Algebraic structures in two-dimensional conformal field theory and a skein theoretic construction of CFT correlators

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Abstract
The **symmetries** of two-dimensional conformal field theories (**CFTs**) can be formalised as chiral algebras, **vertex operator algebras** or nets of observable algebras. Their **representation categories** are abelian categories having additional structures, which are **induced** by properties of **conformal blocks**, i.e. of vector bundles over the moduli space of curves with marked points, which can be constructed from the symmetry structure.
These mathematical notions pertain to the description of chiral CFTs. In a **full local CFT** one deals in addition with correlators, which are specific elements in the spaces of conformal blocks. In fact, a full CFT is the same as a **consistent system of correlators** for arbitrary conformal surfaces with any number and type of field insertions in the bulk as well as on boundaries and on topological defect lines. We present algebraic structures that allow one to construct such systems of correlators.

Key points

- Algebraic structures describing chiral symmetries in two-dimensional conformal field theories: **vertex operator algebras** and their **representations**.
- Structure on the representation categories of nice vertex algebras: **braided tensor category with Grothendieck-Verdier duality**.
- **Conformal blocks** as vector bundles with flat connection; their monodromies are encoded in terms of a modular functor that comes from a modular fusion category.
- Description of **consistent sets of correlators** in terms of three-manifolds with boundary and a **skein-theoretic construction** in terms of **string nets**, for world sheets of arbitrary genus, with arbitrary conformal boundary conditions and with insertions of all types of fields, including generalised defect fields.

Literature:

- These lecture notes are based on `arXiv:2305.02773 [math.QA]`. Older lecture notes with a different focus are available at `arXiv:hep-th/0011109`.
- Background on categorical notions can be found e.g. in the lecture notes for my class Hopf algebras, quantum groups and topological field theory which are available at `https://www.math.uni-hamburg.de/home/schweigert/skripten.html`.

Contents

1 Introduction: chiral and full CFT

Conformal field theory in two dimensions – CFT, for short – is a prime example of the fruitful interplay between mathematics and physics. On the mathematical side it combines concepts and tools from algebraic geometry, higher structures, modular forms, quantum topology and representation theory. **Two-dimensional CFT is arguably the class of quantum field theories for which a maximal mathematical control has been reached. The goal of these lectures is to given an overview over this control.**

For applications to physics, it is crucial to appreciate that there is in fact **no single notion of CFT**; different applications require different theoretical setups.

1. For instance, conformal field theories can be defined on two-dimensional manifolds with a metric of

   - either Euclidean
   - or Lorentzian signature.

   The qualifier *conformal* expresses the fact that the relevant geometry is the one of conformal manifolds, i.e. manifolds endowed with an equivalence class, with respect to local rescalings of metrics
   $$g_{\mu,\nu}(x) \rightsquigarrow \mathrm{e}^{\lambda(x)} g_{\mu,\nu}(x) \ .$$
   Working with **Lorentzian** signature has the advantage that structures familiar from local quantum field theory can be utilized [13, 4]. On the other hand, conformal field theories defined on **Euclidean manifolds** have a broader range of applications and have access to powerful tools from complex geometry. Accordingly we focus our attention on CFTs on compact Euclidean two-manifolds. The relation between the Euclidean and Lorentzian approaches still remains to be understood in full detail; however, remarkably, in both settings similar mathematical structures arise.

2. A further distinction, whose importance cannot be overemphasised, is the one between **chiral** and **full** conformal field theory. Again, these are defined on two different types of two-manifolds:

   - Together with an orientation, a Euclidean conformal structure on a two-manifold amounts to a complex structure. **Chiral CFT** is defined on a **complex curve**. Mathematically, chiral CFT consists of the theory of vertex operator algebras, their representations and their conformal blocks. Their properties are abstracted with the help of the notion of a **modular functor**.
   - In contrast, **full local CFT** is defined on conformal real two-manifolds. Such a manifold may in particular have a non-empty boundary. In fact, one allows for **stratified manifolds**, of which manifolds with boundary are just special instances. For a picture, see (58). The one-dimensional strata of a stratified conformal two-manifold are interpreted as topological line defects which separate different phases of a given CFT. Specific types of such defects encode symmetries and dualities of CFTs [16].

   Given a chiral CFT, a corresponding **full CFT** is a **consistent system of correlators**, i.e. a collection of specific elements in the spaces of conformal blocks of the chiral theory.

Both chiral and full CFTs appear in **physics applications:**

- **Chiral theories** are relevant for the fractional **quantum Hall** effect; in that case, the orientation has a very direct meaning as the direction of an external magnetic field.
- **Full CFTs** arise as world sheet theories of |textbfstring theories, in two-dimensional **critical phenomena** of statistical mechanics, and in effectively one-dimensional systems in condensed matter physics.

In most of these applications one deals with oriented full CFT, i.e. an orientation is chosen on the two-manifolds. But full CFTs can also defined on **unoriented**, and even non-orientable, two-manifolds. **Unoriented** full CFTs arise in particular as world sheet theories of **type-I string** theories.

Remarks 1.1. 1. Best understood among all CFT models is the class of **rational CFTs**, for which the representations of the chiral symmetries form a finitely semisimple modular tensor category, also known as modular fusion category; for a historical review see [18].

2. More recently, the focus of research has shifted to non-rational theories, in particular to **logarithmic CFTs**, for which the conformal blocks may have logarithmic singularities and which are in particular non-unitary; for a review of these, see e.g. the articles in the collection [26].

3. By making use of powerful algebraic structures, CFT can be **formulated without** recourse to a classical **Lagrangian** or any form of perturbation theory. But there do exist models of (full) CFT that are based on a Lagrangian, notably sigma-models with topological terms. Such topological terms are closely related to **bundle gerbes** and other higher geometric structures, compare e.g. [25].

These lectures are organized organized as follows.

- Brief recap of the relevant **two-dimensional geometry**.
- We first review algebraic structures that formalize the chiral symmetries of CFTs. One conceptual framework for these is given by conformal nets of algebras of observables [4]; in this framework unitarity is deeply built in. In view of the importance of non-unitary CFTs in string theory, statistical mechanics and as duals of supersymmetric quantum field theories, we restrict our attention to **vertex operator algebras** [15] – VOAs, for short – which make no assumption on unitarity.
- Afterwards we review aspects **conformal blocks**, which can be constructed from VOAs and which are the building blocks of correlators in full local CFTs. They form vector bundles with projectively flat connections over the moduli space of complex curves. Monodromies of these bundles can be described in terms of a modular functor.
- These data suffice to formulate the concept of a **consistent set of correlators**. We finally exhibit the construction of such consistent sets of correlators, including a recent approach based on a **string-net** formulation of modular functors.

2 Two-dimensional manifolds

Manifolds of (real) dimension two constitute the 'arena' for our studies. In this section we briefly summarize those features that are needed in our discussion. We start with **topological** aspects, then discuss **complex** geometry of these manifolds, and finally describe **Teichmüller** and moduli spaces.

2.1 Topological aspects

Real connected compact two-dimensional topological manifolds Σ are classified by three non-negative integers: The numbers g of handles, b of boundaries, and c of crosscaps.

Remarks 2.1.
1. By cutting out a disc from a given surface one introduces a new boundary component. (We insist that the boundaries obtained this way are *genuine, physical boundaries*. They must never be confused with the small discs one often imagines around insertions of fields, which merely serve to specify local coordinates around the insertion points.)
2. Similarly, a crosscap can be inserted in a surface by first cutting out a disc and then identifying opposite points of the boundary of the disc. The insertion of a crosscap makes a manifold unorientable. Three crosscaps are topologically equivalent to a single crosscap plus a handle, so that it is not necessary to consider more than two crosscaps.
3. World sheets with boundaries play an important role in the description of D-branes; world sheets with crosscaps enter in the construction of string theories of 'type I'.

Another notion we will frequently use is the one of the **mapping class group** Ω of Σ. Consider the group $\mathrm{Homeo}(\Sigma)$ of all homeomorphisms of Σ. It has a normal subgroup $\mathrm{Homeo}_0(\Sigma)$, consisting of those homeomorphisms that are homotopic to the identity. If Σ is orientable, we introduce the subgroup $\mathrm{Homeo}^+(\Sigma)$ of orientation preserving homeomorphisms.

Definition 2.2. For orientable surfaces, we define the **mapping class group** as the quotient

$$\Omega(\Sigma) := \mathrm{Homeo}^+(\Sigma)/\mathrm{Homeo}_0(\Sigma)\,,$$

while for unorientable surfaces we define it to be

$$\Omega(\Sigma) := \mathrm{Homeo}(\Sigma)/\mathrm{Homeo}_0(\Sigma)\,.$$

Remark 2.3. The mapping class group $\Omega(\Sigma)$ acts in particular on the first homology $H_1(\Sigma, \mathbb{Z})$. For orientable surfaces without boundary $H_1(\Sigma, \mathbb{Z})$ is torsion free and comes with a symplectic form from the intersection of one-cycles. (For unorientable surfaces $H_1(\Sigma, \mathbb{Z})$ typically has a torsion part.) The mapping class group preserves the intersection form, and hence we obtain a natural group homomorphism from $\Omega(\Sigma)$ to the corresponding symplectic group; this homomorphism is actually surjective so that we have

$$\Omega(\Sigma) \twoheadrightarrow \mathrm{Sp}(2g, \mathbb{Z})\,.$$

For the torus, we even have identity:

$$\Omega(T^2) = \mathrm{Sp}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})\,.$$

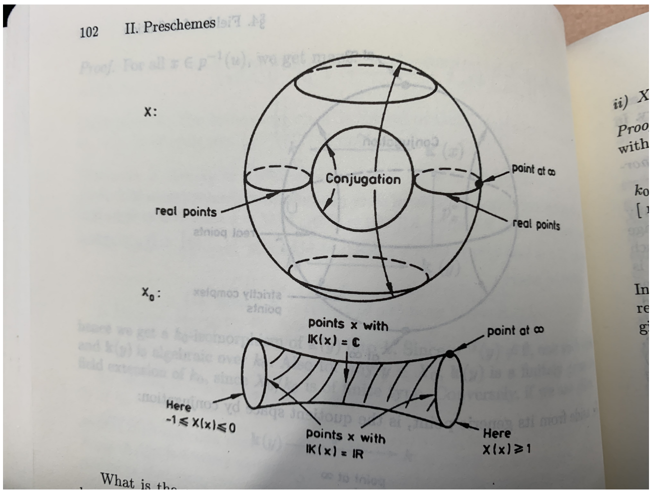
This group is called the **modular group**.

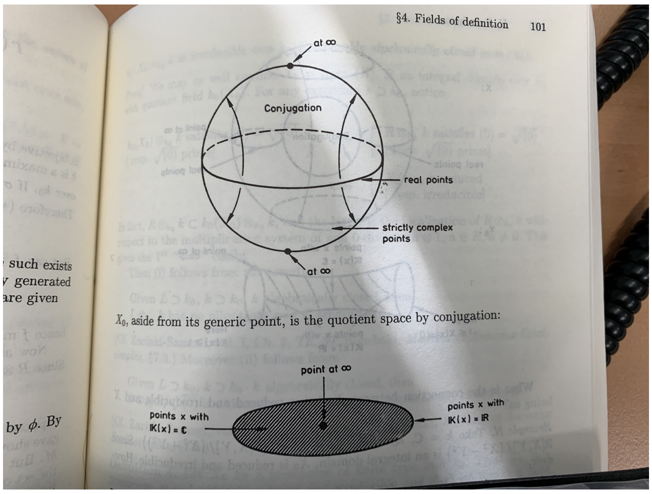
2.2 The Schottky double

We now turn to a construction that allows us to restrict our attention to the case when the manifold Σ is oriented and has no boundary:

Definition 2.4. The **Schottky double** $\hat{\Sigma}$ of Σ. The idea is to double the space, except for the points on the boundary.

- This mimics the method of mirror charges in classical electrodynamics.
- This is also known from the theory real schemes:





Examples 2.5. 1. For Σ a disc, the double $\hat{\Sigma}$ is a sphere, obtained by gluing a disc and its mirror image along their boundaries. Notice that the reflection σ about the equator of this sphere is an orientation-reversing involution, $\sigma^2 = \mathrm{id}$.

2. For Σ the crosscap $\mathbb{RP}^2$, the double is again the sphere, but σ is now the antipodal map.

3. For Σ without boundary, the double is just the total space of the orientation bundle. The orientation bundle is a $\mathbb{Z}_2$-bundle over Σ whose fiber over $p \in \Sigma$ consists of two points, corresponding to the two local orientations at p. Thus for orientable boundaryless Σ it is a trivial bundle, the total space being the disconnected sum of two copies of Σ. An orientation is a global section of this bundle.

Remarks 2.6. 1. The original surface Σ can be obtained as the quotient – or world sheet orbifold, or parameter space orbifold – of the double:

$$\Sigma = \hat{\Sigma}/\sigma\,. \tag{1}$$

The fixed point set of σ gives precisely the boundary of Σ.

2. The total space of the orientation bundle is not only orient*able*, but even naturally orient*ed*. The Euler characteristic $\hat{\chi}$ of the Schottky cover $\hat{\Sigma}$ is related to the Euler characteristic $\chi = 2 - 2g - b - c$ of Σ by $\hat{\chi} = 2\chi$.

The structures we have met so far all belong to the realm of two-dimensional topological manifolds. We now turn to aspects of conformal and complex manifolds that we will need for CFT. Thus let X be a two-dimensional conformal manifold, and $\hat{X}$ its (topological) Schottky double.

We will also need facts from complex geometry; they are the subject of the next subsection.

2.3 Complex geometry

We will be particularly interested in two-dimensional manifolds that are even complex manifolds, i.e. that possess a holomorphic structure.

Definition 2.7. A **holomorphic structure** is the choice of an atlas with maps that take their values in subsets of the complex plane in such a manner that the transition functions are holomorphic.

On such manifolds, all the additional power of complex geometry is at our disposal. In particular, we will be able to construct conformal blocks. The following feature is special to two dimensions:

Proposition 2.8. A holomorphic structure on an orientable manifold X is equivalent to the choice of a conformal structure plus an orientation.

Proof. A classical theorem asserts that every metric on a real two-dimensional differentiable manifold is *locally conformally flat*, i.e. we can find charts U such that the metric is of the form

$$g = \lambda(x,y)\,(\mathrm{d}x^2 + \mathrm{d}y^2)$$

with λ a positive real function on U. When X is oriented, we can choose the charts to be compatible with the orientation. The transformations between different charts are then oriented and conformal diffeomorphisms, i.e. biholomorphic transformations, so we have obtained a complex structure on X; this complex structure depends only on the conformal equivalence class of the metric. Conversely, it is easy to see that a complex structure implies an orientation and a conformal structure. In contrast, higher-dimensional manifolds are not necessarily locally conformally flat. Accordingly, additional integrability conditions must then be met to obtain a complex structure. $\square$

Observation 2.9. Now the double $\hat{X}$ of a conformal manifold X is oriented and inherits a conformal structure from X. In other words, the **double** $\hat{X}$ always has a complex structure, i.e. the double is a complex curve! In particular, at the level of chiral CFT the full power of holomorphy is available, even for the study of conformal field theories on surfaces with boundary.

Example 2.10. In the case of the sphere, $X = S^2$, the cover $\hat{X}$ consists of two spheres with opposite orientation. As quasi-global complex coordinates on the two components of $\hat{X}$, we choose z and $\tilde{z}$. Then the involution σ maps the point with coordinate z to a point on the other component with complex conjugate coordinate: $\tilde{z} = z^*$. But one should be aware of the fact that in chiral CFT $\tilde{z}$ and z are indeed two completely independent complex variables. A frequent description in the literature is to say that "one starts with a single complex variable z and its complex conjugate and then treats the two variables as formally independent". When it comes to concrete calculations this statement amounts to nothing else than working with the Schottky double to treat the chiral aspects of CFT.

2.4 Teichmüller space and moduli space

One and the same oriented topological manifold $\hat{\Sigma}$ without boundary can typically be endowed in different, inequivalent ways with a complex structure. In fact, a lot of important information about CFTs is gained by studying how structures change when one varies the underlying curve. Points at which the underlying curve degenerates in not too bad a way are of special importance; they lead to **factorization constraints**.

We first consider the space $\tilde{\mathcal{M}}(\hat{\Sigma})$ of all complex structures $\mathcal{C}$ on an orientable manifold $\hat{\Sigma}$. On the space $\tilde{\mathcal{M}}(\hat{\Sigma})$ the group $\mathrm{Homeo}(\hat{\Sigma})$ acts as follows. The complex structure $f^*(\mathcal{C})$ is defined by the requirement that the map

$$f: \quad (\hat{\Sigma}, f^*(\mathcal{C})) \to (\hat{\Sigma}, \mathcal{C})$$

is holomorphic if f preserves the orientation and antiholomorphic if f reverses the orientation.

Definition 2.11. One then defines the **Teichmüller space** $\mathcal{T}(\hat{\Sigma})$ as the quotient

$$\mathcal{T}(\hat{\Sigma}) := \tilde{\mathcal{M}}(\hat{\Sigma})/\mathrm{Homeo}_0(\hat{\Sigma})\,.$$

Remarks 2.12. 1. One can show that the Teichmüller space is a complex manifold.

2. For $\hat{g}=0$ it is just a point, while for $\hat{g}=1$ it is isomorphic to the complex upper half plane, $\mathcal{T}_1 = H := \{\tau\in\mathbb{C} \,|\, \mathrm{Im}\tau>0\}$. For every $\hat{g}\geq 2$ the Teichmüller space is isomorphic to $\mathbb{C}^{3\hat{g}-3}$ as a topologiccal manifold, and to $\mathbb{R}^{6\hat{g}-6}$ as a real analytic manifold, but not to $\mathbb{C}^{3\hat{g}-3}$ as a complex analytic manifold. In general, the geometry of Teichmüller spaces is a rich area of geometry.

Definition 2.13. On the Teichmüller space, the mapping class group $\Omega(\hat{\Sigma})$ acts as the full group of holomorphic automorphisms of $\mathcal{T}(\hat{\Sigma})$. The **moduli space** $\mathcal{M}_{\hat{g}}(\hat{\Sigma})$ is obtained from the Teichmüller space by dividing out the action of $\Omega(\hat{\Sigma})$:

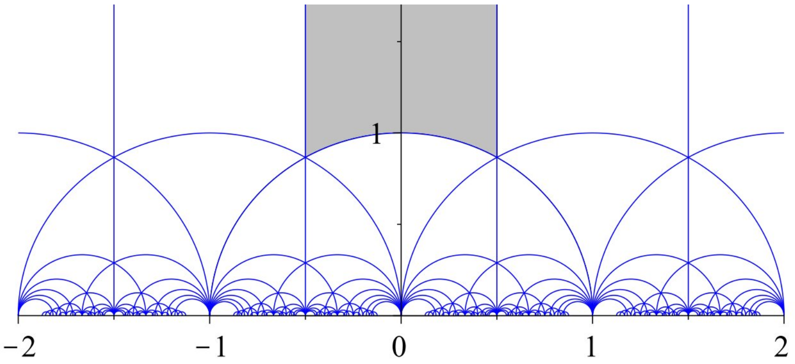
$$\mathcal{M}_{\hat{g}}(\hat{\Sigma}) = \mathcal{T}(\hat{\Sigma})/\Omega(\hat{\Sigma}) = \tilde{\mathcal{M}}(\hat{\Sigma})/\mathrm{Homeo}^+(\hat{\Sigma})\,.$$

Remarks 2.14. 1. If σ is an orientation reversing homeomorphism of $\hat{\Sigma}$, it induces by the same procedure an *anti-holomorphic* involution σ^* on the Teichmüller space. This will be important in the discussion of the double. One can show that for $\hat{\Sigma}$ of genus $\hat{g}$, there are $[\frac{3\hat{g}+4}{2}]$ inequivalent fixed-point free involutions. (For the torus, we obtain in this way the Klein bottle.)

2. The Teichmüller space is simply connected and is in fact the universal covering space of moduli space. This implies that the mapping class group is the fundamental group of the moduli space:
$$\pi_1(\mathcal{M}_{\hat{g}}) = \Omega(\hat{\Sigma})\,. \tag{2}$$
(As one is dealing with singular spaces, one must be careful with the definition of the fundamental group. For details, see [28].)

3. For $\hat{g}=1$, the action of Ω is the standard action
$$\tau \;\mapsto\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau \equiv \frac{a\tau+b}{c\tau+d} \tag{3}$$
of $\mathrm{SL}(2,\mathbb{Z})$ on the upper half-plane H. The action of the mapping class group is not free, and as a consequence the moduli space $\mathcal{M}_{\hat{g}}$ has *orbifold* singularities, which correspond

to curves with non-trivial automorphisms. For genus one, these are the points $\tau = \mathrm{i}$ and $\tau = \exp(2\pi\mathrm{i}/3)$. However, these singular points are not the most interesting ones for conformal field theory. Of far more interest are singularities that are cusps like the point $\tau = \mathrm{i}\infty$ for $\hat{g} = 1$, where the curve degenerates and where factorization constraints can be formulated.



(Source: Wikipedia)

4. The Teichmüller space $\mathcal{T}(\Sigma)$ can be canonically identified with the fixed point set of $\mathcal{T}(\hat{\Sigma})$ under the involution σ^*:
$$\mathcal{T}(\Sigma) \cong \mathcal{T}(\hat{\Sigma})_{\sigma^*} .$$
One can show that this space is real-analytically isomorphic to $\mathbb{R}^{-3\chi}$ for Euler characteristic $\chi(\Sigma) \leq -1$, isomorphic to $\mathbb{R}$ for Euler characteristic 0, and to a point for $\chi(\Sigma) = 1$.

5. Finally, using the action of the mapping class group $\Omega(\Sigma)$ on $\mathcal{T}(\hat{\Sigma})$ one can canonically identify $\Omega(\Sigma)$ with the commutant $\Omega_\sigma(\hat{\Sigma})$ of σ^* in $\Omega(\hat{\Sigma})$. This subgroup is also called the **relative modular group**. We can therefore identify the moduli space $\mathcal{M}(\Sigma)$ with the quotient
$$\mathcal{M}(\Sigma) = \mathcal{T}(\hat{\Sigma})_{\sigma^*}/\Omega_\sigma(\hat{\Sigma}) . \tag{4}$$

3 Vertex operator algebras

This section is devoted to the study of algebraic and representation theoretic aspects of the symmetries of a chiral CFT. The main notion we will introduce is the one of a vertex operator algebra (VOA). It formalizes the notion of a chiral **symmetry algebra**.

3.1 The Virasoro algebra

First, however, we need a few facts about infinitesimal conformal symmetries in two dimensions and its super-extensions. Conformal symmetry is encoded in the **Virasoro algebra**. This is the central extension of the Lie algebra of vector fields on a circle S^1 for which we choose a basis $(\frac{1}{\mathrm{i}}\mathrm{e}^{in\varphi}\partial_\varphi)_{n\in\mathbb{Z}} = (z^{n+1}\partial_z)_{n\in\mathbb{Z}}$ We compute
$$\begin{aligned}[z^{n+1}\partial, z^{m+1}\partial] &= z^{n+1+m}(m+1)\partial - z^{m+1+n}(n+1)\partial \\ &= (n-m)z^{n+m+1}\partial\end{aligned}$$

Definition 3.1. 1. The **Virasoro algebra** is the infinite-dimensional Lie algebra that is spanned by generators L_n with $n \in \mathbb{Z}$ and a central element C, subject to the relations
$$\begin{aligned}[L_n, L_m] &= (n-m)\,L_{n+m} + \frac{1}{12}\,(n^3-n)\,\delta_{n+m,0}\,C\,, \\ [L_n, C] &= 0\,.\end{aligned} \tag{5}$$

2. If v is an eigenvector of L_0 in a representation of the Virasoro algebra, then its eigenvalue is called the **conformal weight** of v and denoted by Δ_v.

Remark 3.2. 1. Using a formal variable z, we can combine the generators into a 'field'

$$T(z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2}\,,$$

called the chiral stress-energy tensor.

2. There are various super-extensions of this construction. The simplest one, the $N=1$ superconformal algebra (or $N=1$ Virasoro algebra), is described by the superfield

$$\mathcal{T}(z,\vartheta) = \frac{1}{2}\, G(z) + \vartheta\, T(z)\,,$$

whose first component $G(z)$ has the expansion

$$G(z) = \sum_{r\in\mathbb{Z}+\epsilon} G_r\, z^{-r-3/2}\,.$$

The parameter $\epsilon \in \{0, \frac{1}{2}\}$ depends on the 'sector': $\epsilon=0$ in the *Ramond* sector, while $\epsilon=1/2$ in the *Neveu–Schwarz* sector. Keep in mind that Ramond and Neveu–Schwarz sector are distinguished by the monodromies of a specific field. The commutation relations among G and T read

$$\begin{aligned} [L_n, G_r] &= (\tfrac{1}{2}n - r)\, G_{n+r}\,, \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{1}{3}(r^2 - \frac{1}{4})\, \delta_{r+s,0}\, C\,. \end{aligned}$$

3.2 Vertex operator algebras

Definition 3.3. The basic data of a **vertex algebra** are
- a complex vector space $\mathfrak{V}$ and
- a linear map $Y\colon \mathfrak{V}\otimes\mathfrak{V} \to \mathfrak{V}((z))$, called the **field map**.

Here z is a formal variable; in the description of conformal blocks below it will be interpreted as a formal local coordinate on a complex curve. $\mathfrak{V}((z))$ denotes $\mathfrak{V}$-valued power series in z and z^{-1} with exponents bounded from below. The evaluation of the field map at a vector $a\otimes b \in \mathfrak{V}\otimes\mathfrak{V}$ is denoted by

$$Y(a,z)\, b = \sum_{n\in\mathbb{Z}} a_n b\, z^{-n-1}. \tag{6}$$

Regarding b as a free variable, this is rewritten as

$$Y(a,z) = \sum_{n\in\mathbb{Z}} a_n\, z^{-n-1} \in \mathrm{End}(\mathfrak{V})[[z,z^{-1}]]\,, \quad a_n \in \mathrm{End}(\mathfrak{V})\,. \tag{7}$$

- The identity element of $\mathfrak{V}$ is called the **vacuum vector** and is frequently denoted by $|0\rangle$; it satisfies

$$Y(|0\rangle, z)\, a = a \qquad \text{and} \qquad Y(a,z)|0\rangle \in a + z\,\mathfrak{V}[[z]] \quad \text{for all } a\in\mathfrak{V}\,, \tag{8}$$

where $z\,\mathfrak{V}[[z]]$ is the subspace of $\mathfrak{V}((z))$ consisting of all $\mathfrak{V}$-valued power series with only positive powers of z. Thus, one recovers states by acting with the corresponding fields on the vacuum and 'sending z to zero',

▪ Finally, we imposie **locality** – is that commutators of fields have poles of at most finite order. More precisely, for any two
$v_1, v_2 \in \mathfrak{V}$ there must exist a number $N = N(v_1, v_2)$ such that

$$(z_1 - z_2)^N \, [Y(v_1, z_1), Y(v_2, z_2)] = 0 \,. \tag{9}$$

This is a kind of weak commutativity. Note that this constraint only makes sense because we consider formal series in the z_i, which can extend to both arbitrarily large positive and negative powers. Had we restricted ourselves to ordinary Laurent series, i.e. series without arbitrarily large negative powers, (9) would already imply that the commutator vanishes.

One now defines an endomorphism, called the shift operator

$$T: \quad \mathfrak{V} \to \mathfrak{V} \,;$$

by the requirement that T implements infinitesimal translations,

$$[T, Y(v, z)] = \partial_z Y(v, z) \,; \tag{10}$$

and that the vacuum is translation invariant, $T v_\Omega = 0$.
We sill need to build in the Virasoro algebra:

Definition 3.4. A **vertex operator algebra** is a vertex algebra with a **conformal structure**. The conformal structure requires the existence of a vector $\omega \in \mathfrak{V}$ such that the coefficients L_n in the expansion

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n \, z^{-n-2} \tag{11}$$

satisfy the relations

$$[L_m, L_n] = (m - n) \, L_{m+n} + \frac{1}{12} \, (m^3 - m) \, \delta_{m+n,0} \, C \tag{12}$$

of the Virasoro Lie algebra. Here C is a central element; it acts on $\mathfrak{V}$ (and on its modules, introduced below) by multiplication with a complex number, called the (conformal) *central charge* and denoted by c.

Remarks 3.5.

1. The axioms of a VOA then imply that $(\mathfrak{V}, Y)$ is essentially a unital commutative associative complex algebra with a derivation and with an additional conformal structure. In particular, unital commutative associative complex algebras with a derivation are examples of VOAs that do not necessarily have a conformal structure. However, owing to the presence of formal variables, in general the commutativity and associativity properties are more involved than in the standard commutative algebra case.

2. The conformal structure endows $\mathfrak{V}$ with an integral grading (usually required to be non-negative):

$$\mathfrak{V} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{V}_n \qquad \text{with} \qquad \mathfrak{V}_n = \{ v \in \mathfrak{V} \mid L_0 v = n v \} \,. \tag{13}$$

3. For a homogeneous vector $b \in \mathfrak{V}_h$ it is common in the physics literature to shift the indexing in the series expansion (7) in such a way that

$$Y(b, z) = \sum_{n \in \mathbb{Z}} b_{(n)} \, z^{-n-h}. \tag{14}$$

The so obtained coefficients $b_{(n)}$ are degree $-n$ maps, that is, $b_{(n)} \mathfrak{V}_m \subset \mathfrak{V}_{m-n}$.

4. Let us also mention that the choice of v_{Vir} for a given VOA and a given shift operator T is not necessarily unique. Ghost systems will provide an explicit example.

VOAs are formulated with the help of formal variables, respectively formal local coordinates. An algebraic structure that captures the symmetries of a chiral CFT on a smooth algebraic curve in a coordinate-independent way is the one of a *chiral algebra* [3]. This structure is formulated with the help of the notion of a D-module. Chiral algebras on the affine line $\mathbb{A}^1$ that are equivariant with respect to translations are in bijection to vertex operator algebras. The study of chiral algebras has triggered further developments, in particular it has led to the notion of a factorisation algebra. For detailed information we refer to [3, 15].

We conclude this section with a word of warning. The notion of a VOA is a mathematical formalization of a physical concept, the algebra of chiral symmetries. It is, however, *not* the only available formalization of that physical concept. A rather different approach, based on the notion of local observables [27], describes the chiral symmetries in terms of nets of von Neumann algebras over a circle S^1 (for details see e.g. [8]). The relation between these different incarnations of chiral symmetries has been clarified in the past few years, but is beyond the scope of these lectures.

3.3 Examples

It is time for presenting examples of VOAs.

Examples 3.6. 1. We start with the chiral CFT for a single free boson; it is based on the Heisenberg algebra $\mathfrak{h}$, which has generators b_n with $n \in \mathbb{Z}$ and relations

$$[b_n, b_m] = n\, \delta_{n+m,0}\,.$$

In this case $\mathcal{H}_\Omega$ is nothing but a Fock space

$$\mathrm{U}(\mathfrak{h}) \otimes_{\mathrm{U}(\mathfrak{h}_+)} \mathbb{C}v_\omega\ ,$$

and v_Ω is the ground state in this Fock space. To define the field-state correspondence one introduces abelian currents

$$J(z) = \sum_{n\in\mathbb{Z}} b_n\, z^{-n-1}$$

and identifies $Y(b_{-1}v_\Omega, z) = J(z)$. More generally, one sets

$$Y(b_{n_1}\cdots b_{n_k}v_\Omega, z) = \frac{1}{(n_1-1)!\cdots(n_k-1)!} : \partial_z^{n_1-1} J(z) \cdots\ \partial_z^{n_k-1} J(z) :,$$

where the colons indicate a normal ordering. This prescription indeed yields the structure of a VOA. (It is not a trivial exercise, though, to check that this works out.)

2. This example has an important generalization. Let L be a lattice, and $V = L\otimes_{\mathbb{Z}}\mathbb{R}$ be the associated real vector space with basis $\{b^{(i)}\}$. Suppose that $\mathfrak{V}$ has a non-degenerate bilinear form κ. To the infinite-dimensional Lie algebra with basis $b_n^{(i)}$, $n \in \mathbb{Z}$, and relations

$$[b_n^{(i)}, b_m^{(j)}] = n\, \kappa(b^{(i)}, b^{(j)})\, \delta_{n+m,0}$$

one associates a Fock space $\mathcal{H}_\Omega$. One checks that it carries again the structure of a VOA. This structure can be further generalized: Suppose that the bilinear form is such that L is an even lattice, i.e. $\kappa(v,w) \in \mathbb{Z}$ and $\kappa(v,v) \in 2\mathbb{Z}$ for all $v, w \in L$. Then the space

$$\mathcal{H}_\Omega \otimes \mathbb{C}[L]\,,$$

where $\mathbb{C}[L]$ is the group algebra of L, has the structure of a VOA, too. It is called the **lattice VOA** for L and describes $\dim V = \operatorname{rank} L$ many compactified chiral free bosons.

3. Similar constructions are possible when choosing other infinite-dimensional Lie algebras in place of the Heisenberg algebra. One can e.g. take the Virasoro algebra itself; then $T(z)$ roughly plays the role of the abelian current $J(z)$. When the chiral algebra is generated solely from the Virasoro algebra, then the model is called a Virasoro *minimal model*. Another important class of examples is furnished by untwisted affine Lie algebras, where non-abelian currents $J^a(z)$ (a an adjoint label of the underlying finite-dimensional simple Lie algebra) take over the role of $J(z)$. The commutation relations are given in a basis $(J^a_n)_{n\in\mathbb{Z},a=1,\dots\dim\bar{\mathfrak{g}}}$:
$$[J^a_n, J^b_m] = f^{ab}_{c} J^c_{n+m} + \kappa^{a,b} n \delta_{n+m,0}$$
The models obtained this way are known as **Wess–Zumino–Witten** (*WZW*) models. The Heisenberg, Virasoro and affine Lie algebras belong to the so-called Lie algebras of formal distributions; these are Lie algebras $\mathfrak{g}$ that are spanned over $\mathbb{C}$ by the coefficients of a collection of $\mathfrak{g}$-valued mutually local formal distributions $\{a^\alpha(z)\}$.

4. Our last example are the so-called **first order systems**. This is a family of VOAs, labelled by two parameters $\lambda\in\mathbb{Z}/2$ and $\eta=\pm1$. One starts with a Lie algebra generated by two formal distributions
$$b(z) = \sum_{n\in\epsilon+\mathbb{Z}} b_n z^{-n-\lambda}\,, \qquad c(z) = \sum_{n\in\epsilon+\mathbb{Z}} c_n z^{-n-(1-\lambda)}$$
of conformal weight λ and $1-\lambda$, respectively. These fields are bosonic for $\eta=-1$ and fermionic for $\eta=1$. In the former case, ϵ is zero, while in the latter it takes the values 0 for the Ramond sector and 1/2 for the Neveu–Schwarz sector. The modes of b and c obey the (anti-)commutation relations $\{c_m, b_n\}_\eta = \delta_{n+m,0}$. The VOA is defined on a Fock space, built on a highest weight vector v_Ω with relations
$$b_n v_\Omega = 0 \ \text{ for } n \geq 1{-}\lambda\,, \qquad b_n v_\Omega = 0 \ \text{ for } n > \lambda\,.$$
As the stress-energy tensor one takes
$$T(z) = -\lambda\, {:}b\,\partial c{:} - (1{-}\lambda)\, {:}\partial b\, c{:}\,;$$
the Virasoro central charge is $c=1{-}3Q^2$ with $Q:=\epsilon(1{-}2\lambda)$. This is an example of a vertex algebra with different conformal structures.
There is a U(1) current with modes $j_n = \sum_{m\in\mathbb{Z}+\epsilon} {:}c_{n-m} b_m{:}$ which is, however, anomalous:
$$[L_n, j_m] = \frac{1}{2}\, Q\, n(n{+}1)\, \delta_{n+m,0} - m\, j_{n+m}\,.$$
First order systems are of particular interest for the following values: $\lambda=2$, $\eta=1$ yields the ghosts for bosonic reparametrizations of the string world sheet, $\lambda=3/2$, $\eta=-1$ gives the ones for gauging the fermionic operators of an $N=1$ superconformal symmetry on the world sheet, and $\lambda=1/2$, $\eta=1$ is just a complex free fermion.

3.4 Representation categories

As a generalisation of associative algebras, VOAs naturally admit modules. Recall that the VOA is formalizing aspects of a chiral symmetry algebra, and symmetries in a quantum theory should be represented on the space of states. We are thus lead to study the representation theory of VOAs.

Definition 3.7. A module over a VOA $(\mathfrak{V}, Y)$ consists of a complex vector space M and a linear map $Y^M : \mathfrak{V} \otimes M \to M((z))$ such that Y^M represents the field map Y (the multiplication on $\mathfrak{V}$) on M in the sense that the representation map

$$Y_M : \quad \mathcal{H}_\Omega \to \mathrm{End}(M)[[z, z^{-1}]] \,,$$

obeys

$$Y_M(v_1, z_1)\, Y_M(v_2, z_2) = Y_M(Y(v_1, z_1{-}z_2)v_2, z_2) \,. \tag{15}$$

Remarks 3.8. 1. Representations of VOAs form a **category** which has many subcategories.

2. The classification of V-modules is in general an extremely hard problem. But for interesting classes of VOAs it can be made tractable by restricting the kind of modules to be considered to those visible to the Zhu algebra formalism; see [23] for more information.

3. One can show that the VOA furnishes a representation of itself; this is called the **vacuum representation**. This implies that the identity (15) is in particular valid for $\mathcal{H}_\Omega$, i.e. (15) remains true when Y_M is replaced by Y. This expresses a kind of associativity of the VOA. Thus for VOAs ‘associativity’ in the sense of (15) is a consequence of ‘commutativity’ in the sense of (9).

4. When the VOA is conformal, every module M is in particular, by restriction, a module over the Virasoro algebra. It follows directly from the definition of a conformal VOA that in each CFT model the central element C of the Virasoro algebra acts as $C = c\,\mathrm{id}$ with one and the same value of the number c in every irreducible representation that occurs in the model. Also recall that when v is an eigenstate of the Virasoro zero mode L_0, its eigenvalue Δ_v is called the *conformal weight* of v. More generally, if the action of L_0 can be described by Jordan blocks, the principal value is called the conformal weight.

5. The conformal weights of different vectors in the same irreducible module differ by integers, or in other words, $\mathrm{e}^{2\pi\mathrm{i}L_0}$ acts as a multiple of the identity on every irreducible module. We will therefore refer, somewhat abusing terminology, to the *fractional part* of the conformal weight of any eigenvector of L_0 in an irreducible module M as the conformal weight Δ_M of the module M.

3.5 Tensor products

Continuing the analogy to commutative algebras, it is natural to study $\mathfrak{V}$-multilinear maps, in particular bilinear ones. Such maps are a crucial ingredient in the construction of conformal blocks and are thus of central importance to chiral CFT, see below.

Definition 3.9. A $\mathfrak{V}$-bilinear map from a pair of $\mathfrak{V}$-modules M_1 and M_2 to a third module M_3 is a linear map

$$\mathcal{Y}\colon \quad M_1 \otimes M_2 \longrightarrow M_3\{z\}[\log(z)] \tag{16}$$

that is compatible with the action of $\mathfrak{V}$ on each of the three modules.

Here $\log(z)$ is a formal variable satisfying $\partial_z \log(z) = z^{-1}$ and $M_3\{z\}[\log z]$ denotes polynomials in $\log(z)$ whose coefficients are M_3-valued power series in z for which the exponents can be arbitrary complex numbers.

We recall the definition of the **tensor product** of vector spaces:

Definition 3.10. The **tensor product** of two K-vector spaces V, W is a pair, consisting of a K-vector space $V \otimes W$ and a bilinear map

$$\begin{array}{cccc} \kappa : & V \times W & \to & V \otimes W \\ & (v, w) & \mapsto & v \otimes w \end{array}$$

with the following universal property: for any K-*bilinear* map

$$\alpha : V \times W \to X$$

there exists a unique *linear* map $\tilde{\alpha} : V \otimes W \to X$ such that

$$\alpha = \tilde{\alpha} \circ \kappa .$$

As a diagram:

$$\begin{tikzcd} V \times W \arrow[r, "\kappa"] \arrow[dr, "\alpha"'] & V \otimes W \arrow[d, dashed, "\exists! \tilde{\alpha}"] \\ & X \end{tikzcd}$$

The existence of bilinear maps allows one to define a $\mathfrak{V}$-tensor product, also called a **fusion product** (see [5] for a summary and [30, 32] for an exhaustive discussion).

Definition 3.11. The fusion product is characterised by a direct generalisation of the universal property for the tensor product over a ring: for any pair of modules M_1 and M_2 the fusion product is a module $M_1 \otimes_{\mathfrak{V}} M_2$ together with an intertwining operator

$$\mathcal{Y}^{M_1,M_2} \colon M_1 \otimes M_2 \to M_1 \otimes_{\mathfrak{V}} M_2\{z\}[\log(z)]$$

such that for any module X and intertwining operator $\mathcal{I} \colon M_1 \otimes M_2 \to X\{z\}[\log(z)]$ there exists a unique module morphism $\phi \colon M_1 \otimes_{\mathfrak{V}} M_2 \to X$ such that the diagram

$$\begin{tikzcd} M_1 \otimes M_2 \arrow[r, "\mathcal{Y}^{M_1,M_2}"] \arrow[dr, "\mathcal{I}"'] & M_1 \otimes_{\mathfrak{V}} M_2\{z\}[\log(z)] \arrow[d, dashed, "\exists! \phi"] \\ & X\{z\}[\log(z)] \end{tikzcd} \tag{17}$$

commutes.

Remarks 3.12.
1. Being defined through a universal property, fusion products are unique if they exist. Moreover, the tensor product is also defined on morphisms.

2. We introduce the notion
$$\mathrm{I}\binom{M_3}{M_1, M_2}$$
for the vector space of intertwiners from $M_1 \otimes M_2$ to M_3. The reader is invited to think about them as **invariant tensors**,

3. As a consequence, well chosen categories of modules (in particular, a morphism ϕ making (17) commutative must exist within the category for every pair of objects) are furnished with the structure of a monoidal category, and even of a balanced braided monoidal category. This structure can be characterised as follows:

   - The tensor unit is given by the VOA $\mathfrak{V}$. The left unit constraint ℓ identifies the intertwining operator $\mathcal{Y}^{\mathfrak{V},M}$ with the action Y^M of $\mathfrak{V}$ on M, that is,

$$\ell_M(\mathcal{Y}^{\mathfrak{V},M}(a,z)\,m) = Y^M(a,z)m \qquad \text{for} \;\; a\in\mathfrak{V}\,,\;\; m\in M\,. \tag{18}$$

   - The associativity constraint A corresponds to identifying compositions of intertwining operators:

$$A_{M_1,M_2,M_3}(\mathcal{Y}^{M_1,M_2\otimes_{\mathfrak{V}}M_3}(m_1,z_1)\mathcal{Y}^{M_2,M_3}(m_2,z_2)m_3) \\ = \mathcal{Y}^{M_1\otimes_{\mathfrak{V}}M_2,M_3}(\mathcal{Y}^{M_1,M_2}(m_1,z_1{-}z_2)m_2,z_2)m_3\,. \tag{19}$$

     Geometrically, thinking of the variables z_i as complex coordinates, the left hand side is expanded in a domain for which the insertion point z_2 of M_2 is close to that of M_3, i.e. 0, while the right hand side is in a domain where M_1 (inserted at z_1) is close to M_2. The existence of associator isomorphisms satisfying pentagon equations is the main obstruction for a chosen category of $\mathfrak{V}$-modules to be monoidal. (The map A_{M_1,M_2,M_3} depends on the insertion points z_1 and z_2. The actual categorical associator is a certain limit; for details see [5, Sect. 3.3]. For notational simplicity we suppress this issue.)

   - The braiding isomorphisms c correspond to identifying the intertwining operator $\mathcal{Y}^{M_1,M_2}$ at z with the intertwining operator $\mathcal{Y}^{M_2,M_1}$ transported to $-z$, that is,

$$c_{M_2,M_1}(\mathcal{Y}^{M_2,M_1}(m_2,z)\,m_1) = \mathrm{e}^{zL_{-1}}\mathcal{Y}^{M_1,M_2}(m_1,-z)\,m_2\,. \tag{20}$$

     The geometric intuition for these isomorphisms is the exchange of the location of the modules M_1 and M_2: Multiplying z by -1 is a rotation of z (where M_1 is located) by the angle π around 0 (where M_2 is located), while the L_{-1}-exponential is a translation of both points by $-z$, so that altogether the locations of the two points are switched. (A choice of logarithm needs to be made for -1 to distinguish between clockwise and counter-clockwise rotations.)

   - There is a twist isomorphism θ given by $\theta_M = \exp(2\pi i L_0)|_M$, which is balanced with respect to the braiding, that is,

$$\theta_{M_1\otimes_{\mathfrak{V}}M_2} = c_{M_2,M_1}\circ c_{M_1,M_2}\circ(\theta_{M_1}\otimes_{\mathfrak{V}}\theta_{M_2})\,. \tag{21}$$

     The twist thus defines a balanced structure on the category. In particular, if the category is in addition rigid, then the twist defines a ribbon structure.

Definition 3.13. 1. Let $\mathcal{C}$ be a category and $\otimes : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ a functor, called a **tensor product**.

Note that this associates to any pair (V,W) of objects an object $V\otimes W$ and to any pair of morphisms (f,g) a morphism $f\otimes g$ with source and target given by the tensor products of the source and target objects. In particular, $\mathrm{id}_{V\otimes W} = \mathrm{id}_V\otimes\mathrm{id}_W$ and for composable morphisms

$$(f'\otimes g')\circ(f\otimes g) = (f'\circ f)\otimes(g'\circ g)\;.$$

2. A **monoidal category** or **tensor category** consists of a category $(\mathcal{C}, \otimes)$ with tensor product, an object $\mathbb{I} \in \mathcal{C}$, called the **tensor unit**, and a natural isomorphism, called the **associator**,

$$a : \otimes(\otimes \times \ \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)\,.$$

of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Explicitly, this means that for any trupe U, V, W of objects in $\mathcal{C}$, we have isomorphisms

$$a_{U,V,W} : \quad (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

compatible with morphisms in the category as well as natural isomorphisms

$$r : \mathrm{id} \otimes \mathbb{I} \to \mathrm{id} \quad \text{ and } \quad l : \mathbb{I} \otimes \mathrm{id} \to \mathrm{id}$$

called **unit constraints** such that the following axioms hold:

- The **pentagon axiom**: for all quadruples of objects $U, V, W, X \in \ \mathrm{Obj}(\mathcal{C})$ the following diagram commutes

$$\begin{tikzcd}
 & (U \otimes V) \otimes (W \otimes X) \arrow[dr, "a_{U,V,W\otimes X}"] & \\
((U \otimes V) \otimes W) \otimes X \arrow[ur, "a_{U\otimes V,W,X}"] \arrow[d, "a_{U,V,W}\otimes \mathrm{id}_X"'] & & U \otimes (V \otimes (W \otimes X)) \\
(U \otimes (V \otimes W)) \otimes X \arrow[rr, "a_{U,V\otimes W,X}"'] & & U \otimes ((V \otimes W) \otimes X) \arrow[u, "\mathrm{id}_U \otimes a_{V,W,X}"']
\end{tikzcd}$$

- The **triangle axiom**: for all pairs of objects $V, W \in \ \mathrm{Obj}(\mathcal{C})$ the following diagram commutes

$$\begin{tikzcd}
(V \otimes \mathbb{I}) \otimes W \arrow[rr, "a_{V,\mathbb{I},W}"] \arrow[dr, "r_V \otimes \mathrm{id}_W"'] & & V \otimes (\mathbb{I} \otimes W) \arrow[dl, "\mathrm{id}_V \otimes l_W"] \\
 & V \otimes W &
\end{tikzcd}$$

Definition 3.14. 1. A **commutativity constraint** for a tensor category $(\mathcal{C}, \otimes)$ is a natural isomorphism

$$c : \otimes \to \otimes^{opp}$$

of functors $\mathcal{C} \times \mathcal{C} \ \to \ \mathcal{C}$. Explicitly, we have for any pair (V, W) of objects of $\mathcal{C}$ an isomorphism

$$c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$$

such that for all morphisms $V \xrightarrow{f} V'$ and $W \xrightarrow{g} W'$ the diagrams

$$\begin{tikzcd}
V \otimes W \arrow[r, "c_{V,W}"] \arrow[d, "f\otimes g"'] & W \otimes V \arrow[d, "g \otimes f"] \\
V' \otimes W' \arrow[r, "c_{V',W'}"] & W' \otimes V'
\end{tikzcd}$$

commute.

2. Let $\mathcal{C}$ be, for simplicity, a *strict* tensor category. A **braiding** is a commutatitivity constraint such that for all objects U, V, W the compatibility relations with the tensor product

$$\begin{array}{rcl} c_{U\otimes V,W} & = & (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}) \\ c_{U,V\otimes W} & = & (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W) \end{array}$$

hold.

If the category is not strict, the following two **hexagon axioms** involving also the associators have to hold:

$$\begin{tikzcd} & U\otimes(V\otimes W) \arrow[r, "c_{U,V\otimes W}"] & (V\otimes W)\otimes U \arrow[dr, "a_{V,W,U}"] & \\ (U\otimes V)\otimes W \arrow[ur, "a_{U,V,W}"] \arrow[dr, "c_{U,V}\otimes \mathrm{id}_W"'] & & & V\otimes(W\otimes U) \\ & (V\otimes U)\otimes W \arrow[r, "a_{V,U,W}"] & V\otimes(U\otimes W) \arrow[ur, "\mathrm{id}_V\otimes c_{U,W}"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} & (U\otimes V)\otimes W \arrow[r, "c_{U\otimes V,W}"] & W\otimes(U\otimes V) \arrow[dr, "a^{-1}_{W,U,V}"] & \\ U\otimes(V\otimes W) \arrow[ur, "a^{-1}_{U,V,W}"] \arrow[dr, "\mathrm{id}_U\otimes c_{V,W}"'] & & & (W\otimes U)\otimes V \\ & U\otimes(W\otimes V) \arrow[r, "a^{-1}_{U,W,V}"] & (U\otimes W)\otimes V \arrow[ur, "c_{U,W}\otimes \mathrm{id}_V"'] & \end{tikzcd}$$

3. A **braided tensor category** is a tensor category together with the structure of a braiding.

4. With c_{UV}, also c^{-1}_{VU} is a braiding. If the identity $c_{U,V} = c^{-1}_{V,U}$ holds, the braided tensor category is called **symmetric**.

An example of a braided category are Drinfeld centers of monoidal categories. The notion is a categorification of the notion of a center of an associative algebra which is a commutative algebra.

Example 3.15. 1. We consider a category $\mathcal{Z}(\mathcal{C})$ whose objects are pairs $(V, c_{-,V})$ consisting of an object V of $\mathcal{C}$ and a natural isomorphism $c_{-,V}: \; -\otimes V \xrightarrow{\sim} V \otimes -$, called a **half-braiding** for V, i.e. isomorphisms for all $X \in \mathcal{C}$

$$c_{X,V}: \quad X\otimes V \to V \otimes X$$

natural in the sense that that for any morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd} X\otimes V \arrow[r, "c_{X,V}"] \arrow[d, "f\otimes \mathrm{id}_V"'] & V\otimes X \arrow[d, "\mathrm{id}_V\otimes f"] \\ Y\otimes V \arrow[r, "c_{Y,V}"'] & V\otimes Y \end{tikzcd}$$

commutes which obey the additional requirement that for all objects X, Y of $\mathcal{C}$ we have

$$c_{X\otimes Y,V} = (c_{X,V}\otimes \mathrm{id}_Y)\circ(\mathrm{id}_X \otimes c_{Y,V}) \; . \qquad \text{[Hex]}$$

2. A morphism $(V, c_{-,V}) \to (W, c_{-,W})$ in $\mathcal{Z}(\mathcal{C})$ is a morphism $f : V \to W$ in $\mathcal{C}$ with the property that for all objects X of $\mathcal{C}$ we have

$$(f \otimes \mathrm{id}_X) \circ c_{X,V} = c_{X,W} \circ (\mathrm{id}_X \otimes f) \ . \qquad (**)$$

It is clear that the identity id_V in $\mathcal{C}$ is a morphism in $\mathcal{Z}(\mathcal{C})$ and that if f, g are morphisms in $\mathcal{Z}(\mathcal{C})$ that are composable in $\mathcal{C}$, then $f \circ g$ is a morphism in $\mathcal{Z}(\mathcal{C})$. Thus $\mathcal{Z}(\mathcal{C})$ is a category with composition and identities inherited from $\mathcal{C}$.

3. Then the category $\mathcal{Z}(\mathcal{C})$ has a natural structure of a strict braided tensor category with

   (a) Monoidal unit $(\mathbb{I}, \mathrm{id}_{\mathbb{I}})$.

   (b) The tensor product of two objects $(V, c_{-,V})$ and $(W, c_{-,W})$ in $\mathcal{Z}(\mathcal{C})$ is given by

   $$(V, c_{-,V}) \otimes (W, c_{-,W}) := (V \otimes W, c_{-,V\otimes W}) \ .$$

   Here, given two objects $(V, c_{-,V})$ and $(W, c_{-,W})$ in $\mathcal{Z}(\mathcal{C})$, we define for any object $X \in \mathcal{C}$ the morphism

   $$c_{X,V\otimes W} : \quad X \otimes V \otimes W \to V \otimes W \otimes X$$

   by

   $$c_{X,V\otimes W} := (\mathrm{id}_V \otimes c_{X,W}) \circ (c_{X,V} \otimes \mathrm{id}_W) \ . \qquad (*)$$

   (c) The braiding on $\mathcal{Z}(\mathcal{C})$ is given by

   $$c_{V,W} : \quad (V, c_{-,V}) \otimes (W, c_{-,W}) \to (W, c_{-,W}) \otimes (V, c_{-,V}) \ .$$

   This braided monoidal category is called the **Drinfeld center** of the monoidal category $\mathcal{C}$.

There are a number of additional conditions that $\mathfrak{V}$ or its chosen of category of modules may satisfy, leading to the existence of additional structure. For example, if the category is closed under taking gradewise duals (also known as contragredient modules), then it is a **ribbon Grothendieck Verdier category** (a type of monoidal category with a notion of duality that can be weaker than rigidity).

Proposition 3.16 (Huang-Lepowsky-Zhang [31, Part I, Theorem 2.34])**.** Let $A \leq B$ be abelian groups, $\mathfrak{V}$ an A-graded VOA, let M be a B-graded weak $\mathfrak{V}$-module and define the vector spaces

$$M' = \bigoplus_{b\in B, h\in\mathbb{C}} \left(M_h^{(\beta)}\right)^*, \qquad \left(M_h^{(\beta)}\right)^* = \mathrm{Hom}_{\mathbb{C}}(M_h^{(\beta)}, \mathbb{C}) \quad . \tag{22}$$

If M is strongly B-graded, then the canonical linear isomorphisms identifying a finite dimensional vector space with its double dual extends to a canonical linear isomorphism $M \cong M''$ of bigraded vector spaces. If, in addition, M is discretely strongly B-graded, then M' is also a discretely strongly B-graded with field map $Y_{M'}$ uniquely characterised by

$$\langle Y_{M'}(v, z)\phi, m\rangle = \langle \phi, Y_M^{opp}(v, z)m\rangle, \qquad v \in \mathfrak{V},\ \phi \in M', m \in M, \tag{23}$$

where Y_M^{opp} is the **opposed field map**

$$Y_M^{opp}(v, z) = Y_M\left(\mathrm{e}^{zL_1}\left(-z^{-2}\right)^{L_0} v, z^{-1}\right). \tag{24}$$

The module M' is called the **contragredient** of M. Opposing the field map is involutive, that is, $Y_M^{opp\, opp} = Y_M$, hence the canonical linear isomorphism $M \cong M''$ above is an isomorphism of V-modules.

It is important to notice that the opposed field map (and thus the GV structure) **depends on the choice of conformal structure**.

The contragredient dual obeys the following relation:

Theorem 3.17 (Huang-Lepowsky-Zhang [31, Part II Proposition 3.46])**.** Let M_1, M_2, M_3 be strongly graded generalised modules over some VOA $\mathfrak{V}$. Then there exists a natural linear isomorphism

$$A : \mathrm{I}\binom{M_3}{M_1, M_2} \to \mathrm{I}\binom{M_2'}{M_1, M_3'} ,$$

which on any intertwining operator $\mathcal{Y} \in \mathrm{I}\binom{M_3}{M_1, M_2}$ evaluates as

$$\langle A(\mathcal{Y})\,(m_1, x)\, m_3', m_2\rangle_{M_2} = \langle m_3', \mathcal{Y}\left(\mathrm{e}^{xL_1}\mathrm{e}^{\mathrm{i}\pi L_0}\left(x^{-L_0}\right)^2 m_1, x^{-1}\right) m_2\rangle_{M_3}, \qquad m_1 \in M_1,\ m_2 \in M_2,\ m_3' \in M_3', \tag{25}$$

where the subscript indicates which module the pairings are evaluated in.

Definition 3.18. A **GV category** is a monoidal category $\mathcal{C}$, together with a distinguished object $K \in \mathcal{C}$, called the **dualising object** satisfying the following conditions.

1. For any object $Y \in \mathcal{C}$, the contravariant functor $\mathrm{Hom}(- \otimes Y, K)$ is representable, that is, there exists an object $DY \in \mathcal{C}$ such that there is a natural isomorphism

   $$\mathrm{Hom}(- \otimes Y, K) \cong \mathrm{Hom}(-, DY). \tag{26}$$

   By Yoneda's Lemma there therefore exists a unique (up to natural isomorphism) contravariant functor D, called *the dualising functor*, which assigns to every $Y \in \mathcal{C}$ the representing object DY, that is $D(Y) = DY$.

2. The contravariant functor D above is an anti-equivalence.

Theorem 3.19. Let $\mathfrak{V}$ be a VOA and $\mathcal{C}$ a choice of category of $\mathfrak{V}$-modules which contains $\mathfrak{V}$ as an object, is closed under taking contragredients and which satisfies the tensor product theory of Huang-Lepowsky-Zhang. Then $\mathcal{C}$ is a ribbon GV category category with dualising object $\mathfrak{V}'$ (the contragredient of the VOA as a module over itself), dualising functor given by the taking of contragredients, and with twist $\theta = \mathrm{e}^{2\pi\mathrm{i}L_0}$.

In a pivotal GV category, we **invariant tensors with cyclic invariance**.

In the rest of this lecture, we will assume that we have the stronger property that the category $\mathcal{C}$ is even rigid. In particular, the dualizing object equals the monoidal unit.

Definition 3.20. 1. Let $\mathcal{C}$ be a tensor category. An object V of $\mathcal{C}$ is called **right dualizable**, if there exists an object $V^\vee \in \mathcal{C}$ and morphisms

   $$b_V : \mathbb{I} \to V \otimes V^\vee \quad \text{and} \quad d_V : V^\vee \otimes V \to \mathbb{I}$$

   such that

   $$\begin{aligned} r_V \circ (\mathrm{id}_V \otimes d_V) \circ a_{V,V^\vee,V} \circ (b_V \otimes \mathrm{id}_V) \circ l_V^{-1} &= \mathrm{id}_V \\ l_{V^\vee} \circ (d_V \otimes \mathrm{id}_{V^\vee}) \circ a^{-1}_{V^\vee,V,V^\vee} \circ (\mathrm{id}_{V^\vee} \otimes b_V) \circ r^{-1}_{V^\vee} &= \mathrm{id}_{V^\vee} \end{aligned}$$

   Such an object $V^\vee$ is called a **right dual** to V.

   The morphism d_V is called an **evaluation**, the morphism b_V a **coevaluation**.

2. A monoidal category is called **right-rigid** or **right-autonomous**, if every object has a right dual.

3. A **left dual** to V is an object ${}^\vee V$ of $\mathcal{C}$, together with two morphisms

$$\tilde{b}_V : \mathbb{I} \to {}^\vee V \otimes V \quad \text{and} \quad \tilde{d}_V : V \otimes {}^\vee V \to \mathbb{I}$$

such that analogous equations hold. A left-rigid or left autonomous category is a monoidal category in which every object has a left dual.

4. A monoidal category is **rigid** or **autonomous**, if it is both left and right rigid or autonomous.

Remark 3.21. In a GV category, we also have evaluation morphisms

$$G(Y) \otimes Y \to K \quad \text{and} \quad Y \otimes G^{-1}(Y) \to K$$

There are, however, not necessarily coevaluations (for the $\otimes$ tensor product.)

Rigidity is a finiteness conditions: in the monoidal category of vector spaces, precisely the finite-dimensional vector spaces are rigid. We will impose further finiteness conditions:

Definition 3.22. 1. Let K be a field. A K-linear category $\mathcal{C}$ is called a **finite category**, if

(a) $\mathcal{C}$ has finite-dimensional spaces of morphisms.

(b) Every object of $\mathcal{C}$ has finite length, i.e. for any object $c \in \mathcal{C}$ there exists a finite filtration

$$0 = c_0 \subset c_1 \subset c_1 \subset \ldots \subset c_n = c$$

by subobjects such that the quotient object c_i/c_{i-1} is a simple object.

(c) $\mathcal{C}$ has enough projectives, i.e. every simple object has a projective cover. (A projective cover of an object $c \in \mathcal{C}$ is a projective object $p(c) \in \mathcal{C}$, together with an epimorphism $\pi : p(c) \to c$ such that if $g : p' \to c$ is an epimorphism from a projective object p' to c, then there exists an epimorphism $h : p' \to p(c)$ such that $\pi \circ h = g$.

(d) There are finitely many isomorphism classes of simple objects.

A K-linear category is finite, if and only if it is equivalent to the category A-mod of finite-dimensional A-modules over a finite-dimensional K-algebra.

2. A **finite tensor category** is a finite rigid monoidal linear category.

3. A semisimple finite tensor category is called a **fusion category**.

Semisimplicity means that any object is a direct sum of simple objects. The category of complex finite-dimensionak representations of a finite group is semisimple. Interesting vertex algebras are not necessarily semisimple.

Remark 3.23. The braiding in conformal field theory is special: the category is **factorizable**:

1. Suppose that A and B are two algebras over the same field K. Then $A \otimes B$ is a K-algebra as well. The **Deligne product** of two finite abelian categories is defined such that

$$A \otimes B\text{-mod} \cong A\text{-mod} \boxtimes B\text{-mod} \ .$$

It can be characterized by a universal property for right exact functors A-mod $\times$ B-mod $\to$ X where X is any finite category. For details, we refer to [9, section 5].

2. Let $\mathcal{C}$ be a braided tensor category. Using the braiding on $\mathcal{C}$ as a half-braiding gives a functor
$$\begin{array}{rcl} \mathcal{C} & \to & \mathcal{Z}(\mathcal{C}) \\ V & \mapsto & (V, c_{-V}) \end{array}$$
which is obviously a braided monoidal functor.

3. Taking the inverse braiding
$$c^{\mathrm{revd}}_{U,V} := c^{-1}_{V,U}$$
on the same monoidal category, gives another structure of braided tensor category $\mathcal{C}^{\mathrm{revd}}$. We get another functor
$$\begin{array}{rcl} \mathcal{C}^{\mathrm{revd}} & \to & \mathcal{Z}(\mathcal{C}) \\ V & \mapsto & (V, c^{\mathrm{revd}}_{-V}) \end{array}$$
which is again a braided monoidal functor.

4. Altogether, we obtain a braided monoidal functor
$$\mathcal{C}^{\mathrm{revd}} \boxtimes \mathcal{C} \to \mathcal{Z}(\mathcal{C}) \ .$$

5. Suppose that $\mathcal{C}$ is the category of representations of a quasi-triangular Hopf algebra (H, R). Then (H, R) is factorizable, if and only if the functor $\mathcal{C}^{\mathrm{revd}} \boxtimes \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ is an equivalence of braided monoidal categories.

6. It can be shown that for any tensor category $\mathcal{C}$ the Drinfeld center $\mathcal{Z}(\mathcal{C})$ is factorizable [10, Proposition 8.6.3].

Definition 3.24. A **modular tensor category** is a finite ribbon category in which the braiding is non-degenerate in the sense that the braided monoidal functor
$$\mathcal{C}^{\mathrm{revd}} \boxtimes \mathcal{C} \to \mathcal{Z}(\mathcal{C})$$
from remark 3.23.7 is an equivalence.

Remark 3.25. A modular tensor category is **pivotal**, i.e. there is a monoidal natural isomorphism
$$\omega : \quad \mathrm{id}_{\mathcal{C}} \to ?^{\vee\vee} \ .$$
A right rigid monoidal category together with a choice of pivotal structure is called a **pivotal category**.

If $\mathfrak{V}$ satisfies a technical condition called C_2-cofiniteness, then there is a natural choice of category (called admissible modules) that is finite, i.e. it has finitely many simple objects, every object has finite length, and all morphism spaces are finite-dimensional.

Proposition 3.26. Finally, if $\mathfrak{V}$ is C_2-cofinite and if the category $\mathrm{Rep}(\mathfrak{V})$ of admissible modules is semisimple, then it is a modular fusion category [29].

Definition 3.27. A VOA having this property is called a **rational VOA.**

Similar results yielding modular fusion categories can also be obtained in the framework of nets of observables [35].

Remarks 3.28. 1. A striking feature of rational VOAs is the behaviour of the characters of their modules under modular transformations. The **character** of a $\mathfrak{V}$-module M is the power series

$$\chi_M(\tau) = \mathrm{tr}_M q^{L_0-c/24} \quad \text{with} \quad q = \exp(2\pi i\tau)\,, \;\; \tau \in \mathbb{H}\,, \tag{27}$$

defined on the complex upper half plane $\mathbb{H}$.

2. For rational $\mathfrak{V}$, the prescriptions

$$S(\chi_M(\tau)) = \chi_M(-\frac{1}{\tau}) \quad \text{and} \quad T(\chi_M(\tau)) = \chi_M(\tau{+}1)\,, \tag{28}$$

called the modular S- and T-transformations, respectively, even give rise (after, if needed, refining the information in the characters by considering dependence on further variables) to an action of the group $\mathsf{SL}(2,\mathbb{Z}) \cong \langle S,T \,|\, S^4 = 1, (ST)^3 = S^2 \rangle$, which is the modular group of the one-punctured torus, on the characters. More specifically, after picking a finite set $\mathrm{I} = \{M_i\}_i$ of representatives for the isomorphism classes of simple modules such that $M_0 = \mathfrak{V}$, the modular transformations (28) are realized by square matrices with complex entries according to

$$S(\chi_{M_i}) = \sum_{j\in\mathrm{I}} S_{i,j}\,\chi_{M_j} \quad \text{and} \quad T(\chi_{M_i}) = T_{i,i}\,\chi_{M_i} \tag{29}$$

for $i \in \mathrm{I}$.

3. On the other hand, as the category $\mathrm{Rep}(\mathfrak{V})$ for a rational VOA $\mathfrak{V}$ is a modular fusion category, via the trace of a double braiding and via the eigenvalues of the ribbon twist, one can define categorical S- and T-matrices which again generate the modular group $\mathsf{SL}(2,\mathbb{Z})$. After a canonical rescaling, these are equal to the modular transformation matrices of the characters defined in (29), and as a consequence the tensor product in $\mathrm{Rep}(\mathfrak{V})$ can be expressed through the Verlinde formula, which depends only on the entries of the modular S-matrix [29]. Thus in the **rational case** the decomposition rules for **fusion products** can be computed from the **modular behaviour of characters**. Determining the modular transformations of characters is significantly more tractable than computing fusion products directly.

   Characters are particular examples of torus chiral one-point correlators (see the section on conformal blocks). Thus their modular behavior is a consequence of the geometry of the curves they are defined on. Accordingly it is believed that the finiteness conditions that make up rationality are not necessary conditions; they are, however, the only fully understood case. Still, various proposals for Verlinde-like formulas beyond rationality have been made, see e.g. [40] and the literature cited there.

4. The information encoded in the modular fusion category $\mathrm{Rep}(\mathfrak{V})$ is often referred to as the **chiral data**, or also Moore-Seiberg data, of a CFT [38]. The **modular data** of the CFT are given by the subset of chiral data consisting of the matrices S and T and the central charge c.

4 Conformal blocks

We now to combine the subjects of the two previous sections, VOAs and complex curves. This will lead us to the central notion of a conformal block.

4.1 Ward identities and conformal blocks

A common theme in quantum field theory is the quest for establishing a correspondence between states and fields. Furthermore, one would like to formulate the theory on a whole class of suitable manifolds. In the present context, this amounts to the idea to associate to every vector $v \in \mathcal{H}_\lambda$ in a VOA-module $\mathcal{H}_\lambda$ a suitable 'field' $\Phi_\lambda(v,p)$ depending on a point p on complex curve X, and to assign, given any complex curve X of genus g and any choice of m pairwise distinct points p_i on X, to an m-fold product of such fields a **'correlator'**

$$\langle \Phi_{\lambda_1}(v_1,p_1) \cdots \Phi_{\lambda_m}(v_m,p_m) \rangle_X \ .$$

As will be seen below, formalizing this physics idea naturally leads to quantities which do not enjoy all properties that one would normally require for the correlators of a local quantum field theory. Accordingly we have put the terms 'field' and 'correlator' in quotation marks.

The **global chiral symmetries** of the CFT that are encoded in a VOA $\mathfrak{V}$ imply linear relations that the 'correlators' must satisfy. These relations are called the chiral **Ward identities**; their solutions form vector spaces $B_{\vec{\lambda}}(X,\vec{p})$, called the spaces of **conformal blocks**. To write the Ward identities in a compact form, we regard a conformal block as a linear form on the vector space tensor product

$$\vec{\mathcal{H}}_{\vec{\lambda}} := \mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_m} \tag{30}$$

of $\mathfrak{V}$-modules that depends on the complex structure of the curve X and on the positions $\vec{p} = (p_1, p_2, \dots, p_m)$ of the field insertions, that is, as a particular vector

$$\beta_{X,\vec{p};\vec{\lambda}} \in \vec{\mathcal{H}}^*_{\vec{\lambda}} \tag{31}$$

in the (algebraic) dual of the tensor product. It depends on the curve $\hat{X}$ and on the positions $\vec{p} = (p_1, p_2, \dots, p_m)$ of the field insertions.

Then the 'correlator' is the scalar

$$\langle \Phi_{\lambda_1}(v_1,p_1) \cdots \Phi_{\lambda_m}(v_m,p_m) \rangle = \beta_{X,\vec{p};\vec{\lambda}}(v_1 \otimes \cdots \otimes v_m) \, . \tag{32}$$

4.2 WZW conformal blocks as invariants

A crucial step in the construction of conformal blocks is to obtain for a given curve X a **'global' variant** of the VOA. For a general vertex operator algebra, this has been analysed in terms of bundles of VOAs in Chapter 6 of [15], and in terms of 'chiral Lie algebras' in [39, 6]. For the present purposes, we content ourselves to convey the idea by spelling it out for the arguably simplest class of CFTs, namely the one of chiral **WZW models**, as expounded in [2].

1. For these the VOA $\mathfrak{V}$ is generated by currents $J^a(z)$ whose zero modes J^a_0 span a finite-dimensional complex simple Lie algebra $\bar{\mathfrak{g}}$, and a $\mathfrak{V}$-module $\mathcal{H}_\lambda$ carries an action of the untwisted affine Lie algebra $\mathfrak{g} = \bar{\mathfrak{g}}^{(1)}$ associated with $\bar{\mathfrak{g}}$.

2. In this situation we can combine the Lie algebra $\bar{\mathfrak{g}}$ with the commutative associative algebra $\mathcal{F}_{X,\vec{p}}$ of holomorphic functions on $X \setminus \vec{p}$ that at each of the points p_i have at most a finite order pole, which yields a Lie algebra $\bar{\mathfrak{g}} \otimes \mathcal{F}_{X,\vec{p}}$ of $\bar{\mathfrak{g}}$-valued functions. This **global Lie algebra** captures the global aspects of the VOA. (For analogous global Lie algebras for general VOAs see [15, Ch. 19.4] and [6, Sect. 3].)

3. An action of $\bar{\mathfrak{g}} \otimes \mathcal{F}_{X,\vec{p}}$ on the tensor product $\vec{\mathcal{H}}_{\vec{\lambda}}$ is obtained as follows: For each i choose a local holomorphic coordinate ξ_i around the insertion point p_i and expand $f \in \mathcal{F}_{X,\vec{p}}$ in a Laurent series
$$f^{(i)}(\xi_i) = \sum_{n \gg -\infty} a_n^{(i)}\, \xi_i^n . \tag{33}$$
Then associate to the element $J_0^a \otimes f$ of $\bar{\mathfrak{g}} \otimes \mathcal{F}_{X,\vec{p}}$ the element $\sum_{n \gg -\infty} a_n^{(i)} J_n^a$ of the affine Lie algebra $\mathfrak{g}^{(1)}$, acting on the $\mathfrak{V}$-module $\mathcal{H}_{\lambda_i}$. The action of $J_0^a \otimes f$ on $\vec{\mathcal{H}}_{\vec{\lambda}}$ is then defined as the sum
$$\sum_{i=1}^{m} \mathbf{1} \otimes \cdots \otimes \Big(\sum_{n \gg -\infty} a_n^{(i)} J_n^a \Big) \otimes \cdots \otimes \mathbf{1} , \tag{34}$$
where in the ith summand the non-trivial tensor factor is located at the ith position.

4. The space of conformal blocks can now be defined as the vector space of invariants for the induced action on the dual space $\vec{\mathcal{H}}^*_{\vec{\lambda}}$:
$$B_{\vec{\lambda}}(X, \vec{p}) := (\vec{\mathcal{H}}^*_{\vec{\lambda}})_0 . \tag{35}$$

5. The action of $\bar{\mathfrak{g}} \otimes \mathcal{F}_{X,\vec{p}}$ depends on the choice of the local coordinates ξ_i, and thus the space $B_{\vec{\lambda}}(X, \vec{p})$ of conformal blocks depends on that choice as well. However, for WZW models, and, more generally, for C_2-cofinite rational VOAs, the Virasoro algebra, obtained from the conformal structure of the VOA, provides a natural action of the group of changes of local coordinates on the VOA-modules, so that the conformal blocks transform covariantly under such choices.

Remarks 4.1. 1. In all rational chiral CFTs, in accordance with the Verlinde formula, the space $V_{\vec{\lambda}}$ is a **finite-dimensional** vector space; it is a subspace of $(\vec{\mathcal{H}}_{\vec{\lambda}})^*$ that depends on the labels $\vec{\lambda}$, on the curve $\hat{X}$, and the positions $\vec{p}$ of the insertion points.

2. These vector spaces also turn out to be the spaces of physical states of certain three-dimensional topological field theories (TFTs), the **Chern–Simons** theories. A mathematically rigorous construction is the **Reshetikhin-Turaev construction**. It must be emphasized, though, that this relation between TFT and chiral CFT does by no means imply that the two theories are equivalent. Indeed, already the respective spaces of physical states are rather different. For chiral CFT, the state space is provided by the infinite-dimensional graded vector spaces $\mathcal{H}_\lambda$ that underly the VOA-modules.

3. The spaces of conformal blocks are of independent interest in mathematics; they provide nonabelian generalizations of **theta functions**, i.e. they are naturally isomorphic to spaces of sections over moduli spaces of (stable equivalence classes of holomorphic) G-bundles over $\hat{X}$, where G is the connected and simply connected complex Lie group whose Lie algebra is $\bar{\mathfrak{g}}$. For more details, we refer to [41].

It is again time for an example.

Example 4.2. 1. Take the sphere $\hat{X} = \mathbb{CP}^1$, with the usual (quasi-)global coordinate z, and two insertions at $z_1 = 0$ and $z_2 = \infty$. As local coordinates, we choose $\xi_1 = z$ and $\xi_2 = 1/z$. The algebra $\mathcal{F}$ is in this case the algebra of all polynomials in z and z^{-1}, $\mathcal{F} = \langle z^n,\, n{\in}\mathbb{Z}\rangle$. The element $J^a \otimes z^n$ acts via $J_n^a {\otimes} \mathbf{1} + \mathbf{1} {\otimes} J_{-n}^a$. The two-point blocks are then functionals $\beta \in (\mathcal{H}_\lambda {\otimes} \mathcal{H}_\mu)^*$ with the property that
$$\beta \circ (J_n^a \otimes \mathbf{1} + \mathbf{1} \otimes J_{-n}^a) = 0 \tag{36}$$

for all $a = 1, 2, \ldots, \dim \bar{\mathfrak{g}}$ and all $n \in \mathbb{Z}$. One can show that non-vanishing functionals β obeying (36) exist only if λ and μ are conjugate $\bar{\mathfrak{g}}$-weights, $\mu = \lambda^+$.

2. Note that these linear functionals are in the *algebraic* dual of $\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^+}$; the Hilbert space dual is too small to contain them. Still, one abuses bra-ket notation and likes to write them as vectors of $\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^+}$. In terms of these "vectors" $|B_\lambda\rangle$, formula (36) is written as

$$(J^a_n \otimes \mathbf{1} + \mathbf{1} \otimes J^a_{-n})\, |B_\lambda\rangle = 0\,. \tag{37}$$

3. The quantities $|B_\lambda\rangle$ show up in various circumstances. In the context of conformally invariant boundary conditions they are also known as **Ishibashi states**. The reader should keep in mind that these are nothing but two-point blocks on the sphere. It is sometimes possible to write down the Ishibashi state $|B_\lambda\rangle$ explicitly; e.g. for theories based on a free boson, it can be written as a **generalized coherent state**,

$$|B_\lambda\rangle = \exp\big(-\sum_{n=1}^{\infty} b_{-n} \otimes b_{-n} \big)\, v_\lambda\,,$$

where v_λ is the highest weight state in the tensor product of Fock spaces. Such a realization is helpful when one is interested in calculating one-point functions on a disc explicitly. It is, however, *not* necessary to know such an explicit realization if one wants to determine the spectrum of boundary fields. In this case, it is sufficient to know how $|B_\lambda\rangle$ behaves under factorization (see below). The crucial information that allows to calculate concretely with boundary states is the following identity that relates two-point blocks and characters:

$$\chi_\lambda(2\tau) = \langle B_\lambda |\, \mathrm{e}^{2\pi\mathrm{i}\tau(L_0 \otimes \mathbf{1} + \mathbf{1} \otimes L_0 - c/12)}\, |B_\lambda\rangle\,.$$

4.3 Bundles of conformal blocks

Next we describe what happens if the shape of the complex curve X and the location of the insertion points $\vec{p}$, which have so far been kept fixed, are varied. These data specify a point in the moduli space $\mathcal{M}_{g,m}$ of curves of genus g with m distinct marked points, and together with choices of local coordinates, a point in a larger moduli space $\widetilde{\mathcal{M}}_{g,m}$.

Proposition 4.3. 1. The vector spaces $B_{\vec{\lambda}}(X, \vec{p})$ define a quasi-coherent sheaf on $\widetilde{\mathcal{M}}_{g,m}$.

2. If the VOA is rational and C_2-cofinite, then [6, Sect. 8.7] the sheaf is coherent and descends to the moduli space $\mathcal{M}_{g,m}$, so that the spaces $B_{\vec{\lambda}}(X, \vec{p})$ fit together into the total space of a vector bundle $\mathcal{B}_{g,\vec{\lambda}}$ over the moduli space $\mathcal{M}_{g,m}$; hereby in particular the dependence on choices of local coordinates is removed.

3. Moreover, the vector bundle $\mathcal{B}_{g,\vec{\lambda}}$ is equipped with the additional structure of a projectively flat connection with regular singularities, which is called the **Knizhnik-Zamolodchikov connection**. This can be traced back to the existence, for any VOA-module $\mathcal{H}_\lambda$, of a flat connection $\nabla = \mathrm{d} + L_{-1} \otimes \mathrm{d}z$ on a vector bundle $\widetilde{\mathcal{H}}_\lambda$ over $\mathcal{M}_{g,m}$ whose fibres are given by $\mathcal{H}_\lambda$.

Remark 4.4. The existence of the Knizhnik-Zamolodchikov connection amounts to a projective action of the fundamental group of $\mathcal{M}_{g,m}$, i.e. of the **mapping class group** $\mathrm{Map}_{g,m}$ of the m-punctured curve X, on each fibre, i.e. on the vector space $B_{\vec{\lambda}}(X, \vec{p})$.

Remarks 4.5. 1. The term 'conformal block' is frequently also used for the sheaves of local horizontal sections in the vector bundles $\mathcal{B}_{g,\vec{\lambda}}$. As the bundles $\mathcal{B}_{g,\vec{\lambda}}$ are generically non-trivial, these sections are multivalued functions of the insertion points. Thus, as already pointed out, they are not ordinary functions, in contrast to the correlators – also called *correlation functions* – of a local quantum field theory. Via the Knizhnik-Zamolodchikov connection, the horizontality of the sections translates to a first order differential equation that the conformal blocks must satisfy, called the **Knizhnik-Zamolodchikov equation**.

2. For genus $g=1$ and one insertion of the vacuum Ω, the (orbifold-)fundamental group is $\mathrm{SL}(2,\mathbb{Z})$ [28]. For rational conformal field theories, one obtains a representation of the modular group $\mathrm{SL}(2,\mathbb{Z})$ on a complex vector space of dimension $|I|$. In a natural basis, the generator T, acting on the complex upper half plane H as $T\colon \tau \mapsto \tau{+}1$ of $\mathrm{SL}(2,\mathbb{Z})$, is represented by a unitary diagonal matrix $T_{\lambda,\mu}$, and S, acting on H like

$$S: \quad \tau \mapsto -1/\tau\,,$$

is represented by a unitary symmetric matrix $S_{\lambda,\mu}$. It is an important conjecture that the modular transformations of the characters are the same as those of the one-point conformal blocks on the torus. This is the core of the *Verlinde conjecture*.

3. There is also a kind of converse of the above derivation of the conformal block spaces: From a system of conformal blocks one can construct a VOA by restricting to insertions of vectors that only produce a meromorphic dependence in correlation functions on the sphere. The axioms of a VOA and its modules as well as further structures can then be extracted from the desired properties of correlations functions. In fact, as demonstrated in [24], to this end it is even sufficient to know the values of the linear forms (35) only on certain finite-dimensional vector spaces, which are then recognized as particular subspaces of the VOA.

4.4 Factorisation

In the considerations above, the curve X has implicitly been assumed to be smooth.

Remarks 4.6. 1. However, many arguments still go through if X is instead allowed to be *stable* and thus can possess a mild form of singularities, known as **ordinary double points**. Such a double point p on X can be 'blown up', which results in a smooth curve X' with a projection onto X under which p has two pre-images $p'_\pm$.

2. A further crucial aspect of conformal blocks is **factorisation**, which describes their behaviour in such a situation. At present, this is thoroughly understood only for C_2-cofinite rational VOAs $\mathfrak{V}$. In this case the conformal blocks even form vector bundles over the moduli spaces $\overline{\mathcal{M}}_{g,m}$ of stable pointed curves, the vector spaces of conformal blocks are finite-dimensional.

Theorem 4.7. In this situation, there exist canonical isomorphisms [6, Thm. 8.4.1]

$$g_{X,X'}\colon \quad B_{\vec{\lambda}}(X,\vec{p}) \xrightarrow{\cong} \bigoplus_{\mu\in I} B_{\vec{\lambda}\cup\{\mu,\mu^\vee\}}(X',\vec{p}\cup\{p'_+,p'_-\}) \tag{38}$$

between the spaces of conformal blocks on X and X', where the (finite) summation is over the isomorphism classes of simple objects of $\mathrm{Rep}(\mathfrak{V})$.

Remarks 4.8. 1. This structure tightly links the system of vector bundles $\mathcal{B}_{g,\vec{\lambda}}$ over the moduli spaces $\overline{\mathcal{M}}_{g,m}$ for all possible values of the genus g and the number m of insertion points.

2. The vector spaces $B_{\vec{\lambda}}(X, \vec{p})$, as well as the sheaves of horizontal sections of the bundles $\mathcal{B}_{g,\vec{\lambda}}$, depend functorially on the VOA-modules at the insertion points, i.e. on the objects λ_i of the category $\mathrm{Rep}(\mathfrak{V})$. From a categorical perspective, it is natural to expect that the factorisation isomorphisms (38) can be generalised to classes of non-rational VOAs, with the direct sum replaced by the categorical notion of a **coend** (similarly as in formula (56) below).

3. An important consequence of factorisation is the **Verlinde formula** for rational VOAs, which expresses the rank of the bundles $\mathcal{B}_{g,\vec{\lambda}}$ for any value of g and m through the entries of the matrix S that according to (28) describes the effect of the modular transformation $\tau \mapsto -1/\tau$ on the characters of the VOA-modules $\mathcal{H}_\mu$. For details about the Verlinde formula we refer to [29] and, for the case of WZW models, to [2, 41]. In the special case $g=0$ and $m=3$, one obtains the statement that the matrix S **diagonalises** the **fusion rules** of the CFT, i.e. the multiplicity matrices for the decompositions of the tensor products $\lambda_i \otimes \lambda_j$ into simple objects. (The matrix S also coincides with the matrix of Hopf link invariants obtained in the three-dimensional surgery topological field theory based on the modular tensor category $\mathrm{Rep}(\mathfrak{V})$; that the latter matrix diagonalises the fusion rules follows quite directly [42, Ch. IV.12], irrespectively of the connection with modular transformations.)

4. One consequence of factorization is that one can express the rank of $\mathcal{V}_{\vec{\lambda},g}$ for all values of m and g in terms of the matrix S that we encountered in the description of the action of the modular group. This results in the famous **Verlinde formula**, which reads

$$\operatorname{rank} \mathcal{V}_{\vec{\lambda},g} = \sum_{\mu \in I} |S_{\Omega,\mu}|^{2-2g} \prod_{i=1}^{m} \frac{S_{\lambda_i,\mu}}{S_{\Omega,\mu}} \,. \tag{39}$$

This implies in particular that the ranks of the bundles of conformal blocks are finite.

5. Using the assumption that the matrix S also describes the modular transformations of the characters, for concrete models the matrix S can be computed explicitly with tools from representation theory. For WZW models, S is given by the Kac–Peterson formula. The combination of the Kac–Peterson formula for S with the general Verlinde formula (39) then gives the Verlinde formula in the sense of algebraic geometry (for reviews see [?, 41]).

4.5 The relation with modular functors

For a rational CFT there is a related more algebraic structure, also referred to as conformal blocks, in which the spaces of conformal blocks are regarded just as finite-dimensional vector spaces that are endowed with a representation of the mapping class group, obey factorisation, and depend functorially on the representation-theoretic data at the insertion points. These data are captured by the notion of a **modular functor**. Various versions of this notion have been studied in the literature; roughly speaking, a modular functor is a three-dimensional topological field theory that is only defined on a particular subclass of three-manifolds. For our purposes, and in particular for the construction of CFT correlators further below, we work with the following variant:

Definition 4.9. An (anomaly-free) **open-closed modular functor** is a symmetric monoidal pseudofunctor

$$\mathrm{Bl}\colon \quad \mathrm{Bord}^{\mathrm{or}}_{2,\mathrm{o/c}} \longrightarrow \mathrm{Prof}_{\mathbb{C}} \tag{40}$$

from the symmetric monoidal bicategory $\mathrm{Bord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ to the symmetric monoidal bicategory $\mathrm{Prof}_{\mathbb{C}}$.

Remarks 4.10.

1. The bicategory $\mathrm{Bord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ has compact oriented 1-manifolds as objects, open-closed oriented two-bordisms as 1-morphisms, and isotopy classes of homeomorphisms (relative to boundary parametrisation) as 2-morphisms.

2. The bicategory $\mathrm{Prof}_{\mathbb{C}}$ has (small) $\mathbb{C}$-linear categories as objects, $\mathbb{C}$-linear profunctors as 1-morphisms, and $\mathbb{C}$-linear natural transformations as 2-morphisms (a linear profunctor between linear categories $\mathcal{A}$ and $\mathcal{B}$ is a linear functor $P\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Vect}_{\mathbb{C}}$.) The horizontal compositions in $\mathrm{Prof}_{\mathbb{C}}$ are given by coends.

3. It is widely believed that any rational chiral CFT, with associated modular fusion category $\mathcal{C}$, gives rise to a modular functor $\mathrm{Bl} = \mathrm{Bl}_{\mathcal{C}}$ such that the mapping class group representations are related by a Riemann-Hilbert correspondence to the monodromies of conformal blocks that come from the projectively flat connection on the vector bundles $\mathcal{B}_{g,\vec{\lambda}}$. Accordingly, yet another use of the term conformal blocks refers to the vector spaces that are furnished by a modular functor. **We will use modular functors to keep track of the monodromies of conformal blocks.**

5 Full CFT

We are now finally in a position to address full CFT. The most important point of this section is that the construction of a full CFT from a chiral CFT can be **formulated** in completely **model-independent terms**.

5.1 Consistent systems of correlators

As already stated, a full local conformal field theory is a consistent system of correlators – to be precise, of correlators for the class of world sheets one is considering.

Remarks 5.1.

1. The dependence on the conformal structure of a world sheet is already incorporated in the modular functor (40). Accordingly, from now on by a world sheet we mean a **topological world sheet** $\mathcal{S}$, which has an underlying compact oriented topological 2-manifold $\Sigma_{\mathcal{S}}$ with possibly non-empty boundary.

2. While in a chiral CFT the symmetries are encoded in a vertex operator algebra $\mathfrak{V}$, in a full CFT we deal with the combined action of two VOAs $\mathfrak{V}_L$ and $\mathfrak{V}_R$, which encode holomorphic and anti-holomorphic chiral symmetries, respectively, Accordingly the relevant representation category is $\mathrm{Rep}(\mathfrak{V}_L \otimes_{\mathbb{C}} \mathfrak{V}_R)$, which under suitable finiteness conditions is equivalent to the Deligne product $\mathcal{C}_L \boxtimes \mathcal{C}_R$. This is referred to as the **combination of left and right movers**, or as **holomorphic factorization**. In the following we restrict our attention to rational CFTs for which the left and right movers are governed by the same VOA, such that $\mathcal{C}_L = \mathcal{C}$ and $\mathcal{C}_R = \mathcal{C}^{\mathrm{rev}}$, where $\mathcal{C}^{\mathrm{rev}}$ equals $\mathcal{C}$ as a monoidal category, but has reversed braiding. For a modular tensor category there is a canonical equivalence

$$\Xi\colon \quad \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C}) \tag{41}$$

of ribbon categories between the Deligne product $\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}$ and the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$. (This relationship can be extended to heterotic theories, for which $\mathcal{C}_L$ and $\mathcal{C}_R^{\mathrm{rev}}$ are different [7, Cor. 3.30].) This observation will be the basis for the use of **string net** and **state-sum constructions**.

Definition 5.2. We can then define a full CFT as a **consistent system of correlators**, i.e. as an assignment

$$\mathcal{S} \longmapsto \mathrm{Cor}_{\mathcal{C}}(\mathcal{S}) \in \mathrm{Bl}_{\mathcal{C}}(\mathcal{S}) \tag{42}$$

that specifies for every world sheet $\mathcal{S}$ a correlator $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})$ as an element in the pertinent vector space $\mathrm{Bl}_{\mathcal{C}}(\mathcal{S})$ of conformal blocks.

Consistency of the system means that

- $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})$ is required to be invariant under the action of the mapping class group $\mathrm{Map}(\mathcal{S})$ of the world sheet,
- and that the assignment is compatible with the sewing of world sheets.

(In the presence of defects, $\mathrm{Map}(\mathcal{S})$ is typically a subgroup of the mapping class group $\mathrm{Map}(\Sigma_{\mathcal{S}})$ of the oriented topological surface $\Sigma_{\mathcal{S}}$.)

Remarks 5.3. 1. The mapping class group invariance of a correlator reflects the fact that the corresponding correlation function is a global section of the pertinent bundle of conformal blocks.

2. Given the category $\mathcal{C}$ of chiral data, the problem of constructing a full CFT is of **algebraic nature** and can be solved in a **model-independent way**.

Claims 5.4. 1. Let us suppose for the moment that $\mathcal{C}$ is a fusion category. For oriented world sheets that are allowed to have physical boundaries, but no defects, a consistent system of correlators (subject to the extra requirements of non-degeneracy of the sphere and disk two-point correlators and the uniqueness of the closed state vacuum) is uniquely determined [11, 12] by a **simple special symmetric Frobenius algebra** A internal to the modular fusion category $\mathcal{C}$, i.e. by a unital algebra object in $\mathcal{C}$ that has non-zero quantum dimension, is simple as a bimodule over itself and is endowed with a counital coalgebra structure such that the comultiplication is a morphism of bimodules and is right inverse to the multiplication.

2. For unoriented world sheets, the algebra A must in addition be endowed with a so-called Jandl structure.

3. The category mod-A of A-modules provides the conformal boundary conditions for such a CFT, and the algebra A describes, up to Morita equivalence, the operator product expansions for boundary fields that do not change the boundary condition. Equivalently, such an open-closed full theory corresponds to an indecomposable semisimple $\mathcal{C}$-module category $\mathcal{M}$ of boundary conditions.

We will need the following definition:

Definition 5.5. Let (A, μ, η) be a unital associative algebra in a monoidal category $\mathcal{C}$. A κ-Frobenius structure on A is a pairing $\kappa \in \mathrm{Hom}_{\mathcal{C}}(A\otimes A, \mathbb{I})$ that is invariant (or associative) i.e. satisfies

$$\kappa \circ (\mu \otimes \mathrm{id}_A) = \kappa \circ (\mathrm{id}_A \otimes \mu)\,,$$

and that is **non-degenerate** in the sense that

$$\Phi_\kappa := (\mathrm{id}_{^\vee\! A} \otimes \kappa) \circ (\tilde{b}_A \otimes \mathrm{id}_A) \;\in \mathrm{Hom}(A, {}^\vee\! A)$$

is an isomorphism. Equivalently, a Frobenius structure on an associative algebra can be defined as the structure of a coassociative, counital coalgebra (Δ, ϵ) such that $\Delta : A \to A \otimes A$ is a morphism of A-bimodules.

Claims 5.6. 1. A larger class of world sheets is obtained when one also allows for topological defects, i.e. when one works on stratified manifolds such as (43). Defect conditions for topological defects of arbitrary codimension comprise a pivotal bicategory $\mathcal{F}r(\mathcal{C})$.

2. The objects of $\mathcal{F}r(\mathcal{C})$ are simple symmetric special Frobenius algebras; they serve as labels for the **phases** of the full CFT which live on the two-dimensional strata of the world sheets. The 1- and 2-morphisms of $\mathcal{F}r(\mathcal{C})$ are bimodules and bimodules morphisms, which provide the defect conditions for line and point defects, respectively.

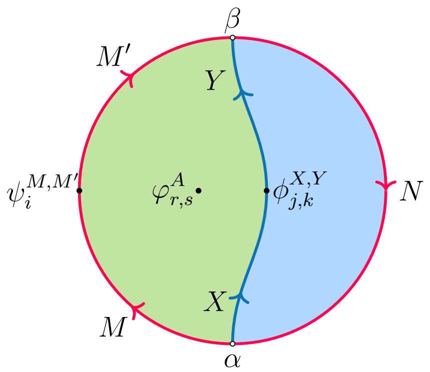
3. The pivotal bicategory of defects can be equivalently realised as the bicategory $\mathcal{C}\text{-}\mathcal{M}\mathrm{od}^{\mathrm{tr}}$ of indecomposable semisimple $\mathcal{C}$-module categories admitting a module trace, module functors and module natural transformations.

5.2 RCFT Correlators from RT TFT and three-manifolds

One way to construct a consistent system of correlators [17, 11] for a rational CFT is to exploit the connection with the three-dimensional topological field theory (TFT) of Reshetikhin-Turaev (RT) type that is associated to the modular fusion category $\mathcal{C}$. In this connection, the state spaces of the TFT functor $\mathrm{RT}_\mathcal{C}$ provide the spaces of conformal blocks of the CFT: this is an instance of the **holographic principle** in quantum field theories and subsumes the duality between Chern-Simons TFTs and Wess-Zumino-Witten models as a special case.

In this approach, called the **TFT-construction**, one considers world sheets $\mathfrak{S}$ with field insertions. An illustrative **example** of such a world sheet is

$\mathfrak{S}_0 \;:=$



(43)

This world sheet $\mathfrak{S}_0$ has the following attributes:

- **two phases**, labeled by Frobenius algebras A and B, respectively, which we also indicate by using two different colors;
- three **physical boundaries**, with boundary conditions given by right A-modules M and M' and a right B-module N;
- two **line defects**, with defect conditions given by A-B-bimodules X and Y,
- and two **point defects**, with defect conditions $\alpha \in \mathrm{Hom}_{\text{mod-}B}(N, M \otimes_A X)$ and $\beta \in \mathrm{Hom}_{\text{mod-}B}(M' \otimes_A Y, N)$.

Finally, $\mathfrak{S}_0$ has three **field insertions**:

1. A **boundary field** $\psi_i^{M,M'}$ which separates the two physical boundaries labeled by M and M'. It has one chiral label $i \in \mathrm{I}(\mathcal{C})$ given by a simple object in $\mathcal{C}$ together with an element of the degeneracy space
$$\mathrm{Hom}_{\text{mod-}A}(i \otimes M, M') \,. \tag{44}$$

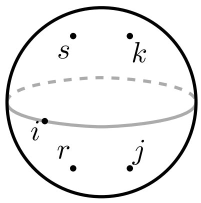
2. A **defect field** $\phi_{j,k}^{X,Y}$ which separates two line defects labeled by X and Y. It has chiral labels $j, k \in \mathrm{I}(\mathcal{C})$ given by two simple objects in $\mathcal{C}$, combined with an element of the degeneracy space
$$\mathrm{Hom}_{A\text{-mod-}B}(j \otimes^- X \otimes^+ k, Y) \,. \tag{45}$$
(The A-B-bimodule structure on $j \otimes^- X \otimes^+ k$ is defined with the help of the braiding of $\mathcal{C}$; for details see e.g. [17].)

3. A **bulk field** $\varphi_{r,s}^{A}$ inserted in the region with phase A. It carries two chiral labels $r, s \in \mathrm{I}(\mathcal{C})$ and an element of the degeneracy space $\mathrm{Hom}_{A\text{-mod-}A}(r \otimes^- A \otimes^+ s, A)$.

Remarks 5.7. 1. As suggested by the form of their degeneracy spaces, a bulk field is a special case of a defect field: it separates two invisible defects, i.e. defect lines labeled by the Frobenius algebra (as a bimodule over itself).

2. As an additional datum, the insertion point p of any field carries an arc-germ, i.e. an equivalence class of curves passing through p (two such curves are equivalent if they coincide in an open neighbourhood containing p). The arc-germ is what remains on the topological world sheet from the germ of local coordinates on the original conformal world sheet.

In the TFT-construction, holomorphic factorization is implemented topologically via the notion of the **double** $\widehat{\Sigma}_{\mathfrak{S}}$ of a world sheet $\mathfrak{S}$: $\widehat{\Sigma}_{\mathfrak{S}}$ is a compact oriented topological surface with marked points (together with arc-germs) that is obtained as a quotient of the orientation bundle over the surface $\Sigma_{\mathfrak{S}}$:
$$\widehat{\Sigma}_{\mathfrak{S}} := \mathrm{Or}(\Sigma)/\!\sim \qquad \text{with} \;\; (p,\mathrm{or}) \sim (p,-\mathrm{or}) \quad \text{for } p \in \partial\Sigma_{\mathfrak{S}}. \tag{46}$$
For instance, the double of the world sheet (43) is given by a sphere with five marked points:

$\widehat{\Sigma}_{\mathfrak{S}_0} \;=$  (47)

The labels of the marked points come from the chiral labels of the field insertions. Boundary points on Σ have a unique pre-image on $\hat{\Sigma}$. Thus, in order to define conformal blocks, we need to attach one chiral label to each boundary insertion. A bulk point, in contrast, has two pre-images on $\hat{\Sigma}$, and hence requires two chiral labels.

The space of conformal blocks for a world sheet $\mathfrak{S}$ with field insertions is given by evaluating the TFT functor on the double of $\mathfrak{S}$, i.e.
$$\mathrm{Bl}_{\mathcal{C}}(\mathfrak{S}) = \mathrm{RT}_{\mathcal{C}}(\widehat{\Sigma}_{\mathfrak{S}}) \;.$$

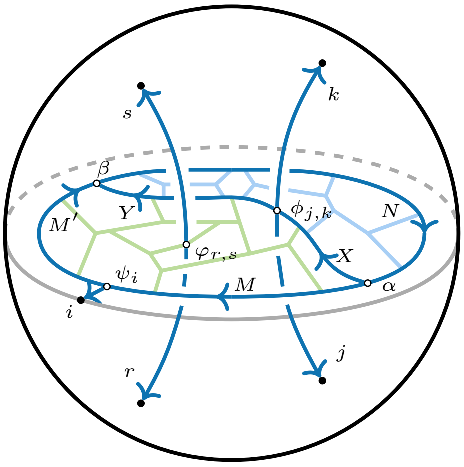
In the TFT-construction, the correlator $\mathrm{Cor}_{\mathcal{C}}(\mathfrak{S})$ is obtained by evaluating $\mathrm{RT}_{\mathcal{C}}$ on a specific three-bordism $M_{\mathfrak{S}}\colon \emptyset \to \widehat{\Sigma}_{\mathfrak{S}}$ with an embedded $\mathcal{C}$-colored ribbon link;

$$\begin{aligned} \mathrm{RT}_{\mathcal{C}}(M_{\mathfrak{S}})\colon \mathrm{RT}_{\mathcal{C}}(\emptyset) = \mathbb{C} \quad &\to \quad \mathrm{RT}_{\mathcal{C}}(\widehat{\Sigma}_{\mathfrak{S}}) \\ 1 \quad &\mapsto \quad \mathrm{Cor}_{\mathcal{C}}(\mathfrak{S}) \end{aligned}$$

This is a holographic description of correlators.

As an illustration, the correlator of the world sheet (43) is obtained from the three-bordism $\emptyset \stackrel{B_3}{\to} S^2$

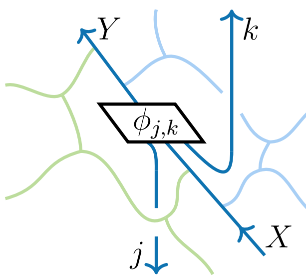
$$M_{\mathfrak{S}_0} \quad = \tag{48}$$



Remarks 5.8. 1. The world sheet $\mathfrak{S}$ is a deformation retract of $M_{\mathfrak{S}}$; it can be seen sitting inside the solid ball.

2. Each of its two-dimensional regions is replaced by a network of ribbons located along a sufficiently fine dual triangulation of the region, having trivalent vertices labeled by structural morphisms of the corresponding Frobenius algebra.

3. Each field insertion is replaced by its associated (bi)module morphism, with the protruding legs colored with the respective chiral labels and attached to the marked points on $\partial M_{\mathfrak{S}} = \widehat{\Sigma}_{\mathfrak{S}}$. For example, zooming in on the defect field insertion $\phi_{j,k} \equiv \phi_{j,k}^{X,Y}$ reveals the ribbon network



(49)

(This explains the particular braidings appearing in (45).)

Theorem 5.9. 1. Making use of the defining properties of the bicategory $\mathcal{F}r(\mathcal{C})$, one shows that the correlator $\mathrm{Cor}_{\mathcal{C}}(\mathfrak{S}) = \mathrm{RT}_{\mathcal{C}}(M_{\mathfrak{S}})(1) \in \mathrm{RT}_{\mathcal{C}}(\widehat{\Sigma}_{\mathfrak{S}})$ does not depend on the choice of triangulations and is invariant under the mapping class group $\mathrm{Map}(\mathfrak{S})$ of the world sheet, which is the subgroup of $\mathrm{Map}(\Sigma_{\mathfrak{S}})$ containing those elements which fix the physical boundaries, defects and field insertions up to isotopies.

2. One also shows that factorization holds.

Remarks 5.10. 1. One insight resulting from the TFT-construction is that in the absence of field insertions the correlators for a rational CFT can be expressed in terms of surface defects that separate Reshetikhin-Turaev type TFTs [34, 20]. When field insertions are present, there are in addition the ribbons carrying the chiral labels attached to the surface defect at the insertion points.

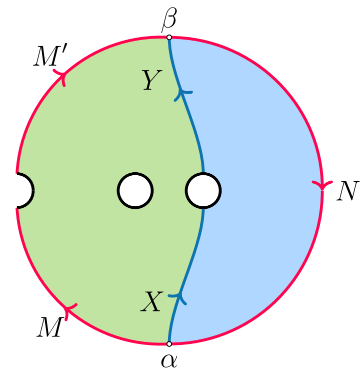
2. Also, instead of working with the doubles of world sheets, the holomorphic factorization can alternatively be implemented by using the Reshetikhin-Turaev TFT for the Drinfeld center $\mathcal{Z}(\mathcal{C})$. Here, one uses

$$\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}(\Sigma) = \mathrm{RT}_{\mathcal{C}\boxtimes\mathcal{C}^{rev}}(\Sigma) = \mathrm{RT}_{\mathcal{C}}(\Sigma) \otimes \mathrm{RT}_{\mathcal{C}}(\overline{\Sigma}) = \mathrm{RT}(\hat{\Sigma}) \; .$$

where we first used that $\mathcal{C}$ is modular, $\mathcal{Z}(\mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C}^{rev}$, then a property of the Deligne product and then the definition of the double. **This observation will allow to use state-sum constructions for the description of correlators.** This is important, since a three-dimensional topological field theory that is defined on all three-manifolds can be constructed only for **semisimple** modular tensor categories.

5.3 Sewing boundaries and field contents

When studying modular functors in the framework of functorial field theory, it is convenient to describe a field not via an insertion point carrying an arc-germ, but instead via a **sewing boundary**. Instead of world sheets $\mathfrak{S}$ with field insertions, we then deal with world sheets $\mathcal{S}$ with sewing boundaries. For instance, the world sheet $\mathfrak{S}_0$ in (43) gets replaced by the world sheet

$\mathcal{S}_0 =$  (50)

$\mathcal{S}_0$ has

- one **sewing interval** which separates two physical boundaries
- and two *sewing circles* obtained from cutting out disks around the defect and bulk field insertions.

World sheets can be sewn along sewing boundaries with matching boundary data. For every sewing boundary there is a field content, i.e. a space of fields associated to it. For instance, the field for the sewing boundaries of the world sheet (50) are:

1. The **boundary field content** $\mathbb{B}^{M,M'} := \bigoplus_{i\in \mathrm{I}(\mathcal{C})} \mathrm{Hom}_{\text{mod-}A}(i\otimes M, M')\otimes_{\mathbb{C}} \; \in \mathcal{C}\,.$

2. The **defect field content** $\mathbb{D}^{X,Y} := \bigoplus_{i,j\in \mathrm{I}(\mathcal{C})} \mathrm{Hom}_{A\text{-mod-}B}(i\otimes^- X\otimes^+ j, Y)\otimes_{\mathbb{C}}\Xi(i\boxtimes j) \; \in \mathcal{Z}(\mathcal{C})\,.$

3. The **bulk field content** $\mathbb{D}^{A,A} := \bigoplus_{i,j\in \mathrm{I}(\mathcal{C})} \mathrm{Hom}_{A\text{-mod-}A}(i \otimes^- A \otimes^+ j, A) \otimes_{\mathbb{C}} \Xi(i \boxtimes j) \;\in \mathcal{Z}(\mathcal{C})$.

   Here Ξ is the ribbon equivalence (41) between $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ and the Drinfeld center $\mathcal{Z}(\mathcal{C})$. A boundary field content is an object in $\mathcal{C} = \mathrm{Rep}(\mathfrak{V})$ because we require the boundary conditions to preserve the chiral VOA $\mathfrak{V}$. In contrast, while the spaces of defect and bulk fields are naturally modules over $\mathfrak{V} \otimes_{\mathbb{C}} \mathfrak{V}$, hence their field contents are objects in $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$.

Remarks 5.11. 1. The field contents can also be expressed as **internal Homs**. For a left module category $\mathcal{M}$ over a monoidal category $\mathcal{A}$ and a pair of objects $m, n \in \mathcal{M}$, the internal Hom $\underline{\mathrm{Hom}}_{\mathcal{M}}(m, n)$ (if it exists, which is the case especially when $\mathcal{M}$ and $\mathcal{A}$ are finitely semisimple) is an object in $\mathcal{A}$ that is defined up to unique isomorphisms by the adjunction

$$\mathrm{Hom}_{\mathcal{M}}(c \triangleright m, n) \cong \mathrm{Hom}_{\mathcal{A}}(c, \underline{\mathrm{Hom}}_{\mathcal{M}}(m, n)) \ .$$

2. Since mod-A is a finitely semisimple left module category over the fusion category $\mathcal{C}$, we have

$$\begin{aligned}
\mathbb{B}^{M,M'} &= \bigoplus_{i\in \mathrm{I}(\mathcal{C})} \mathrm{Hom}_{\text{mod-}A}(i \otimes M, M') \otimes_{\mathbb{C}} i = \int^{c\in\mathcal{C}} \mathrm{Hom}_{\text{mod-}A}(c \triangleright M, M') \otimes_{\mathbb{C}} c \\
&\cong \int^{c\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(c, \underline{\mathrm{Hom}}_{\text{mod-}A}(M, M')) \otimes_{\mathbb{C}} c = \underline{\mathrm{Hom}}_{\text{mod-}A}(M, M') \;\in \mathcal{C}\,.
\end{aligned} \tag{51}$$

   Here the integral sign denotes a categorical coend. We first rewrote the direct sum as a coend, then the definition of an internal Hom and finally the Yoneda lemma.

3. If $M = M'$ and the quantum dimension $\dim(M)$ (of M as an object in $\mathcal{C}$) is non-zero, then $\mathbb{B}^{M,M} = \underline{\mathrm{Hom}}_{\text{mod-}A}(M, M)$ is a **simple special symmetric Frobenius algebra** that is Morita equivalent to A.

4. By a similar, albeit less straightforward, procedure for the finite left $\mathcal{Z}(\mathcal{C})$-module category $\mathcal{F}un_{A,B}$ of $\mathcal{C}$-module functors from mod-A to mod-B it follows [19] that defect field contents are **internal natural transformations**

$$\mathbb{D}^{X,Y} = \underline{\mathrm{Nat}}(- \otimes_A X, - \otimes_A Y) := \underline{\mathrm{Hom}}_{\mathcal{F}un_{A,B}}(- \otimes_A X, - \otimes_A Y) \;\in \mathcal{Z}(\mathcal{C})\,. \tag{52}$$

5. The bulk field content $\mathbb{D}^{A,A} = \underline{\mathrm{Nat}}(\mathrm{id}_{\text{mod-}A}, \mathrm{id}_{\text{mod-}A}) =: Z(A) \in \mathcal{Z}(\mathcal{C})$, called the full center of the simple special symmetric Frobenius algebra $A \in \mathcal{C}$, is a commutative symmetric Frobenius algebra in the Drinfeld center $\mathcal{Z}(\mathcal{C})$; $Z(A)$ is *Lagrangian* in the sense of [7, Def. 4.6].

5.4 Modular functor, field maps and correlators

We now formulate the full local rational CFT with given modular fusion category $\mathcal{C}$ of chiral data with the help of an open-closed modular functor $\mathrm{Bl}_{\mathcal{C}}$, as defined in (40). $\mathrm{Bl}_{\mathcal{C}}$ provides the conformal blocks of the CFT.

Remarks 5.12. 1. For obtaining correlators, $\mathrm{Bl}_{\mathcal{C}}$ must in addition fulfill the following requirements:

- The categories assigned to the interval $I = [0,1] \subset \mathbb{R}$ and to the circle $S^1 = \{|z| = 1\} \subset \mathbb{C}$ are equipped with equivalences

$$\Phi_I\colon \quad \mathrm{Bl}_{\mathcal{C}}(I) \xrightarrow{\simeq} \mathcal{C} \qquad \text{and} \qquad \Phi_{S^1}\colon \quad \mathrm{Bl}_{\mathcal{C}}(S^1) \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C}) \tag{53}$$

of $\mathbb{C}$-linear categories.

- The closed sector of the modular functor $\mathrm{Bl}_{\mathcal{C}}$ is canonically equivalent to the Reshetikhin-Turaev modular functor $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$.

The equivalences (53) extend uniquely to equivalences $\Phi_\ell\colon \mathrm{Bl}_{\mathcal{C}}(\ell) \xrightarrow{\simeq} \mathcal{C}^{\times p} \times \mathcal{Z}(\mathcal{C})^{\times q}$ for all one-manifolds $\ell = (I)^{\sqcup p} \sqcup (S^1)^{\sqcup q}$.

2. A second ingredient needed for the construction of correlators is a collection of **field maps** which encode the field contents of all types of fields, including multi-pronged generalizations of defect fields, which are not considered traditionally.

   To describe these we need the notion of an $\mathcal{F}r(\mathcal{C})$**-boundary datum** b on a compact oriented one-manifold ℓ; this is a finite set $O \subset \mathrm{int}(\ell)$ of points in the interior of ℓ, together with a labeling of the connected components of the complement $\ell \setminus O$ by the objects in $\mathcal{F}r(\mathcal{C})$, i.e. by simple special symmetric Frobenius algebras, and a labeling of the elements of O by the 1-morphisms in $\mathcal{F}r(\mathcal{C})$, i.e. by bimodules. Denote by $\mathcal{F}r(\mathcal{C})_\ell$ the set of $\mathcal{F}r(\mathcal{C})$-boundary data on ℓ. Given a world sheet $\mathcal{S}$, a structure of an open-closed two-bordism $\Sigma_{\mathcal{S}}\colon \ell_{\mathrm{in}} \to \ell_{\mathrm{out}}$ on its underlying surface uniquely determines $\mathcal{F}r(\mathcal{C})$-boundary data $\mathsf{b}_{\mathrm{in}} \in \mathcal{F}r(\mathcal{C})_{\ell_{\mathrm{in}}}$ and $\mathsf{b}_{\mathrm{out}} \in \mathcal{F}r(\mathcal{C})_{\ell_{\mathrm{out}}}$.

   Field maps are a collection

$$\{\mathbb{F}_\ell\colon \mathcal{F}r(\mathcal{C})_\ell \to \mathrm{obj}(\mathrm{Bl}_{\mathcal{C}}(\ell))\}_\ell$$

   of maps defined for every compact oriented one-manifold ℓ with any numbers p of intervals and q of circles as connected components, such that for any world sheet $\mathcal{S}$ with its underlying surface viewed as an open-closed bordism, the objects $\Phi_{\ell_\varepsilon} \circ \mathbb{F}_{\ell_\varepsilon}(\mathsf{b}_\varepsilon) \in \mathcal{C}^{\times p_\varepsilon} \times \mathcal{Z}(\mathcal{C})^{\times q_\varepsilon}$, for $\varepsilon \in \{\mathrm{in}, \mathrm{out}\}$, are given by the correct field contents.

3. Given the open-closed modular functor $\mathrm{Bl}_{\mathcal{C}}$ and the field maps $\{\mathbb{F}_\ell\}$, we obtain the vector spaces of conformal blocks as follows. The space $\mathrm{Bl}_{\mathcal{C}}(\mathcal{S})$ of conformal blocks for a world sheet $\mathcal{S}$ with underlying bordism $\Sigma_{\mathcal{S}}\colon \ell_{\mathrm{in}} \to \ell_{\mathrm{out}}$ is the vector space

$$\mathrm{Bl}_{\mathcal{C}}(\mathcal{S}) := \mathrm{Bl}_{\mathcal{C}}(\Sigma_{\mathcal{S}}; \mathbb{F}_{\ell_{\mathrm{in}}}(\mathsf{b}_{\mathrm{in}}), \mathbb{F}_{\ell_{\mathrm{out}}}(\mathsf{b}_{\mathrm{out}}))\,, \tag{54}$$

   where $\mathrm{Bl}_{\mathcal{C}}(\Sigma_{\mathcal{S}}, -; \sim)\colon \mathrm{Bl}_{\mathcal{C}}(\ell_{\mathrm{in}})^{\mathrm{op}} \times \mathrm{Bl}_{\mathcal{C}}(\ell_{\mathrm{out}}) \to \mathrm{Vect}_{\mathbb{C}}$ is the profunctor obtained by evaluating the modular functor on $\Sigma_{\mathcal{S}}$.

   - By functoriality, the vector space $\mathrm{Bl}_{\mathcal{C}}(\mathcal{S})$ carries an action of the **mapping class group** $\mathrm{Map}(\Sigma_{\mathcal{S}}) = \mathrm{End}_{\mathrm{Bord}^{\mathrm{or}}_{2,\mathrm{o/c}}}(\Sigma_{\mathcal{S}})$.
   - Also by functoriality, for any sewing of two world sheets $\mathcal{S}$ and $\mathcal{S}'$ along a sewing boundary ℓ we get a **sewing map**

$$s\colon \quad \mathrm{Bl}_{\mathcal{C}}(\mathcal{S}) \otimes_{\mathbb{C}} \mathrm{Bl}_{\mathcal{C}}(\mathcal{S}') \to \mathrm{Bl}_{\mathcal{C}}(\mathcal{S} \cup_\ell \mathcal{S}') \tag{55}$$

     of conformal blocks. These maps endow $\mathrm{Bl}_{\mathcal{C}}(\mathcal{S} \cup_\ell \mathcal{S}')$ with the structure of a coend,

$$\mathrm{Bl}_{\mathcal{C}}(\mathcal{S} \cup_\ell \mathcal{S}') = \int^{\mathsf{b} \in \mathrm{Bl}_{\mathcal{C}}(\ell)} \mathrm{Bl}_{\mathcal{C}}(\Sigma_{\mathcal{S}}; \mathbb{F}_{\ell_{\mathrm{in}}}(\mathsf{b}_{\mathrm{in}}), \mathsf{b}) \otimes_{\mathbb{C}} \mathrm{Bl}_{\mathcal{C}}(\Sigma_{\mathcal{S}}; \mathsf{b}, \mathbb{F}_{\ell'_{\mathrm{out}}}(\mathsf{b}'_{\mathrm{out}}))\,. \tag{56}$$

Definition 5.13. A **consistent system of correlators** is then an assignment $\mathcal{S} \mapsto \mathrm{Cor}_{\mathcal{C}}(\mathcal{S}) \in \mathrm{Bl}_{\mathcal{C}}(\mathcal{S})$, as in (42), such that $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})$ is invariant under the action of $\mathrm{Map}(\mathcal{S}) \subset \mathrm{Map}(\Sigma_{\mathcal{S}})$ and such that the sewing maps (55) take correlators to correlators.

5.5 Correlators from string nets

Another approach to the construction of correlators, called the **string-net construction** [21], uses a realization of the open-closed modular functor $\mathrm{Bl}_{\mathcal{C}}$ via string nets. String-net models arose in the study of topologically ordered phases of matter [37] and were later formulated as two-dimensional skein theories [36].

Remarks 5.14. 1. The basic **input** datum of a string-net model is a spherical **fusion category**. In our context this is the modular fusion category $\mathcal{C}$ of chiral data.

2. For any compact oriented surface Σ and $\mathcal{C}$-boundary datum B° on $\partial\Sigma$ (defined in the same way as an $\mathcal{F}r(\mathcal{C})$-boundary datum, by regarding $\mathcal{C}$ as a one-object bicategory), the string-net model for $\mathcal{C}$ defines a finite-dimensional vector space $\mathrm{SN}^\circ_{\mathcal{C}}(\Sigma, \mathsf{B}^\circ)$, called the bare string-net space associated to $(\Sigma, \mathsf{B}^\circ)$, as a **quotient of the free vector space generated by $\mathcal{C}$-colored string diagrams** drawn on the surface Σ with boundary datum B°, by a **subspace that encodes the local graphical calculus** of $\mathcal{C}$. Thus a **string net**, i.e. an element of $\mathrm{SN}^\circ_{\mathcal{C}}(\Sigma, \mathsf{B}^\circ)$, is a linear combination of equivalence classes of string diagrams on Σ, where two string diagrams are equivalent if they can be transformed into each other by applying the graphical calculus of $\mathcal{C}$ within disk-shaped regions.

3. For every compact oriented one-manifold ℓ one can then define a linear category $\mathrm{Cyl}^\circ(\mathcal{C}, \ell)$, called the **bare cylinder category** over ℓ, whose objects are $\mathcal{C}$-boundary data on ℓ and whose **morphisms are string nets** on the cylinder $\ell \times I$. Composition in $\mathrm{Cyl}^\circ(\mathcal{C}, \ell)$ is given by sewing the cylinders and concatenating the string diagrams.

Definition 5.15. 1. The **cylinder category** $\mathrm{Cyl}(\mathcal{C}, \ell)$ over ℓ is the idempotent completion of $\mathrm{Cyl}^\circ(\mathcal{C}, \ell)$, whose objects are idempotents in $\mathrm{Cyl}^\circ(\mathcal{C}, \ell)$. We have canonical equivalences $\mathrm{Cyl}(\mathcal{C}, \ell) \xrightarrow{\simeq} \mathcal{C}$ and $\mathrm{Cyl}(\mathcal{C}, S^1) \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C})$ of linear categories.

2. Accordingly one defines the string-net space $\mathrm{SN}_{\mathcal{C}}(\Sigma, \mathsf{B})$, which takes an object B in $\mathrm{Cyl}(\mathcal{C}, \ell)$ as its boundary datum, as the subspace of the bare string-net space that consists of elements which are invariant under sewing with the string net on a cylinder that is given by B.

Remarks 5.16. 1. The so defined string-net spaces carry a mapping class group action obtained by pushforward.

2. When Σ is equipped with the structure of an open-closed bordism, one obtains a profunctor

$$\mathrm{SN}_{\mathcal{C}}(\Sigma)\colon \quad \mathrm{Cyl}(\mathcal{C}, \ell_{\mathrm{in}})^{\mathrm{op}} \times \mathrm{Cyl}(\mathcal{C}, \ell_{\mathrm{out}}) \longrightarrow \mathrm{Vect}\,. \tag{57}$$

The assignments $\ell \mapsto \mathrm{Cyl}(\mathcal{C}, \ell)$ and $\Sigma \mapsto \mathrm{SN}_{\mathcal{C}}(\Sigma)$ define an open-closed modular functor $\mathrm{SN}_{\mathcal{C}}$.

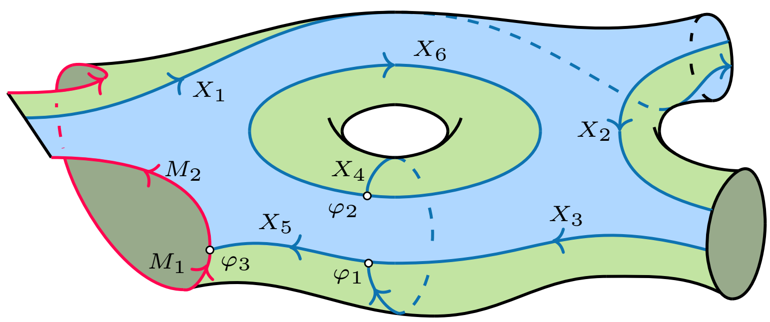
3. Moreover, the **closed sector of** $\mathrm{SN}_{\mathcal{C}}$ **extends** to a once-extended three-dimensional TFT that is equivalent to the **Turaev-Viro TFT** $\mathrm{TV}_{\mathcal{C}}$ which, in turn, is equivalent to $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$. Thus one can use the string-net modular functor $\mathrm{SN}_{\mathcal{C}}$ as the model for conformal blocks.

In the **string-net approach**, the construction of correlators is fairly straightforward.

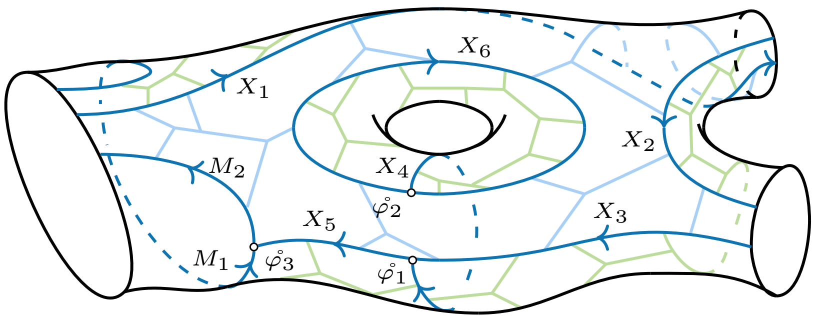
The correlator $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})$ is a $\mathcal{C}$-colored string net that is represented by a string diagram $\Gamma_{\mathcal{C}}(\mathcal{S})$ that is obtained

- by replacing each two-dimensional stratum of $\mathcal{S}$ with a network of strings labeled by the relevant Frobenius algebra according to a fine triangulation,
- and, when $\mathcal{S}$ has physical boundaries, by adding two-dimensional strata labeled by the trivial Frobenius algebra $\mathbf{1}$ which turn a physical boundary (a right A-module) into a defect line (a $\mathbf{1}$-A-bimodule).

As an illustration, a string diagram for the world sheet

$\mathcal{S}_1 \;=$  (58)

is given by

$\Gamma_{\mathcal{C}}(\mathcal{S}_1) \;=$  (59)

Theorem 5.17. Due to the defining properties of the underlying algebraic structures, the so obtained string net is well defined and is invariant under the action of $\mathrm{Map}(\mathcal{S})$. Also, it is in the string-net space $\mathrm{SN}_{\mathcal{C}}(\Sigma_{\mathcal{S}}, \mathbb{F}_{\partial\Sigma_{\mathcal{S}}}(\mathsf{b}_{\mathcal{S}}))$, which is taken to be the space $\mathrm{Bl}_{\mathcal{C}}(\mathcal{S})$ of conformal blocks, where the boundary datum $\mathbb{F}_{\partial\Sigma_{\mathcal{S}}}(\mathsf{b}_{\mathcal{S}}) \in \mathrm{Cyl}(\mathcal{C}, \partial\Sigma_{\mathcal{S}}) \simeq \mathcal{C}^{\times p} \times \mathcal{Z}(\mathcal{C})^{\times q}$ produces the correct field contents.

Remarks 5.18.

1. The string-net description of correlators for world sheets of particular interest, such as those that correspond to **operator products**, provides explicit expressions which match results proposed in the literature. For instance, operator products of defect fields correspond to the vertical and horizontal compositions of internal natural transformations, and in the special case of a bulk field they reduce to the commutative product of the full center.

2. The torus partition function $\mathrm{Cor}_{\mathcal{C}}(\mathcal{T}_A) \in \mathrm{SN}_{\mathcal{C}}(\mathrm{T})$, where the world sheet $\mathcal{T}_A$ is a torus T without defect lines whose phase is given by a Frobenius algebra $A \in \mathcal{F}r(\mathcal{C})$, decomposes as $\mathrm{Cor}_{\mathcal{C}}(\mathcal{T}_A) = \sum_{i,j\in \mathrm{I}(\mathcal{C})} Z_{i,j}(A)\, e_{i,j}$, with the integers

$$Z_{i,j}(A) = \dim_{\mathbb{C}}(\mathrm{Hom}_{A\text{-mod-}A}(i \otimes^- A \otimes^+ j, A)) \tag{60}$$

being dimensions of degeneracy spaces for bulk fields (compare (45)). Here $\{e_{i,j}\}_{i,j\in \mathrm{I}(\mathcal{C})}$ is a distinguished basis of $\mathrm{SN}_{\mathcal{C}}(\mathrm{T})$ that corresponds to holomorphic/antiholomorphic pairs of characters (27) of VOA-modules and transforms under the modular group $\mathrm{Map}(\mathrm{T}) = SL(2,\mathbb{Z})$ via conjugation by the modular S- and T-matrices of $\mathcal{C}$. The mapping class group invariance of the string net $\mathrm{Cor}_{\mathcal{C}}(\mathcal{T}_A)$ immediately implies the well known modular invariance relations $[Z(A),S] = 0 = [Z(A),T]$.

Remarks 5.19. 1. For the implementation of **categorical symmetries** [16, 14] via **topological defects**, a desirable property of the system of correlators is that one can modify the defect network in accordance with the local graphical calculus for the category $\mathcal{C}$ on any world sheet without altering its correlator.

2. The string-net correlators indeed enjoy this property; this can be shown with the help of generalised string-net models which take, instead of a spherical fusion category, a **pivotal bicategory** as input datum [22].

   Taking this input bicategory to be $\mathcal{F}r(\mathcal{C})$, one defines for any compact oriented surface Σ and $\mathcal{F}r(\mathcal{C})$-boundary datum b a vector space $\mathrm{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\Sigma,\mathsf{b})$ of $\mathcal{F}r(\mathcal{C})$-colored string nets on Σ.

3. The string-net space $\mathrm{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\Sigma,\mathsf{b})$ can be interpreted as the space of equivalence classes of world sheets with underlying surface Σ and boundary datum b, with the equivalence relation given by the local graphical calculus of the pivotal bicategory $\mathcal{F}r(\mathcal{C})$. It is thus appropriate to call an element of the space $\mathrm{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\Sigma,\mathsf{b})$ a **quantum world sheet**.

4. The assignment of the correlators provides a linear map

$$\mathrm{Cor}_{\mathcal{C}}(\Sigma,\mathsf{b})\colon \mathbb{C}\mathrm{G}_{\mathcal{F}r(\mathcal{C})}(\Sigma,\mathsf{b}) \to \mathrm{SN}_{\mathcal{C}}(\Sigma,\mathbb{F}_{\partial\Sigma}(\mathsf{b}))\ ,$$

   where $\mathbb{C}\mathrm{G}_{\mathcal{F}r(\mathcal{C})}(\Sigma,\mathsf{b})$ is the vector space generated by the set of $\mathcal{F}r(\mathcal{C})$-colored string diagrams on Σ. This map factors through the canonical quotient map

$$\mathrm{q}(\Sigma,\mathsf{b})\colon \mathbb{C}\mathrm{G}_{\mathcal{F}r(\mathcal{C})}(\Sigma,\mathsf{b}) \twoheadrightarrow \mathrm{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\Sigma,\mathsf{b})$$

   i.e. there is a unique linear map

$$\mathrm{U}(\Sigma,\mathsf{b})\colon \quad \mathrm{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\Sigma,\mathsf{b}) \longrightarrow \mathrm{SN}_{\mathcal{C}}(\Sigma,\mathbb{F}_{\partial\Sigma}(\mathsf{b})) \tag{61}$$

   such that $\mathrm{Cor}_{\mathcal{C}}(\Sigma,\mathsf{b}) = \mathrm{U}(\Sigma,\mathsf{b}) \circ \mathrm{q}(\Sigma,\mathsf{b})$. As a consequence, the correlator of a world sheet only depends on the quantum world sheet it represents, and is therefore unchanged under modifications via the local graphical calculus. The map $\mathrm{U}(\Sigma,\mathsf{b})$ is called a **universal correlator**. It intertwines with the action of the mapping class group.

6 Conclusion

We have presented various algebraic structures that play a role in conformal field theory, considering both chiral and full local CFTs. It is worth pointing out that all of these are perfectly customary mathematical structures and that they can be analysed with standard mathematical tools.

Needless to say, our exposition is highly biased by the authors' taste and restricted knowledge. Many interesting aspects and important developments as well as a substantial portion of pertinent literature had to be omitted owing to limitations of length and of expertise of the authors.

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