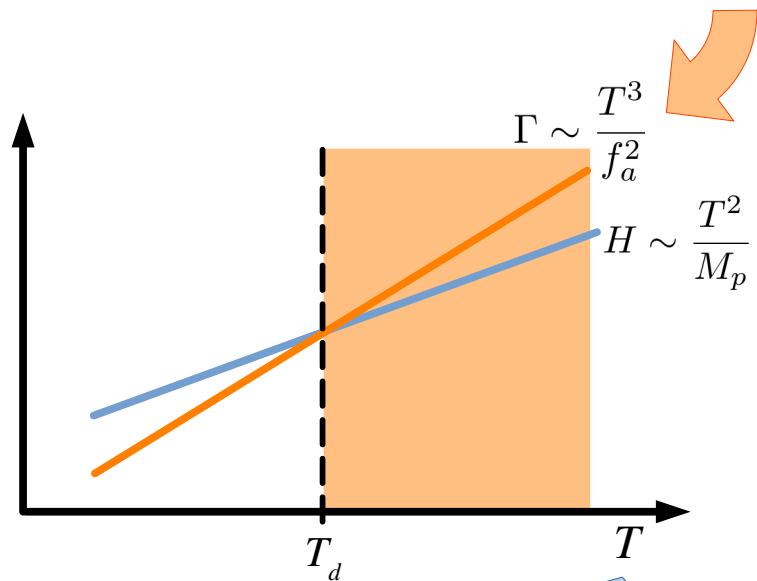


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$$\Delta N_{\text{eff}} \equiv N_{\text{eff}} - N_\nu = \frac{8}{7} \left(\frac{11}{4} \right)^{\frac{4}{3}} \left(\frac{\rho_a}{\rho_\gamma} \right)_{\text{CMB}} \simeq 0.027 [106.75/g_{*,S}(T_d)]^{4/3}$$

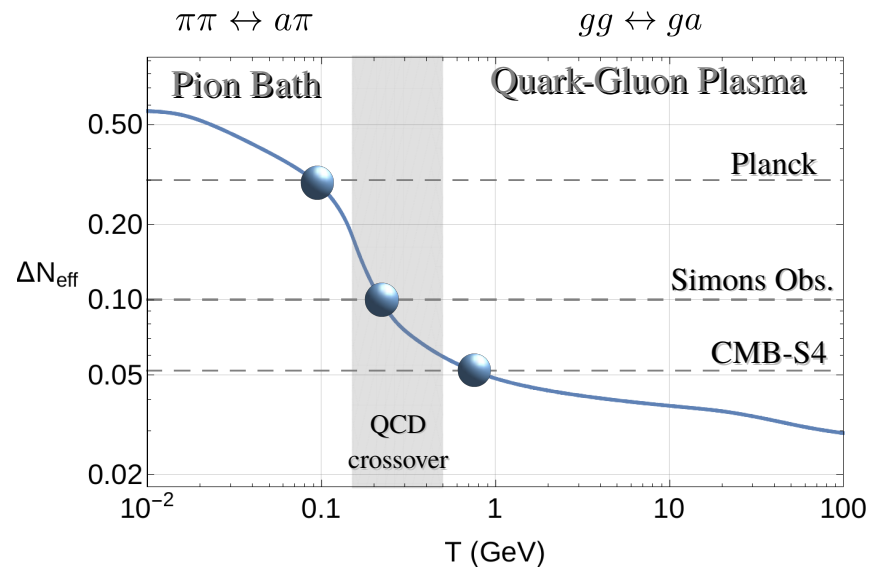
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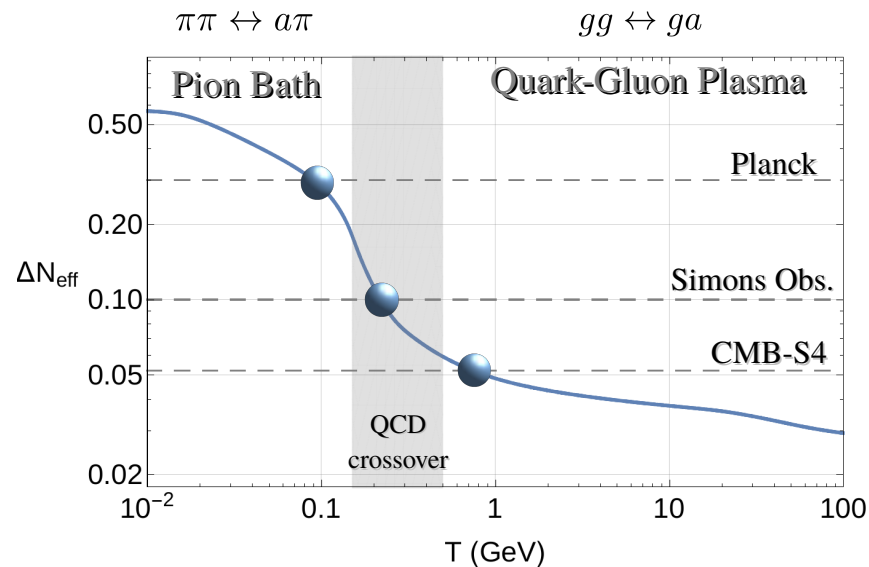
Axion ΔN_{eff} has long history:



Arias-Aragon, Baumann, Bernal,
Berezhiani, Chang, Choi, D'Eramo, Di
Luzio, Di Valentino, Dunskey, Ferreira,
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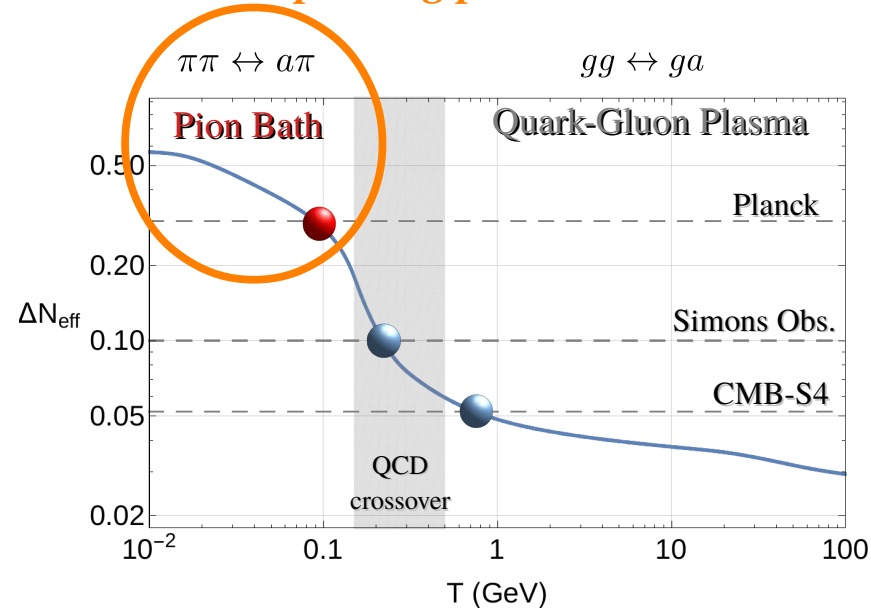


Boltzmann Eq.

$$\frac{dY}{d \log x} = (Y^{\text{eq}} - Y) \frac{\bar{\Gamma}}{H} \left(1 - \frac{1}{3} \frac{d \log g_{*,S}}{d \log x} \right)$$

Axion ΔN_{eff} has long history:

Improving present bounds



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Boltzmann Equation and Thermalization Rate Γ

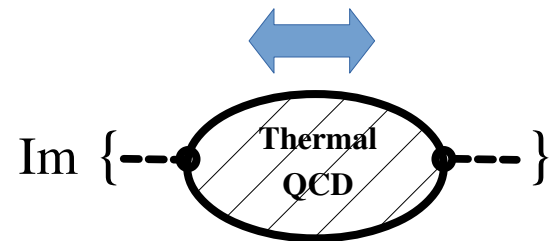
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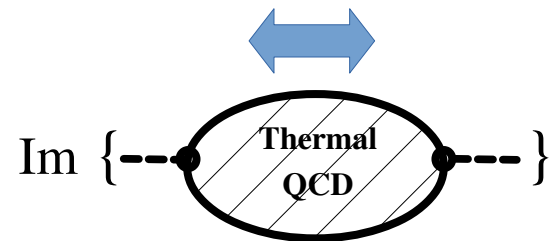


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$$\Gamma^{<} = \frac{1}{2E} \int \left(\prod_{i=1}^3 \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2E_i} \right) f_1^{\text{eq}} f_2^{\text{eq}} (1 + f_3^{\text{eq}}) (2\pi)^4 \delta^{(4)}(k_1^\mu + k_2^\mu - k_3^\mu - k^\mu) |\mathcal{M}|_{2 \leftrightarrow 2}^2$$

1. The Thermalization Rate Γ

$$\pi\pi \leftrightarrow a\pi$$

LO χ PT rate
(Chang Choi '93)

$$|\mathcal{M}^{\text{LO}}|^2 = \theta_{a\pi}^2 \frac{s^2 + t^2 + u^2 - 3m_\pi^4}{f_\pi^4}$$

$$\theta_{a\pi} = \frac{f_\pi}{2f_a} \left(\frac{m_d - m_u}{m_d + m_u} + c_u - c_d \right)$$

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$\rightarrow$ breaks down at $T \sim 60 \text{ MeV}$

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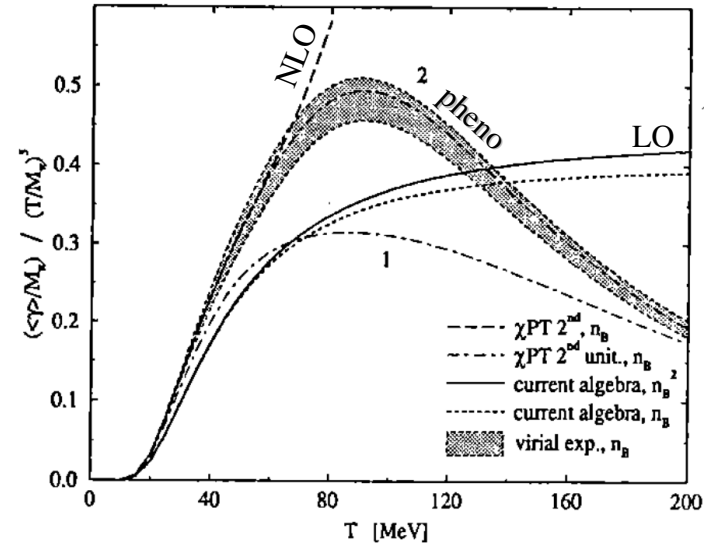
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Schenk '94

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General form of low energy axion QCD Lagrangian:

$$\mathcal{L} = \bar{q} \left(i \not{\partial} + \frac{c_0}{2f_a} \not{\partial} a \gamma_5 \right) q - \bar{q}_L M_a q_R + h.c., \quad M_a \equiv \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} e^{i \frac{a}{2f_a} (1 + c_3 \sigma^3)}$$

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@ all orders in χPT

$$\mathcal{M}_{a\pi^i \rightarrow \pi^j \pi^k} = \theta_{a\pi} \cdot \mathcal{M}_{\pi^0 \pi^i \rightarrow \pi^j \pi^k} + \mathcal{O}\left(\frac{m_\pi^2}{s}\right)$$

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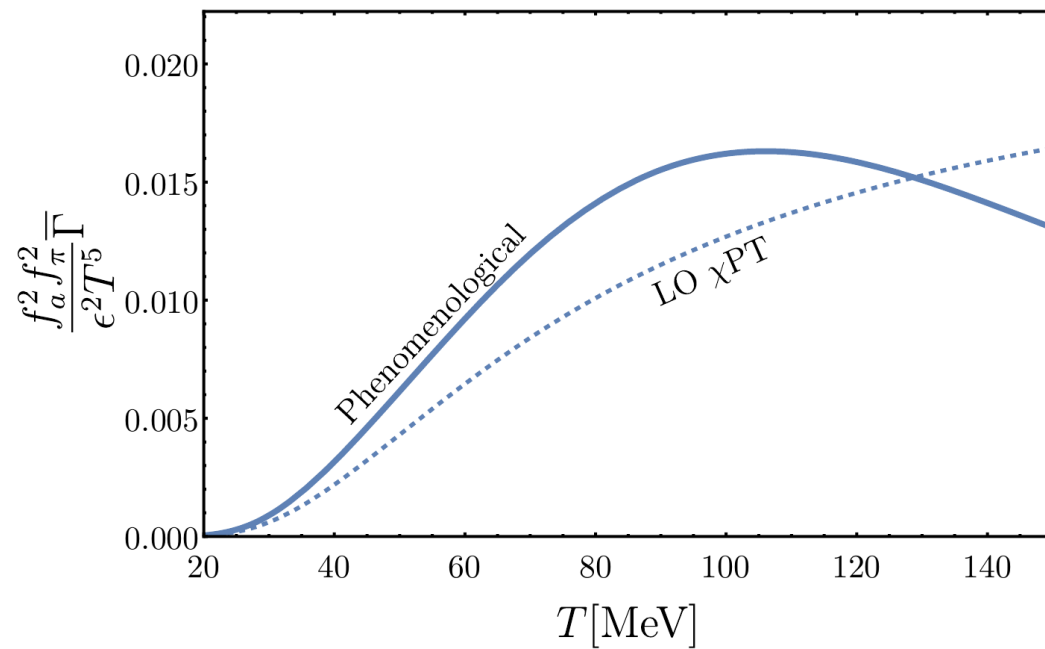
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$\lesssim 10\%$

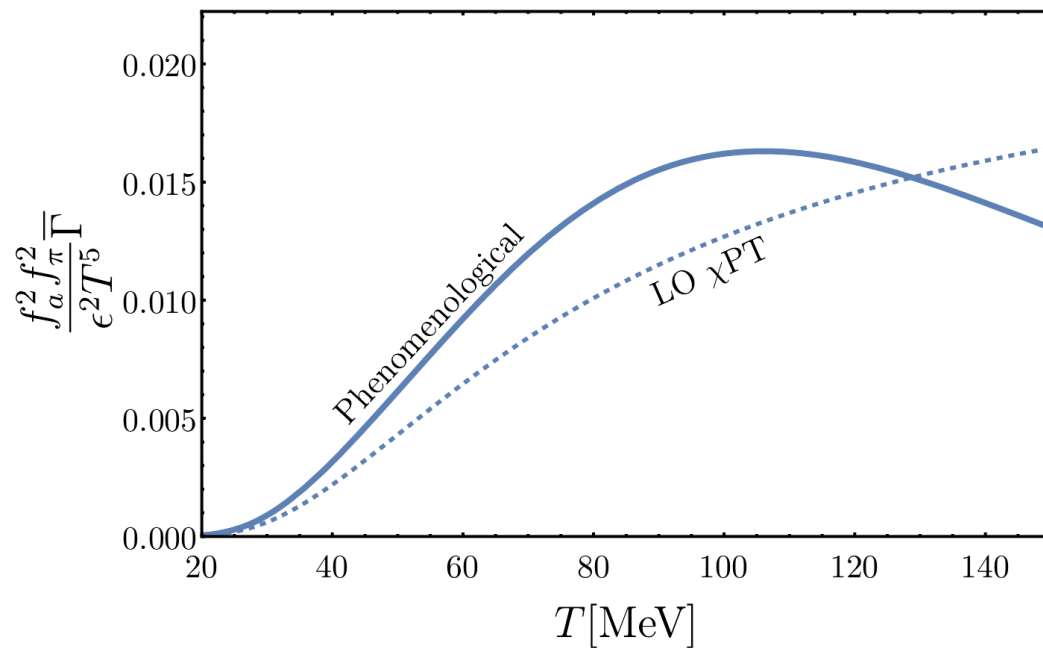
e.g. @ LO

$$|\mathcal{M}^{\text{LO}}|^2 = \theta_{a\pi}^2 \frac{s^2 + t^2 + u^2 - 3m_\pi^4}{f_\pi^4} \quad |\mathcal{M}_{\pi\pi}^{\text{LO}}|^2 = \frac{s^2 + t^2 + u^2 - 4m_\pi^4}{f_\pi^4}$$

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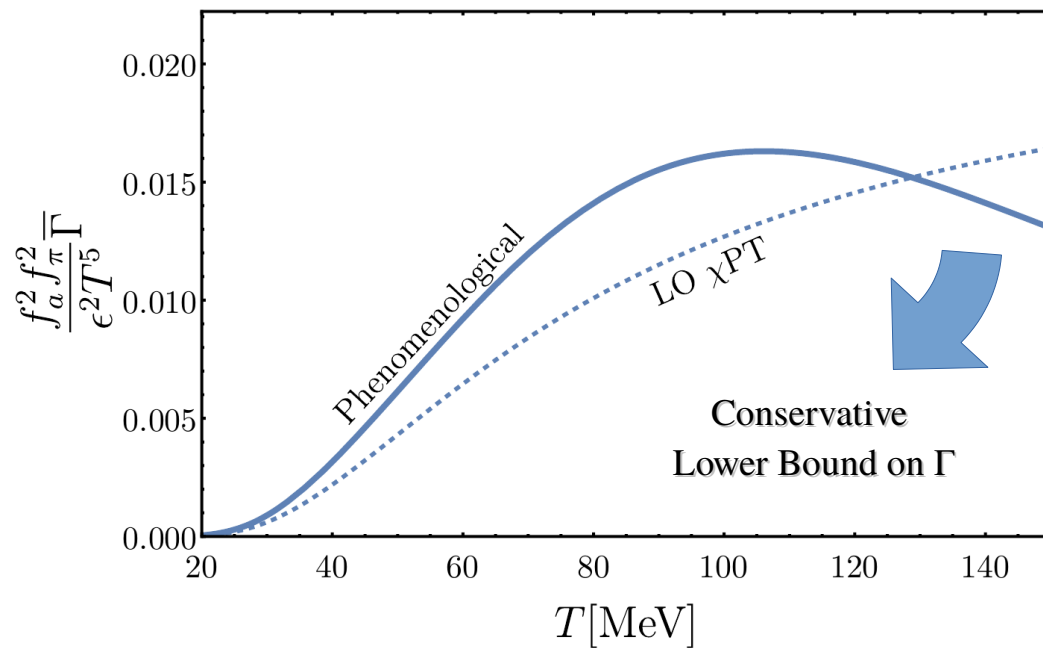


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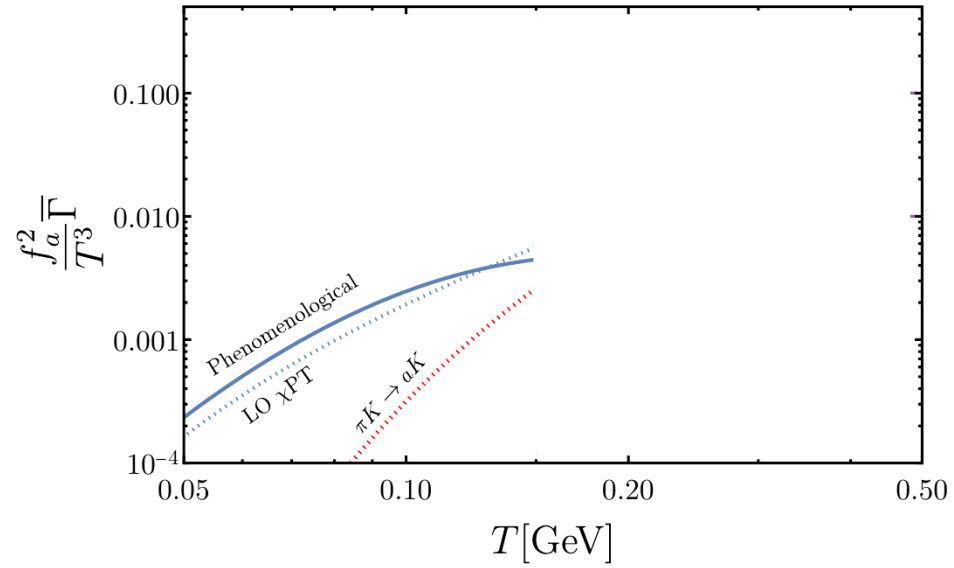
In reasonable agreement with:
Di Luzio, Camalich, Martinelli,
Oller, Piazza '22
(using NLO+unitarization)

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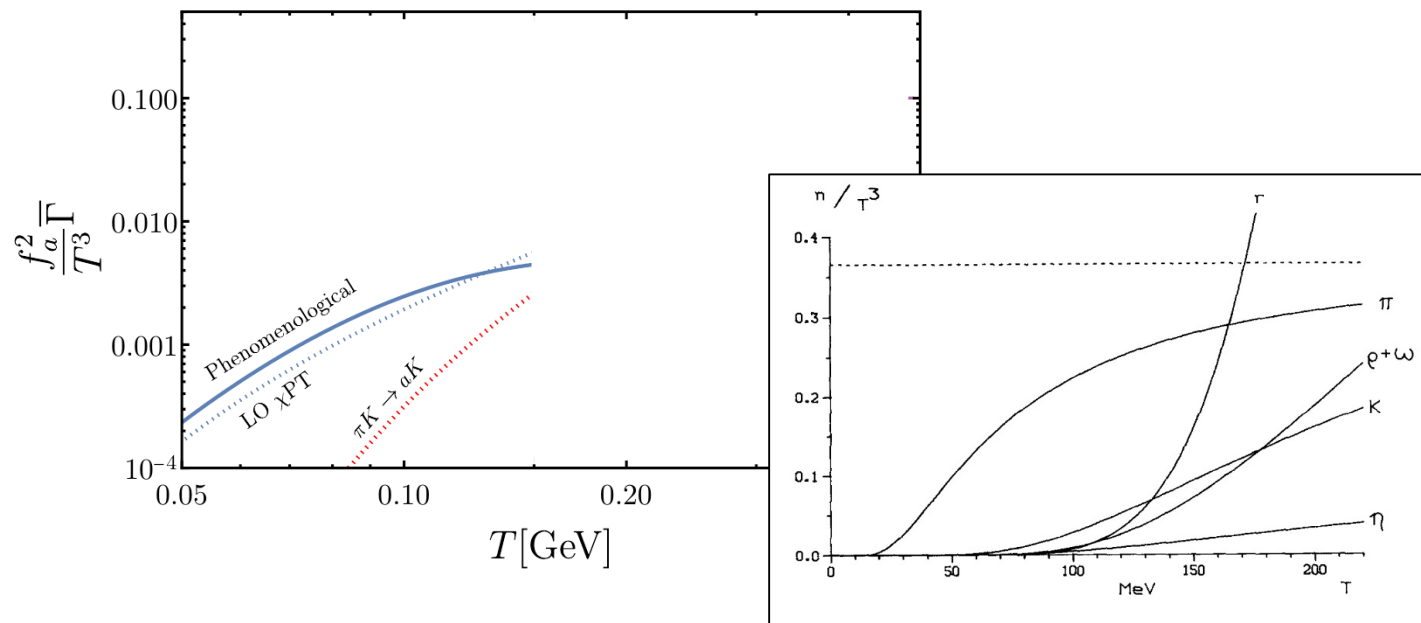


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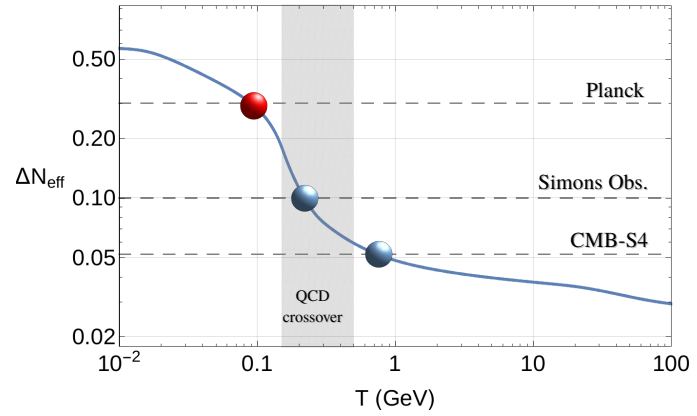


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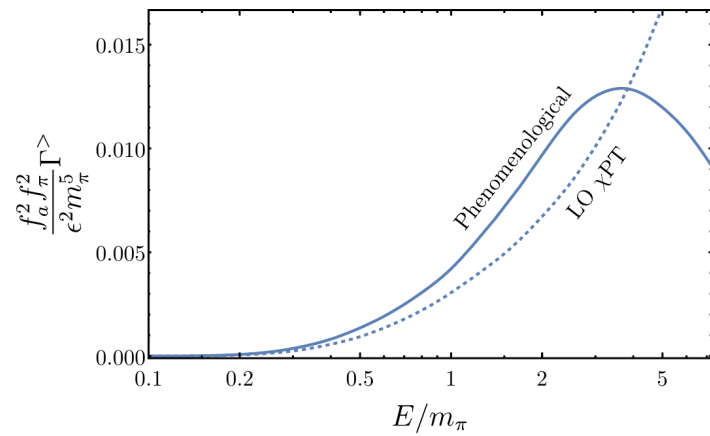
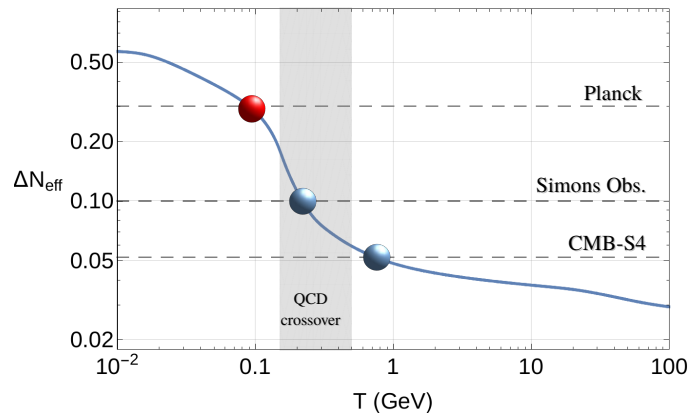


Gerber Leutwyler '89

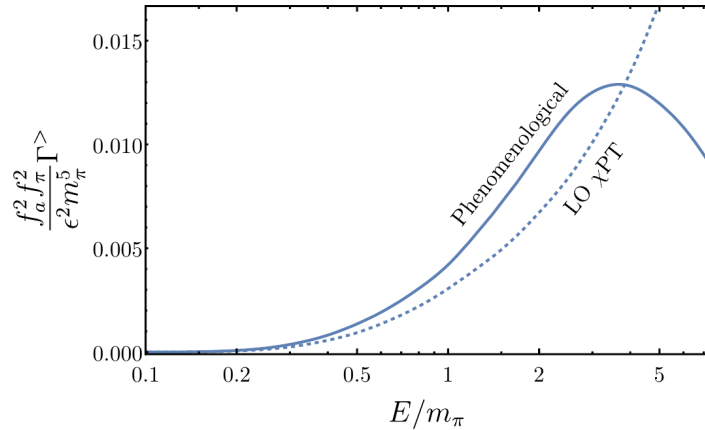
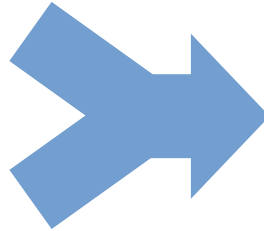
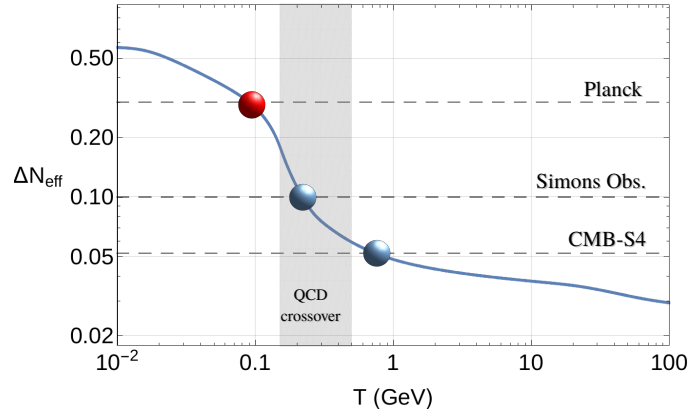
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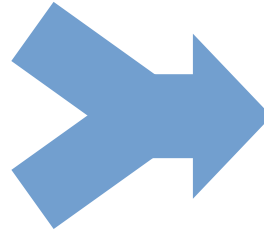
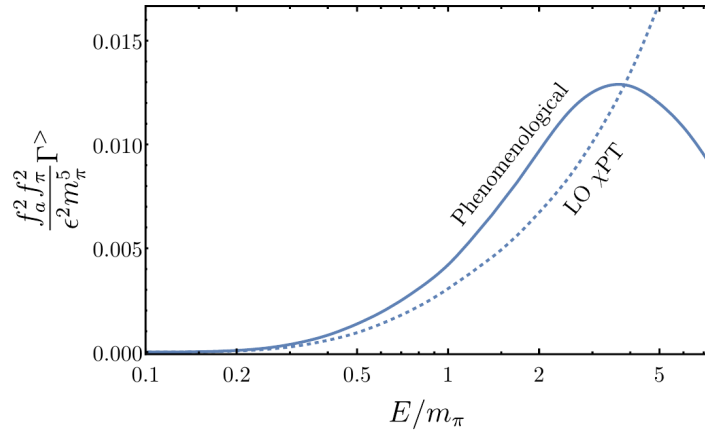
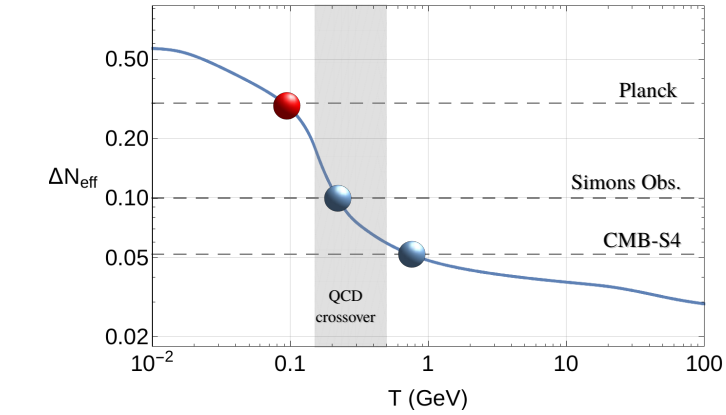
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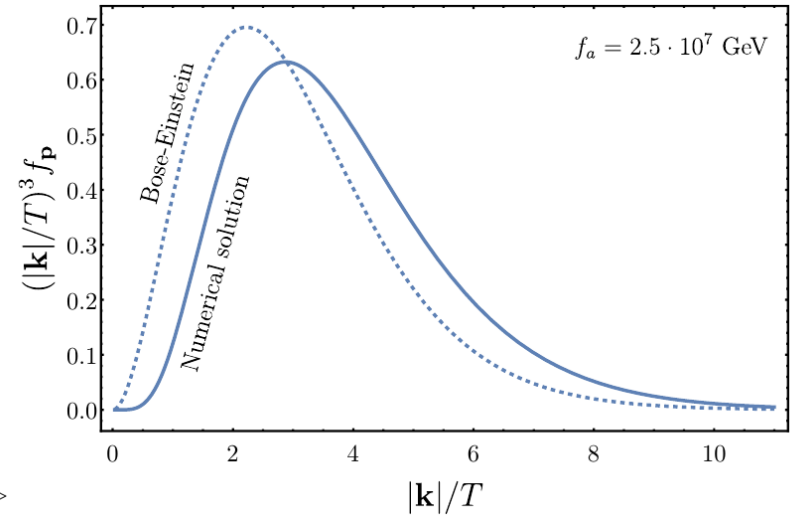
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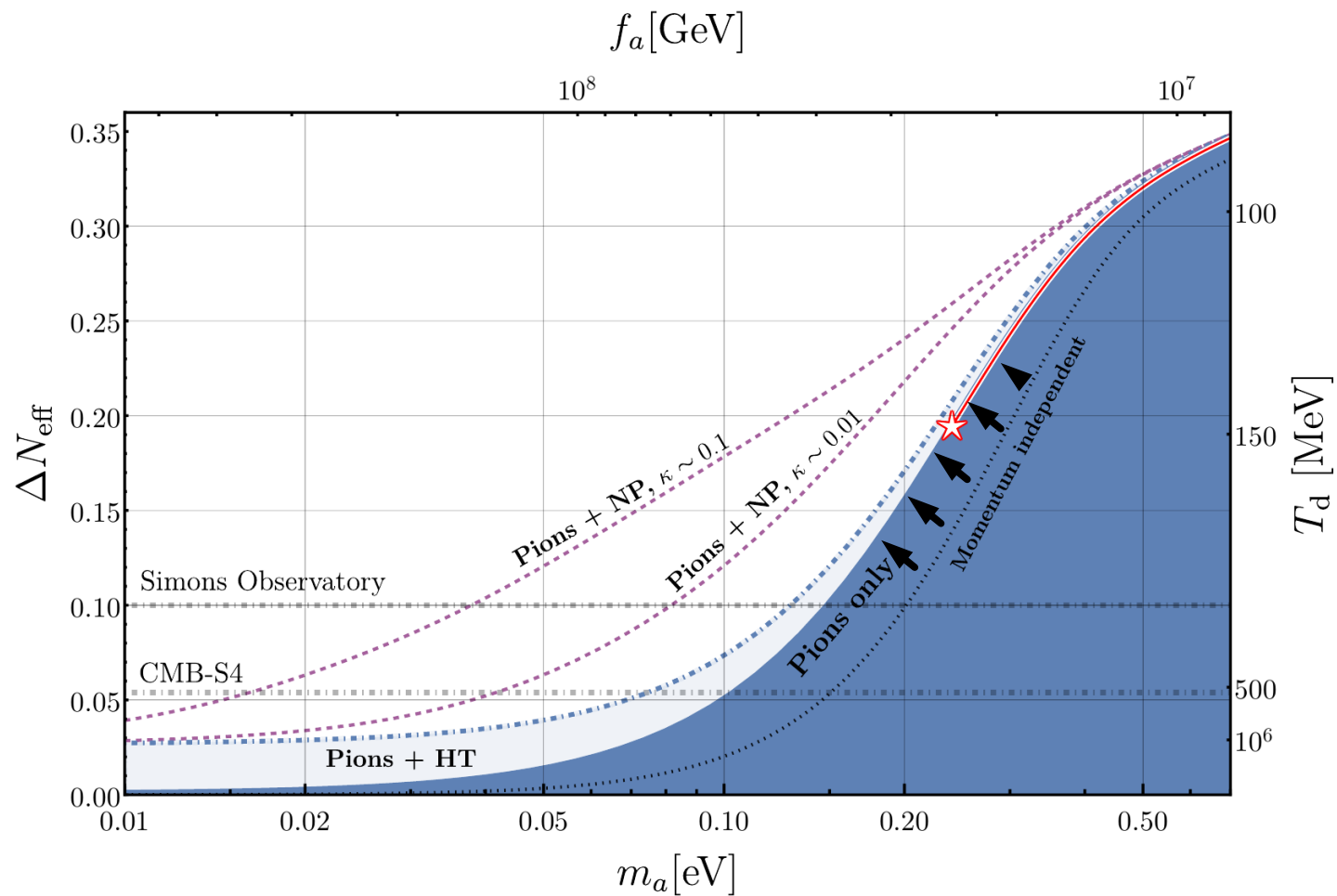
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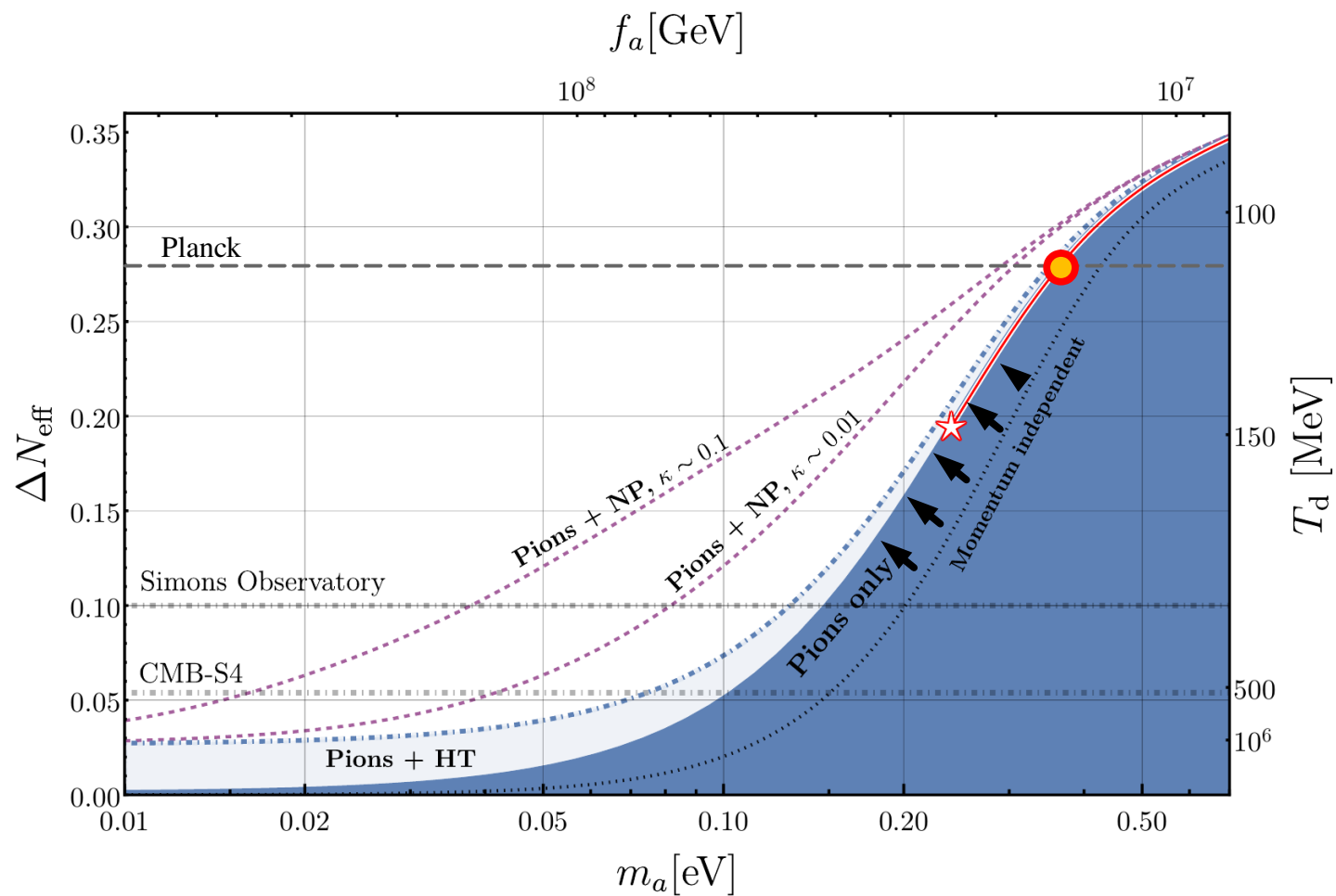


$\sim 40\%$ enhancement

Future Reach

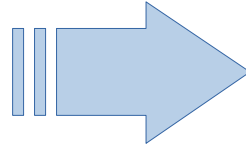
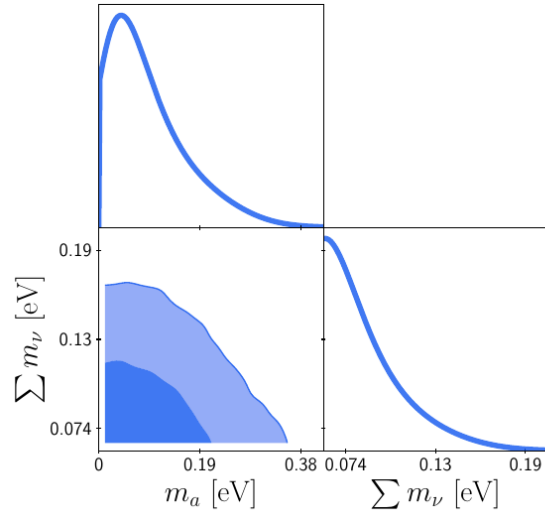


Future Reach



3. Cosmological Fit ($\Lambda_{\text{CDM}} + \Sigma m_{\nu} + m_a$)

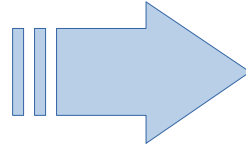
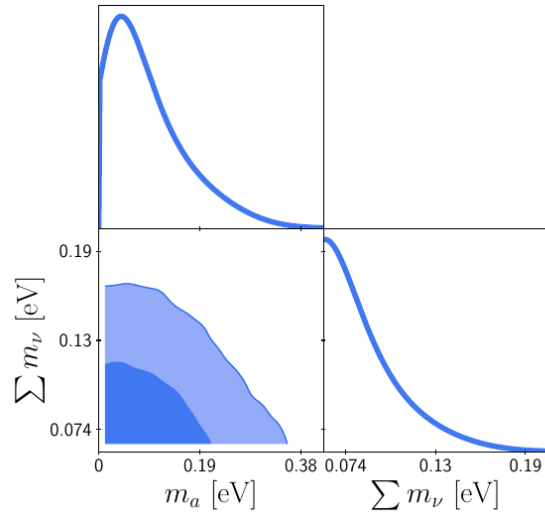
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$$m_a \leq 0.24 \text{ eV}$$

$$f_a \geq 2.4 \cdot 10^7 \text{ GeV}$$

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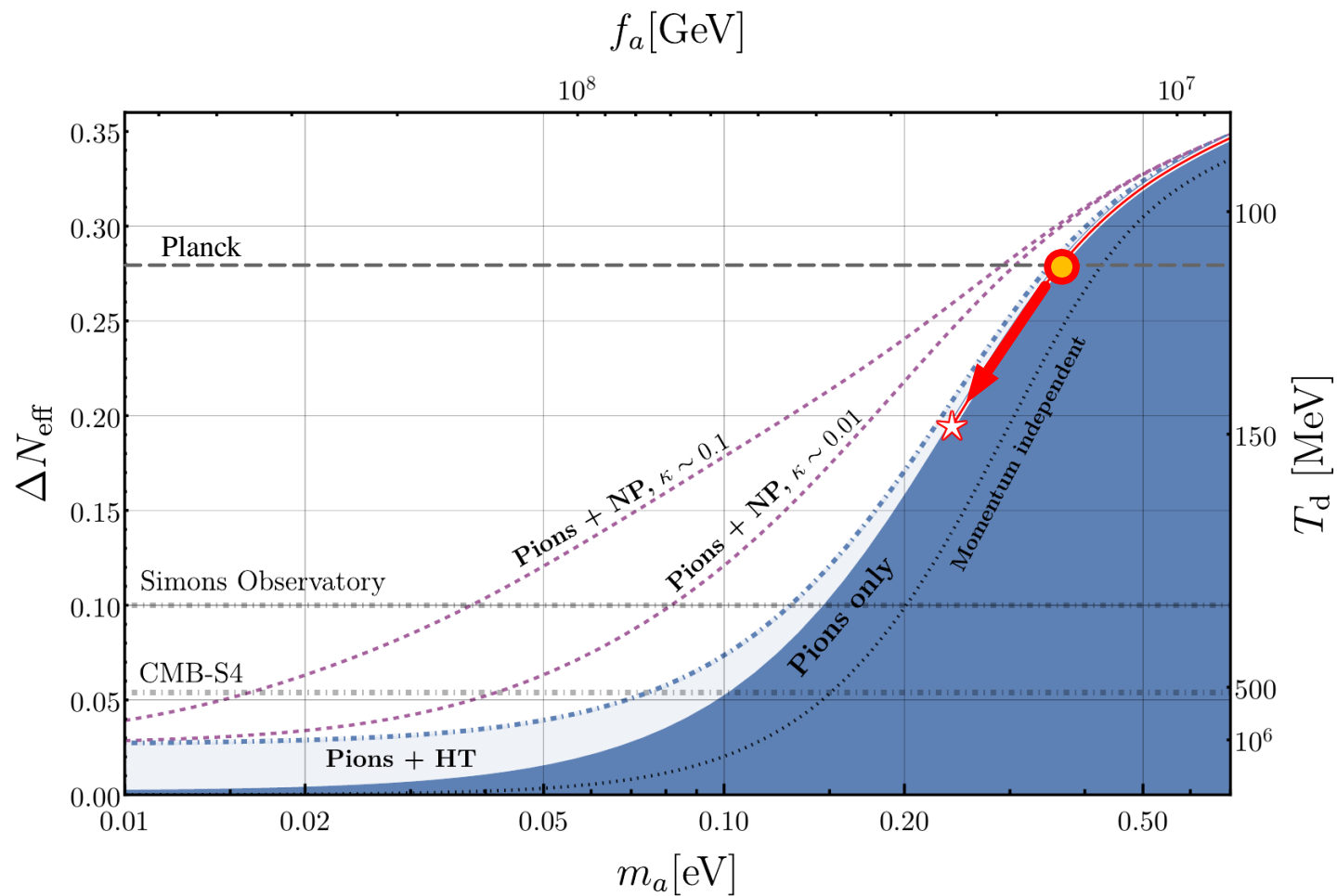
$\Leftrightarrow$

$$\Delta N_{\text{eff}} \lesssim 0.19$$

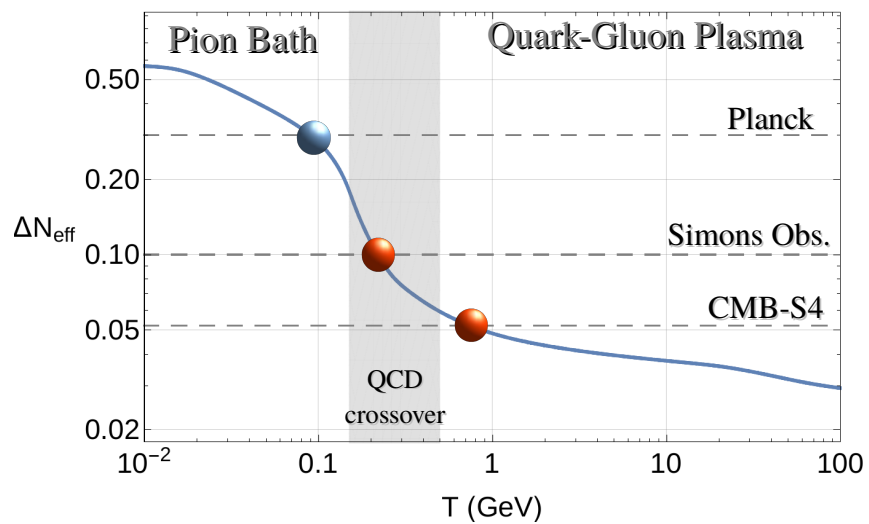
finite mass effect

$\sim 40\%$ enhancement

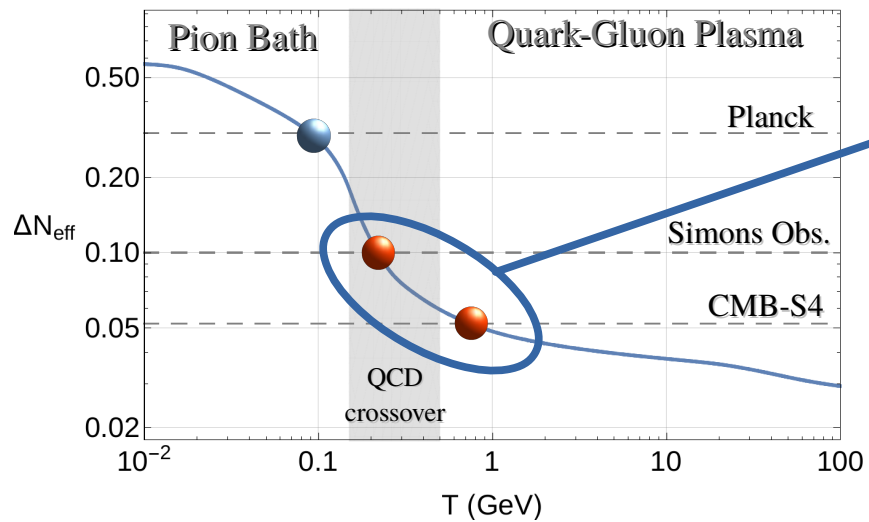
Future Reach



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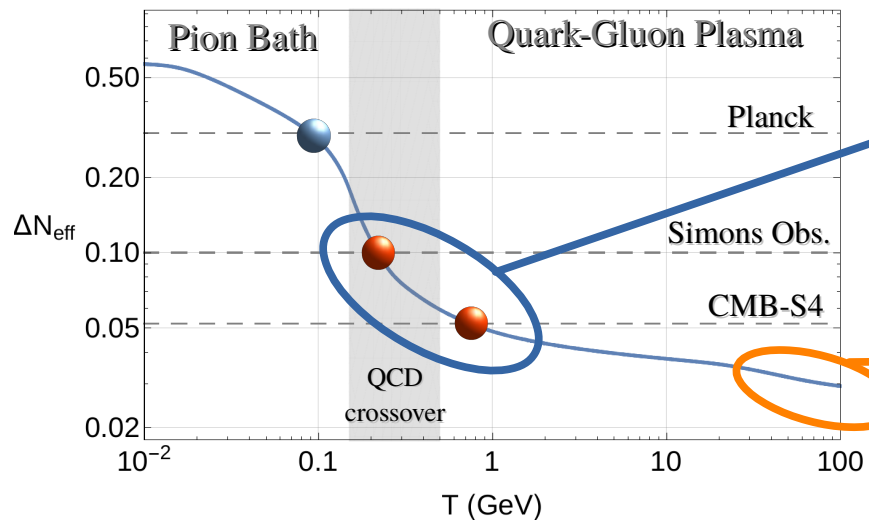
Future Reach



$$\Gamma_{\text{top}}^{\geq} \equiv \int d^4x e^{ik^\mu x_\mu} \left\langle \frac{\alpha_s}{8\pi} G\tilde{G}(x^\mu) \frac{\alpha_s}{8\pi} G\tilde{G}(0) \right\rangle$$

Non-Perturbative

Future Reach



$$\Gamma_{\text{top}}^{\geq} \equiv \int d^4x e^{ik^\mu x_\mu} \left\langle \frac{\alpha_s}{8\pi} G\tilde{G}(x^\mu) \frac{\alpha_s}{8\pi} G\tilde{G}(0) \right\rangle$$

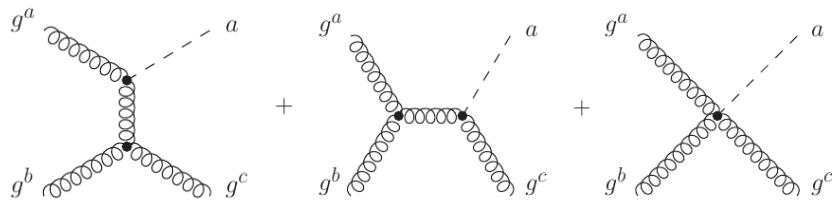
Non-Perturbative

$gg \leftrightarrow ga$

Perturbative ?

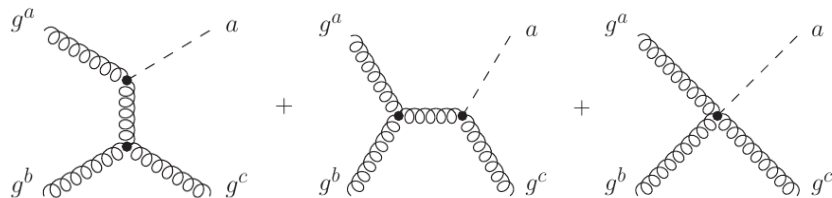
$$\bar{\Gamma}_{\text{pert}} = \frac{\alpha_s^2 T^3}{4\pi^3 f_a^2} F_3$$

High Temperatures Regime



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High Temperatures Regime

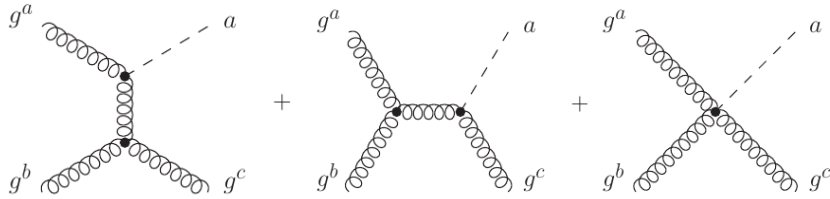


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Masso, Rota, Zsembinski '02
Graf, Steffen '10

$$F_3 = g_s^2 \log \left(\frac{3T^2}{2m_g^2} \right) = g_s^2 \log \left(\frac{3}{2g_s} \right)^2 \quad \text{for } g_s \ll 1$$

High Temperatures Regime

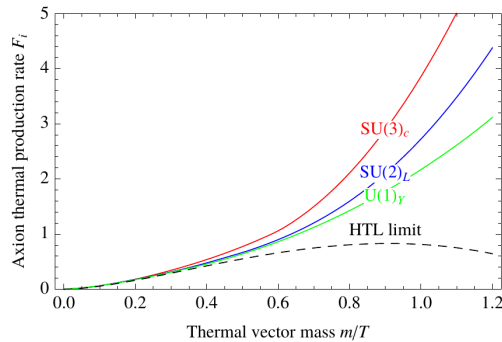
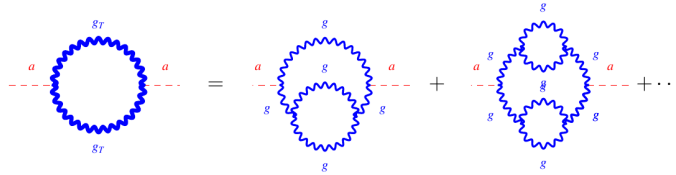


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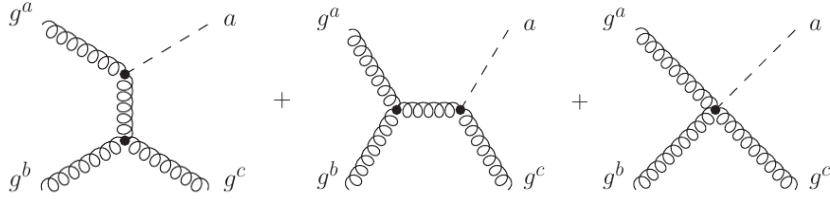
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Salvio, Strumia, Xue '13



High Temperatures Regime

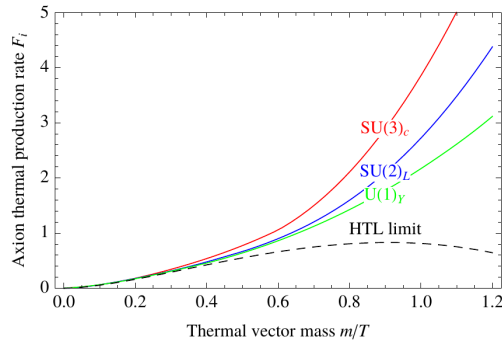
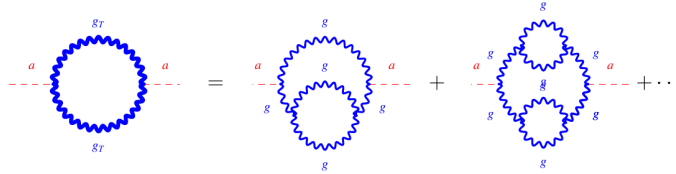


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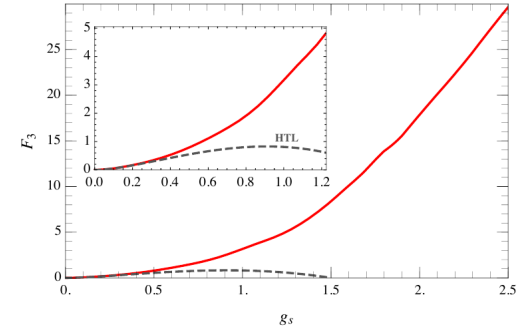
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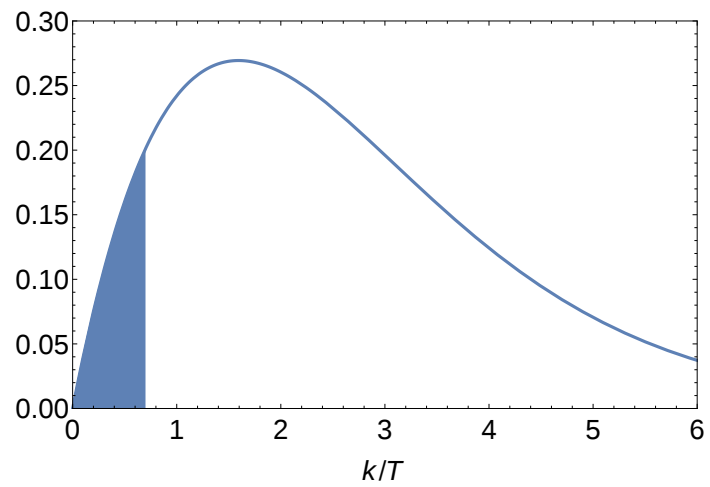
Salvio, Strumia, Xue '13



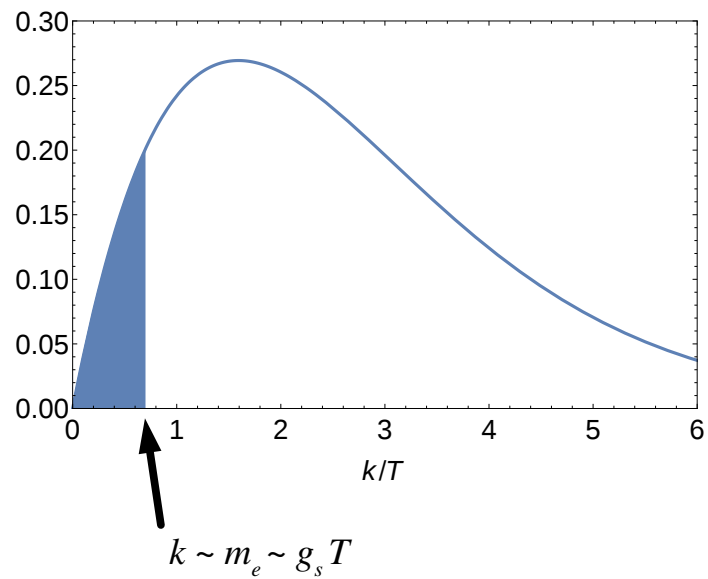
D'Eramo, Hajkarim, Yun '21



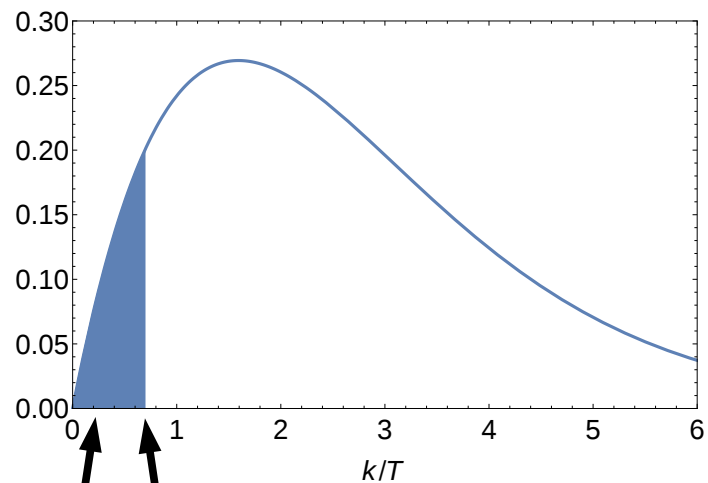
High Temperatures Regime



High Temperatures Regime



High Temperatures Regime



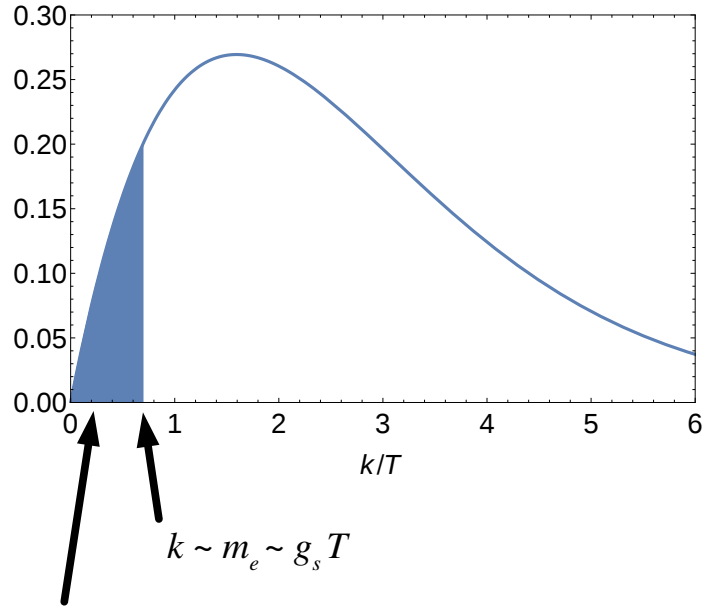
$$k \sim m_m \sim g_s^2 T$$

$$\# \sim 1 / g_s^2$$

Linde '80

Gross, Pisarski, Yaffe '81

High Temperatures Regime



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@ $g_s \ll 1$:

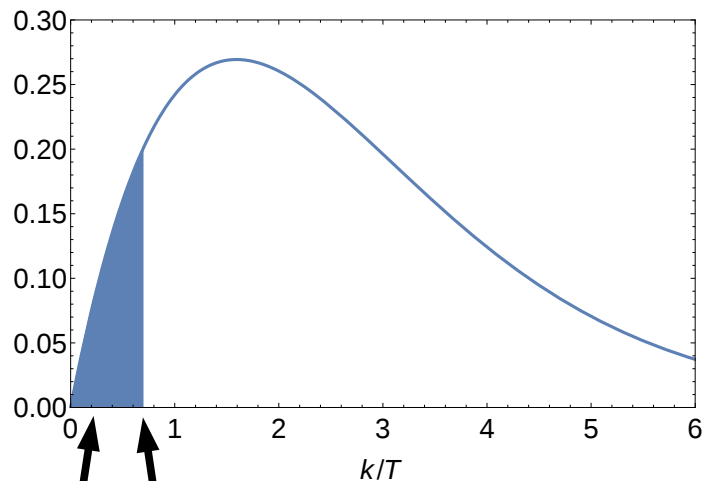
collective effects are phase-space suppressed $O(g_s^n)$

[e.g. for free energy $O(g_s^6)$]

large occupation numbers $\rightarrow$ dominated by semi-classic

[non-linear YM equations - e.g. strong sphalerons]

High Temperatures Regime



$$k \sim m_e \sim g_s T$$

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Linde '80

Gross, Pisarski, Yaffe '81

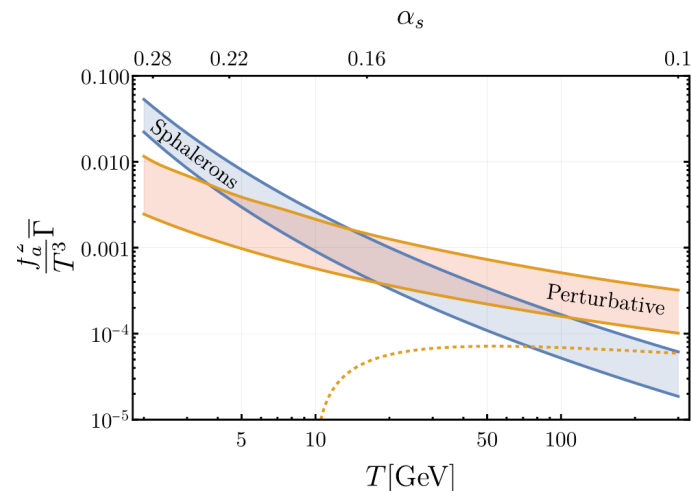
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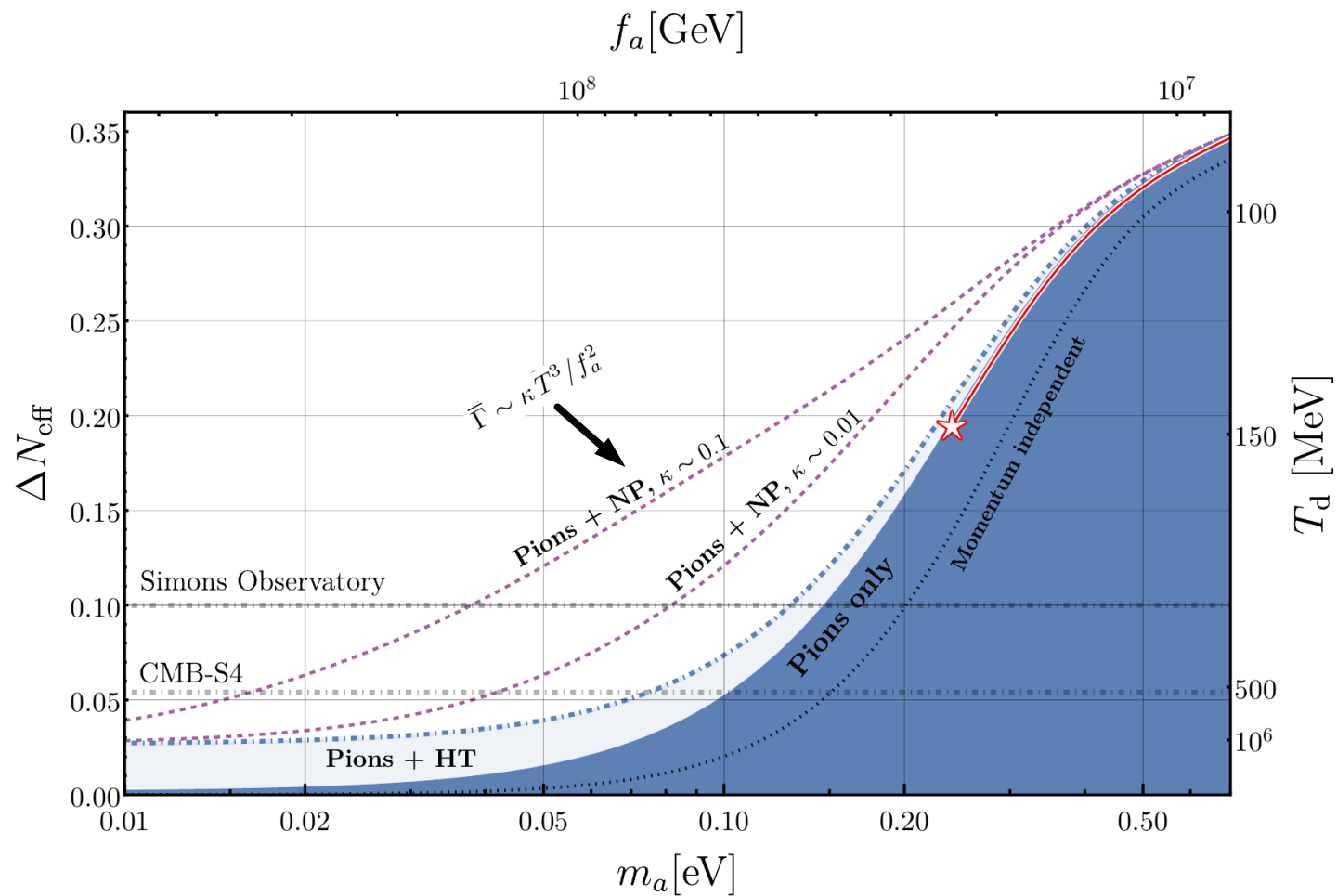
[non-linear YM equations - e.g. strong sphalerons]



$$\Gamma_{\text{sphal}} \simeq (N_c \alpha_s)^5 T^4$$

Moore, Tassler '10

Future Reach



Conclusions:

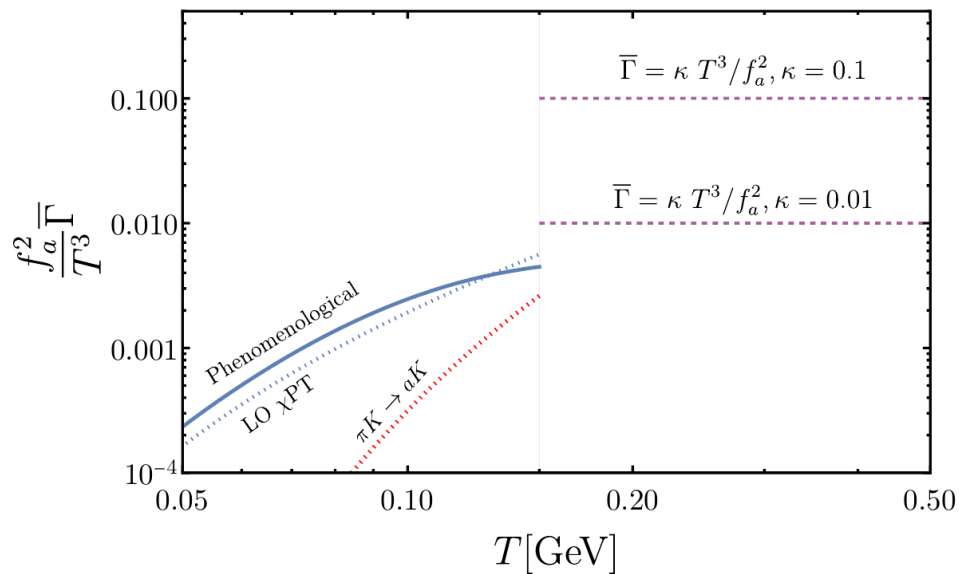
- More reliable upper bound on m_a (< 0.24 eV) from cosmology (for minimal KSVZ-like axions)
- Importance of momentum dependence on Boltzmann equation @ around QCD scale
- Doubts about reliability of perturbative rates above T_c
- Non-perturbative rates crucial for interpreting upcoming CMB experiments

Thanks!

Back Up

Strong Sphaleron-like contribution to Axion rate

$$\bar{\Gamma}_{\text{sphal}} = \frac{1}{n^{\text{eq}}} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E} \frac{\Gamma_{\text{sphal}}}{f_a^2} e^{-E/T} = \frac{(N_c \alpha_s)^5 T^3}{4\zeta_3 f_a^2} \left(1 - \left(1 + \frac{|\mathbf{k}_s|}{T} \right) e^{-|\mathbf{k}_s|/T} \right)$$



$$\Gamma_{\text{top}}^>(E = |\mathbf{k}| < |\mathbf{k}_s|) \simeq \Gamma_{\text{sphal}} \simeq (N_c \alpha_s)^5 T^4$$

$$|\mathbf{k}_s| \sim N_c \alpha_s T$$

The Thermal Width:

Challenge for Lattice QCD: Compute Γ_k for $T > T_c$

Existing Attempts (at $k=0$) e.g.

Moore, Tassler '10 : Classical SU(N) simulations

Kotov '18 : Quantum Euclidean (anal. cont.)

Altenkort et al. '20 : Quantum Euclidean (anal. cont.)

Mancha, Moore '22 : Quantum Euclidean (plus modeling)

$$\left. \begin{array}{l} \text{Moore, Tassler '10 : Classical SU(N) simulations} \\ \text{Kotov '18 : Quantum Euclidean (anal. cont.)} \\ \text{Altenkort et al. '20 : Quantum Euclidean (anal. cont.)} \\ \text{Mancha, Moore '22 : Quantum Euclidean (plus modeling)} \end{array} \right\} \begin{array}{l} \Gamma_{\text{sphal}} = 2T \lim_{\omega \rightarrow 0} \frac{\rho(\omega)}{\omega} \\ G(\tau) = \int d^3x \langle q(\vec{0}, 0) q(\vec{x}, \tau) \rangle \\ = - \int_0^\infty \frac{d\omega}{\pi} \rho(\omega) \frac{\cosh[\omega(1/2T - \tau)]}{\sinh(\omega/2T)} \end{array}$$

Important to exploit upcoming experiments!