GGI mini-course

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"Intro to the BV-BFV formalism"

[initial construction: A. Cattaneo, PM, N. Reshetikhin
further examples: worked on: M. Schiavina, K. Wernli, N. Moshayedi, R. Iraso, G. Canepa, ...]

[also: A. Alekseev]

> ✓Quick intro to BV & examples → ✓CS, ✓Yang-Mills, ✓BF, ?p-form EM
  ↪ ✓AKSZ theories, ✓ex: PSM

> ✓classical FT with bdry → ✓scalar field, ✓CS

> ✓Class. BV-BFV.

> ? quantum BV-BFV → ✓fiber BV integrals ← quantum BV
  ↪ ✓ex: simplicial BF
  ↪ ex: simplicial 1d CS

[✓aside: Segal's axioms]

✓ex: abelian BF, A-B polarization

[possible aside: $p(\varphi)_2$ theory]
↑ gluing f-lae

> ✓BF-like theories w/ bdry; conf. space integrals

[possible aside: scalar theory on a graph]

> ? corners → 2d YM
  ↪ P_∞ phase spaces (ex: CS)

>? CS on a cylinder → WZW
[7d ab CS → BCOV]

Intro to BV-BFV formalism

BV-BFV: BV formalism, Segal's axioms

Goal: perturbative quantization of (gauge theories) compatible (w/ gluing-cutting)

Idea: Z(manifold cut into "chunks" – e.g. top. cells) = Gluing(Z(chunks)) — easy to compute

e.g. (i) assemble 2d YM on surfaces from pairs of pants and disks

(ii) assemble 3d CS on a lens space $L(p,q)$ from two solid tori.

- BV formalism – a quick intro / reminder

Input: A classical gauge system – action $S_{cl}(\varphi) \in C^\infty(F)$ (space of fields), $v_a \in \mathfrak{X}(F)$ symmetries, $\mathcal{L}_{v_a} S_{cl} = 0$

Step I: (a) replace $F \to \mathcal{F}_{BV}$ (space of BV fields)

$\mathcal{F}_{BV}$ " $= T^*[-1](\mathfrak{g}[1] \times F)$ ($\mathfrak{g}$: Lie algebra of v. fields v_a)

– $\mathbb{Z}$-graded super-mfd with (-1)-symplectic form $\omega_{BV} = \sum_i \delta\varphi^i \wedge \delta\varphi_i^+ + \sum_a \delta c^a \wedge \delta c_a^+$

coordinates:	c_a^+	φ_i^+	c^a	φ^i
gh-degree	-2	-1	$+1$	0
	anti-ghosts	anti-fields	ghosts	cl. fields
	Φ_I^+		Φ^I	

$|\Phi_I^+| = -1 - |\Phi^I|$

(b) replace $S_{cl}(\varphi) \longrightarrow S(\Phi, \Phi^+)$ (or: S_{BV})

ansatz:

$$S(\Phi,\Phi^+) = \underbrace{S_{cl}(\varphi) + \varphi_i^+ v_a^i(\varphi) c^a + \tfrac{1}{2} f^a_{bc}(\varphi) c_a^+ c^b c^c}_{\text{"initial condition"}} + \underbrace{\sum_{k\geq 2} \Phi^+_{I_1} \cdots \Phi^+_{I_k} S^{I_1 \ldots I_k}(\Phi)}_{\text{corrections needed to satisfy CME}}$$

such that $\{S, S\} = 0$ – CME ($(+1)$-Poisson bracket generated by ω_{BV})

$$\Leftrightarrow \sum_I S \frac{\overleftarrow{\partial}}{\partial \Phi^I} \frac{\overrightarrow{\partial}}{\partial \Phi_I^+} S = 0$$

Rem: quadratic (and higher) terms in Φ^+ in S_{BV} are needed if $[v_a, v_b] \in \mathrm{span}(v_c)$ only mod EL. ("open" or "on-shell" gauge symmetry algebra)

- define $Q := \{S, -\} \in \mathfrak{X}(\mathcal{F})_{+1}$ – BRST(-BV) operator

CME $\Leftrightarrow$ $Q^2 = 0$, i.e. Q is a cohomological vector field.

Step II Quantization

Replace $S \longrightarrow S_\hbar = \sum_{k\geq 0} \hbar^k S^{(k)}$, $S^{(0)} = S$

s.t. $\underbrace{\Delta}_{\frac{\partial^2}{\partial\Phi^I \partial\Phi^+_I}} e^{\frac{i}{\hbar} S_\hbar} = 0$ (QME)

$\frac{\partial^2}{\partial\Phi^I \partial\Phi^+_I}$ - BV Laplacian

$\Leftrightarrow \frac{1}{2}\{S_\hbar, S_\hbar\} - i\hbar \Delta S_\hbar = 0$ $\Big(\Leftrightarrow \frac{1}{2}\{S^{(0)}, S^{(0)}\} = 0$ CME

$QS^{(1)} = \{S^{(0)}, S^{(1)}\} = i\Delta S^{(0)}$

$QS^{(2)} = \{S^{(0)}, S^{(2)}\} = -\frac{1}{2}\{S^{(1)}, S^{(1)}\} + i\Delta S^{(1)}$ $\Big)$

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Then: partition function is

$$Z_{\mathcal{L}} := \int_{\mathcal{L}\subset\mathcal{F}} e^{\frac{i}{\hbar} S_\hbar(\Phi,\Phi^+)}$$

$\mathcal{L}$ 'gauge-fixing Lagrangian'

Thm (Batalin-Vilkovisky)
$Z_{\mathcal{L}}$ is invariant under Lagrangian homotopy of $\mathcal{L}$.

E.g. $\mathcal{L} = \text{graph}(d\Psi) = \{(\Phi,\Phi^+) \mid \Phi^+_I = \frac{\partial\Psi}{\partial\Phi^I}\}$

for $\Psi(\Phi)$ a function with gh $= -1$ ("gauge-fixing fermion")

Geometric viewpoint (A. Schwarz "Geometry of BV quantization", 92)

$(\mathcal{F}, \omega)$ odd-symplectic supermanifold ←

Thm (Schwarz)
an odd-symp mfd can be written as
$(\mathcal{F},\omega) = (\Pi T^* N, \omega_{can})$
for some <u>even</u> mfd N

$f \in C^\infty(\mathcal{F}) \longrightarrow X_f \in \mathfrak{X}(\mathcal{F})$ s.t. $\iota_{X_f}\omega = df$.

X_f hamiltonian v.f.

$\{f,g\} = X_f(g)$ - odd-Poisson bracket (BV- or anti-)

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given μ a volume elt. on $\mathcal{F}$ compatible with ω (i.e. locally $\exists$ "special" Darboux charts (x^α, ξ_α) s.t. $\mu = Dx^\alpha D\xi_\alpha$)

$\Delta_\mu f := \frac{1}{2}\text{div}_\mu X_f$

Then: · $\Delta_\mu = \sum \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial \xi_\alpha}$ ← in special Darboux chart

· $\Delta_\mu^2 = 0$

· $\Delta_\mu(fg) = \Delta_\mu f g + (-1)^{|f|} f \Delta_\mu g + (-1)^{|f|}\{f,g\}$

Thm (Schwarz): (a) $\int_{\mathcal{L} \subset_{\text{Lagr}} \mathcal{F}} \Delta_\mu g \, \underbrace{\sqrt{\mu}|_{\mathcal{L}}} = 0$ canonical density on $\mathcal{L}$ induced by $\sqrt{\mu}$.

(b) $\int_{\mathcal{L} \subset_{\text{Lagr}} \mathcal{F}} f \sqrt{\mu}|_{\mathcal{L}} = \int_{\mathcal{L}' \subset_{\text{Lagr}} \mathcal{F}} f \sqrt{\mu}|_{\mathcal{L}'}$ if $\Delta_\mu f = 0$ and $\mathcal{L} \sim \mathcal{L}'$ Lagr. homotopic

Rem (Khudaverdian)

Consider the space of $\frac{1}{2}$-densities on $\mathcal{F}$. There exists a canonical BV Laplacian - an operator $\Delta^{can} \circlearrowright \mathrm{Dens}^{1/2}(\mathcal{F})$ s.t.

$$\begin{array}{ccc} C^\infty(\mathcal{F}) & \xrightarrow{\cdot\sqrt{\mu}} & \mathrm{Dens}^{1/2}(\mathcal{F}) \\ \circlearrowleft \Delta_\mu & & \circlearrowleft \Delta^{can} \end{array}$$

- intertwines Δ_μ and Δ^{can}

Examples of Lagrangians $\mathcal{L} \subset \mathcal{F} = \Pi T^* \mathcal{N}$

(i) conormal bundle $\mathcal{L}_C = \Pi N^* C$ for some $C \subset \mathcal{N}$ submfd

$$= \left\{ (x, \xi) \,\middle|\, \begin{array}{l} x \in C \\ \xi \in \mathrm{Ann}(T_x C) \end{array} \right\}$$

(ii) graph: choose $\psi \in C^\infty(\mathcal{N})_{odd}$,

$$\mathcal{L}_\psi = \mathrm{graph}(d\psi) = \left\{ (x, \xi) \,\middle|\, \xi_\alpha = \frac{\partial \psi(x)}{\partial x^\alpha} \right\}$$

Thm (Schwarz): every Lagrangian in $(\mathcal{F}, \omega)$ [$= \Pi T^* N$] can be obtain by a combination of these two constructions.

Rem (Witten) Let N be a mfd with v a vol. form. Then BV Laplacian on $T^*[-1]N$ corresponds to de Rham on N by "odd Fourier transform"

$$f(x, \xi) \longmapsto \int D\xi \, e^{\langle \psi, \xi \rangle} f(x, \xi) = \tilde{f}(x, \psi) \quad \text{"}\psi = dx\text{"}$$

$$v \longmapsto \iota_v v$$

$$C^\infty(T^*[-1]N), \Delta_{BV} \rightleftarrows \Omega^\bullet(N), d$$

$C^\infty(T^*[-1]N) \cong \mathcal{V}^{-\bullet}(N)$ polyvectors

Examples

Yang-Mills theory (M,g) Riemannian mfd (n-dim. (oriented, closed)), $\mathfrak{g} = \mathrm{Lie}(G)$ (cpt Lie group)

classical action $S_{cl}(A) = \int_M \frac{1}{2} \langle F_A \overset{\wedge}{,} * F_A \rangle$, $A \in \mathrm{Conn}\begin{pmatrix} M\times G \\ \downarrow \\ M \end{pmatrix} \simeq \Omega^1(M, \mathfrak{g}) = F$

(infinitesimal) gauge transformations $A \longrightarrow A + \varepsilon \underbrace{d_A \gamma}_{d\gamma + [A,\gamma]}$, $\gamma \in \Omega^0(M, \mathfrak{g}) = \mathcal{G}$

BV fields:
$$\mathcal{F} = \{(c, A, A^+, c^+)\} = \Omega^0[1] \oplus \Omega^1 \oplus \Omega^{n-1}[-1] \oplus \Omega^n[-2]$$
$\Omega^k = \Omega^k(M, \mathfrak{g})$

BV action
$$S = S_{BV} = \int_M \frac{1}{2}\langle F_A \overset{\wedge}{,} * F_A \rangle + \langle A^+ \overset{\wedge}{,} d_A c \rangle + \langle c^+, \tfrac{1}{2}[c,c] \rangle$$

BV 2-form: $\omega = \int_M \langle \delta A \overset{\wedge}{,} \delta A^+ \rangle + \langle \delta c, \delta c^+ \rangle$

- $\{S, S\} = 0 \iff$
  - S_{cl} is $\mathcal{G}$-invariant ($\mathcal{O}(c)$ term)
  - $\gamma \longmapsto d_A \gamma$ is a Lie algebra homomorphism $\mathcal{G} \longrightarrow \mathfrak{X}(F)$ ($\mathcal{O}(c^2)$ term)
  - Jacobi identity in $\mathcal{G}$ ($\mathcal{O}(c^3)$ term)

- Q:
  - $A \longmapsto d_A c$ (gauge transf.)
  - $c \longmapsto \frac{1}{2}[c,c]$ (commutator of gauge transf)
  - $A^+ \longmapsto d_A * F_A$ (YM eom) $+ [c, A^+]$
  - $c^+ \longrightarrow d_A A^+ + [c, c^+]$

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gauge-fixing: $\mathcal{F} = T^*[-1](\underbrace{\mathfrak{g}[1] \times F}_{\mathcal{F}_{BRST}})$ $\leftarrow$ nonneg.-graded

- cannot find a gauge-fixing fermion $\Psi \in C^\infty(\mathcal{F}_{BRST})_{-1}$

- $\mathcal{F}^{non\text{-}min.} := \mathcal{F} \times \underbrace{\underset{b}{\Omega^n[-1]} \times \underset{\lambda}{\Omega^n} \times \underset{b^+}{\Omega^0} \times \underset{\lambda^+}{\Omega^0[-1]}}_{\mathcal{F}^{aux}} = T^*[-1](\overbrace{\underset{c}{\mathfrak{g}[1]} \times \underset{A}{F} \times \underset{b,\lambda}{F^{aux}}}^{\mathcal{F}_{BRST}^{non\text{-}min}})$

$S^{non\text{-}min} := S + \int_M \langle \lambda, b^+ \rangle$, choose e.g. $\Psi = \int_M \langle b, d^* A \rangle$

g.f. Lagrangian $\mathcal{L}_\Psi$:
$$A^+ = d^* b, \quad b^+ = d^* A, \quad c^+ = 0, \quad \lambda^+ = 0$$

$$S^{non\text{-}min}\big|_{\mathcal{L}_\Psi} = \int_M \frac{1}{2}\langle F_A, * F_A \rangle + \langle \lambda, d^* A \rangle + \langle b, d^* d_A c \rangle$$
= Faddeev-Popov action for Y-M in Lorenz gauge. (λ: Lagr. multiplier; b, c: F-P ghosts)

Example: Chern-Simons. Fix M closed oriented 3-mfld, $\mathfrak{g} = Lie(G)$

$$S_{cl}(A) = \int_M \frac{1}{2}\langle A, dA\rangle + \frac{1}{6}\langle A,[A,A]\rangle, \quad F = \Omega^1(M,\mathfrak{g}), \quad \text{symmetries - as in YM}$$

$$A \mapsto A + \varepsilon d_A \gamma$$

$$S_{BV} = S_{cl}(A) + \int_M \langle A^+, d_A c\rangle + \langle c^+, \tfrac{1}{2}[c,c]\rangle = \int_M \tfrac{1}{2}\langle \mathcal{A}, d\mathcal{A}\rangle + \tfrac{1}{6}\langle \mathcal{A},[\mathcal{A},\mathcal{A}]\rangle$$

"superfield"

$$\mathcal{F}_{BV} = \Omega^0[1] \oplus \Omega^1 \oplus \Omega^2[-1] \oplus \Omega^3[-2] = \Omega^\bullet(M,\mathfrak{g})[1] \ni \mathcal{A} = c + A + A^+ + c^+$$

non-homogeneous diff. form

Example: AKSZ construction (topological σ-model)

source: M - oriented closed n-manifold $\rightsquigarrow T[1]M$, $d = \psi^\alpha \frac{\partial}{\partial u^\alpha}$ - cohomological vector field on $T[1]M$.

ψ^α = "du^α", u^α

target: $\mathcal{N}$, $\omega_\mathcal{N} = \delta\alpha_\mathcal{N} \in \Omega^2(\mathcal{N})_{n-1}$,

$\mathcal{N}$: $\mathbb{Z}$-graded manifold; $\omega_\mathcal{N}$: symplectic form

$\Theta_\mathcal{N} \in C^\infty(\mathcal{N})_n$ with $\{\Theta_\mathcal{N}, \Theta_\mathcal{N}\}_{\omega_\mathcal{N}} = 0$

$(\Rightarrow Q_\mathcal{N} = \{\Theta_\mathcal{N}, -\}$ satisfies $Q_\mathcal{N}^2 = 0)$

} "Hamiltonian dg manifold" of degree $n-1$.

space of (BV) fields: $\mathcal{F} = Map(T[1]M, \mathcal{N})$

using a local chart X^i on the target, one has

$$X^i = \sum_{p=0}^{n} \underbrace{\frac{1}{p!}\psi^{\alpha_1}\cdots\psi^{\alpha_p} \underbrace{X^{i(p)}_{\alpha_1\cdots\alpha_p}(u^1,\ldots,u^n)}_{\text{coords on } \mathcal{F}}}_{X^{i(p)}}$$

$X^{(0)} \in Map(M,\mathcal{N})$

for $p \geq 1$ $X^{(p)} \in \Omega^p(M, X^{(0)*}T\mathcal{N})$

· for $\mathcal{N}$ a gr. vector space, $\mathcal{F} = \Omega^\bullet(M) \otimes \mathcal{N}$

BV action: $$S = \int_M \alpha_i(X)\, dX^i + \Theta(X), \qquad \omega = \int_M \tfrac{1}{2}\omega_{ij}(X)\,\delta X^i \wedge \delta X^j$$

$$Q = d_M^{\text{lifted}} + Q_\mathcal{N}^{\text{lifted}}, \quad Q(X^i) = dX^i + Q^i_\mathcal{N}(X) \qquad (\text{on target}, Q_\mathcal{N} = Q^i_\mathcal{N}(x)\tfrac{\partial}{\partial x^i})$$

$$\begin{array}{ccc} \mathcal{F} \times T[1]M & \xrightarrow{ev} & \mathcal{N} \\ p\downarrow & & \\ \mathcal{F} & & \end{array}$$

"transgression" $\tau: \Omega^\bullet(\mathcal{N}) \to \Omega^{\bullet-p}(\mathcal{F})$, $\tau = p_* ev^*$ $\rightsquigarrow \omega_\mathcal{F} = \tau(\omega_\mathcal{N})$

$$S = \iota_{d_M^{\text{lifted}}}\tau(\alpha_\mathcal{N}) + \tau(\Theta_\mathcal{N})$$

(kinetic/source term; target/potential term)

Thm S_{AKSZ} satisfies CME $\{S, S\} = 0$.

Examples: CS $n=3$, $\mathcal{N} = \mathfrak{g}[1]$ (quadratic Lie alg.), $\Theta_{\mathcal{N}} = \frac{1}{6}\langle \xi, [\xi, \xi]\rangle = \frac{1}{6} f_{abc}\, \xi^a \xi^b \xi^c$ (structure constants; coords on $\mathfrak{g}[1]$ (rel to an o/n basis T_a in $\mathfrak{g}$))

$\xi : \mathfrak{g}[1] \to \mathfrak{g}$ coord. on $\mathcal{N}$

$$\alpha_{\mathcal{N}} = \frac{1}{2}\xi^a \delta \xi^a$$

superfield $X^a = \mathcal{A}^a = \sum_{p=0}^{3} \mathcal{A}^{(p)a} \in \Omega^\bullet(M)$ (p-form, $gh = 1-p$)

$$S_{CS} = \int_M \frac{1}{2}\langle \mathcal{A}, d\mathcal{A}\rangle + \frac{1}{6}\langle \mathcal{A}, [\mathcal{A}, \mathcal{A}]\rangle \quad \text{- Chern-Simons theory (in BV)}$$

BF n any, $\mathcal{N} = \overset{a}{\mathfrak{g}[1]} \oplus \overset{b}{\mathfrak{g}^*[n-2]}$, $\Theta_{\mathcal{N}} = \frac{1}{2}\langle b, [a,a]\rangle$, $\alpha_{\mathcal{N}} = \langle b, \delta a\rangle$

Lie alg. (we don't need quadratic form)

$$S_{BF} = \int_M \langle \mathcal{B}, d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]\rangle$$

$$\mathcal{A} = \sum_{p=0}^{n} \mathcal{A}^{(p)} \in \Omega^\bullet(M) \otimes \mathfrak{g}[1]$$
$$\mathcal{B} = \sum_{p=0}^{n} \mathcal{B}^{(p)} \in \Omega^\bullet(M) \otimes \mathfrak{g}^*[n-2]$$

} AKSZ superfields.

PSM $n=2$, $\mathcal{N} = T^*[1]N$ (coords ξ_i, x^i; N Poisson manifold, π - Poisson bivector), $\Theta_{\mathcal{N}} = \frac{1}{2}\xi_i \xi_j \pi^{ij}(x)$, $\alpha_{\mathcal{N}} = \xi_i \delta x^i$

(Schaller-Strobl; Kontsevich, Cattaneo-Felder)

$$S_{PSM} = \int_M \tilde{\eta}_i d\tilde{X}^i + \frac{1}{2}\tilde{\eta}_i \tilde{\eta}_j \pi^{ij}(\tilde{X})$$

$\tilde{X}^i = \sum_{p=0}^{2} X^{i(p)}$ - superfield for x^i $= X + \eta^+ + \beta^+$ (cl. field X)

$\tilde{\eta}_i = \sum_{p=0}^{2} \eta_i^{(p)}$ - " - for ξ_i $= \beta + \eta + X^+$ (β ghost, η cl. field)

$$= \underbrace{\int_M \eta_i dX^i + \frac{1}{2}\eta_i \eta_j \pi^{ij}(X)}_{S_{cl}} + \eta^{+k}(d\beta_k + \partial_k \pi^{ij}\eta_i \beta_j) + X_i^+ \pi^{ij}\beta_j + \beta^{+k}\partial_k \pi^{ij}\beta_i\beta_j + \frac{1}{4}\eta_k^+ \eta_l^+ \partial_k \partial_l \pi^{ij}\beta_i \beta_j$$

(last term: ← quadratic term in anti-fields)

Fiber BV integrals (a.k.a. BV pushforwards, a.k.a. effective BV actions)

Assume one has a (symplectic)-orthogonal splitting of BV fields, $\mathcal{F} = \mathcal{F}' \times \mathcal{F}''$ ("IR fields" $\mathcal{F}'$, "UV fields" $\mathcal{F}''$)

$$\omega = \begin{pmatrix} \omega' & 0 \\ 0 & \omega'' \end{pmatrix}$$

$\mu = \mu' \otimes \mu''$ – vol. element

Given a sol. of QME S on $\mathcal{F}$ one can "pushforward" it to an effective action S' on $\mathcal{F}'$

$$e^{\frac{i}{\hbar} S'(\varphi')} = \int_{\mathcal{L} \subset \mathcal{F}''} e^{\frac{i}{\hbar} S(\varphi', \varphi'')} \sqrt{\mu''}\Big|_{\mathcal{L}}$$

($\mathcal{L}$ – Lagrangian)

OR: can consider the pushforward of $\frac{1}{2}$-densities $P_*^{\mathcal{L}} : \mathrm{Dens}^{\frac{1}{2}} \mathcal{F} \to \mathrm{Dens}^{\frac{1}{2}} \mathcal{F}'$

$$\int_{\mathcal{L} \subset \mathcal{F}''} \text{"} \qquad e^{\frac{i}{\hbar} S} \sqrt{\mu} \longmapsto e^{\frac{i}{\hbar} S'} \sqrt{\mu'}$$

Thm:

(i) $P_*^{\mathcal{L}} \Delta = \Delta' P_*^{\mathcal{L}}$ ($\Rightarrow$ QME for S implies QME for S')

(ii) if $\mathcal{L} \sim \tilde{\mathcal{L}}$ homotopic Lagrangians and $\Delta Z = 0$, then $P_*^{\tilde{\mathcal{L}}} Z = P_*^{\mathcal{L}} Z + \Delta'(\ldots)$ ($\Rightarrow$ changing $\mathcal{L}$, changes S' by a canonical transformation)

Def: two sols of QME $S, \tilde{S}$ differ by a canonical transf. if

$$e^{\frac{i}{\hbar} \tilde{S}} = e^{\frac{i}{\hbar} S} + \Delta_\mu (e^{\frac{i}{\hbar} S} \cdot R)$$

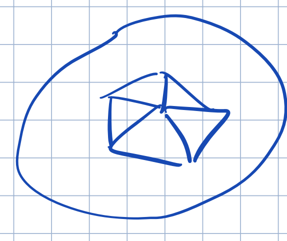
(R – generator)

infinitesimal can. transf.:

$$\tilde{S} = S + \{S, R\} - i\hbar \Delta_\mu R$$

Example (simplicial BF)

BF theory on a mfld M with a triangulation T.



$$\Omega^\bullet(M) \underset{i}{\overset{p}{\rightleftarrows}} C^\bullet(T)$$

(K acts on $\Omega^\bullet(M)$; $C^\bullet(T)$ – cell cochains)

i – Whitney embedding of cochains into dif. forms

$$p : \alpha \longmapsto \sum_{\sigma \in T} \left(\int_\sigma \alpha \right) e_\sigma$$

K – Dupont's chain homotopy operator; $dK + Kd = \mathrm{id} - i \circ p$

Then: $\Omega^\bullet(M) = \underbrace{\mathrm{im}(i)}_{\Omega'} \oplus \underbrace{\ker(p)}_{\Omega''}$ $(*)$ – splitting of the complex into def. retract and acyclic complement

$$\rightsquigarrow \underset{(A,B)}{\mathcal{F}} = \underbrace{\Omega^\bullet(M, \mathfrak{g})[1]}_{\text{decomp }(*)} \oplus \underbrace{\Omega^\bullet(M, \mathfrak{g})[n-2]}_{\text{dual decomp}} = \mathcal{F}' \oplus \mathcal{F}''$$

$\mathcal{F}' \cong C^\bullet(T)[1] \oplus C_\bullet(T)[-2] \ni (A_T, B_T)$

$\mathcal{F}'' \supset \mathcal{L} = \mathrm{im}(K) \oplus \mathrm{im}(K^\vee)$

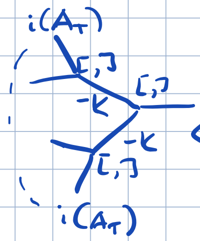
Effective action on cell cochains:

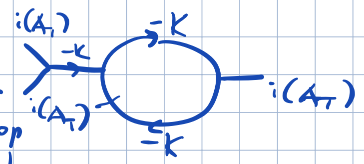
$$e^{\frac{i}{\hbar}S_T(A_T,B_T)} = \int_{\mathcal{L}\subset\mathcal{F}''} e^{\frac{i}{\hbar}S_{BF}(i(A_T)+A'',\, p^\vee(B_T)+B'')}$$

$A_T=\sum_{\sigma\in T} A^\sigma e_\sigma$, $A^\sigma\in\mathfrak{g}$

$B_T=\sum_{\sigma\in T} B_\sigma e^\sigma$, $B_\sigma\in\mathfrak{g}^*$

$\rightsquigarrow$ perturbative computation

$S_T = \sum_{\text{trees}}$ 

$\langle B_T, p(-)\rangle \; - i\hbar \sum_{\text{1-loop graphs}}$ 

,

$\mathrm{Str}_{\Omega^\bullet(M)}\, -K[i(A_T), -K[-K[i(A_T), i(A_T)], \bullet]]$

due to simplicial locality of i, p, K:

$$S_T = \sum_{\sigma\subset T} \bar{S}_\sigma(\{A^{\sigma'}\}_{\sigma'\subset\sigma}, B_\sigma)$$

given by a univ. f-la depending only on $\dim(\sigma)$

E.g.: $M = \overset{0}{\bullet}\!\!\underset{01}{-\!\!-\!\!-}\!\!\overset{1}{\bullet}$

$\left(= \langle B_{01}, F_+(\mathrm{ad}_{A_{01}})\circ A_1 + F_-(\mathrm{ad}_{A_{01}})\circ A_0\rangle\right.$, $F_+(x)=\frac{x}{1-e^{-x}}$, $\left.F_-(x)=\frac{x}{1-e^{x}}\right)$

Bernoulli number

$$\sum_{k\geq 0}\frac{B_k}{k!}\langle B_{01}, \underbrace{[A_{01},\cdots[A_{01}}_{k}, A_1-A_0]\cdots]\rangle$$

$$S_T = \langle B_0, \tfrac12[A_0,A_0]\rangle + \langle B_1, \tfrac12[A_1,A_1]\rangle + \langle B_{01}, F(\mathrm{ad}_{A_{01}})\circ(A_1-A_0) \;+[A_{01}, \tfrac{A_0+A_1}{2}]\rangle$$

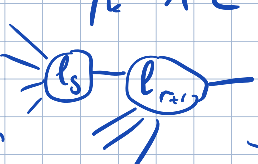
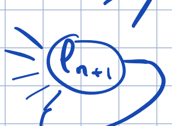
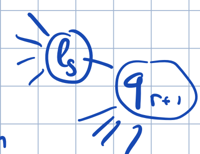
$$- i\hbar \underbrace{\log\det_{\mathfrak{g}} G(\mathrm{ad}_{A_{01}})}_{\sum_{k\geq 0}\frac{B_k}{k\cdot k!}\mathrm{tr}_{\mathfrak{g}}\underbrace{[A_{01},\cdots[A_{01}}_{k},-]\cdots]},$$

$$F(x) = \frac{x}{2}\coth\frac{x}{2} = 1 + \frac{x^2}{2!\cdot 6} - \frac{x^4}{4!\cdot 30} - \cdots$$

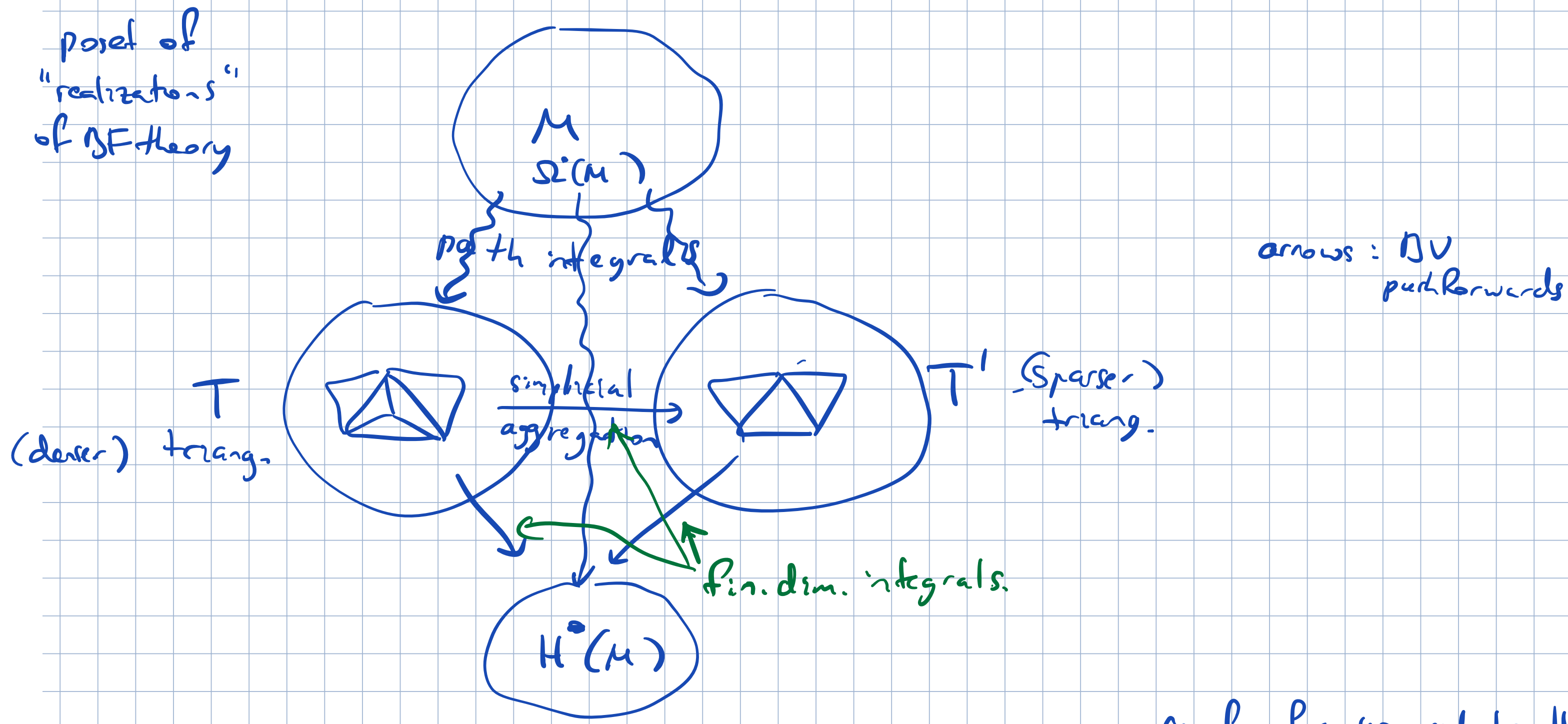
$$G(x) = \frac{\sinh\frac{x}{2}}{\frac{x}{2}} = \exp\left(\frac{x^2}{2\cdot 2\cdot 6} - \frac{x^4}{4\cdot 4!\cdot 30} + \cdots\right)$$

By construction, S_T satisfies QME ($\approx$ nontrivial relation on Bernoulli numbers)

Rem:

BV field theory	algebra
BF theory on M $S = \int_M \langle B, dA + \frac12[A,A]\rangle$	dgLa $\Omega^\bullet(M,\mathfrak{g})$, d, $[\,,]$
QME $\longleftrightarrow$	structure equations on operations $d^2=0$, Leibniz, Jacobi, $\mathrm{Str}\,\mathrm{ad}_x=0$
fiber BV integral effective theory on T $S_T = \sum_{k\geq 0}\frac{1}{k!}\langle B_T, \ell_k(A_T,\ldots,A_T)\rangle \xrightarrow{\text{Taylor expansion}}$ $-i\hbar\sum_{k\geq 0}\frac{1}{k!} q_k(A_T,\ldots,A_T)$	L_∞ algebra structure on $C^\bullet(T,\mathfrak{g})$ operations $\ell_k: \Lambda^k C \to C$ of $\deg = 2-k$ + "quantum operations" $q_k: \Lambda^k C\to \mathbb{R}$, $\deg = -k$ homotopy transfer of L_∞ algebras
QME $\longleftrightarrow$	L_∞ relations $\sum_{r+s=n}$  $=0$ + 1-loop relations  $+\sum_{r+s=n}$  $=0$

One can pushforward further, to cohomology



E.g. $M = S^1$, $\quad S_{H^\bullet(M)}(A_1, A_{d\theta}, B_{pt}, B_{S^1}) = \langle B_{pt}, \frac{1}{2}[A_1, A_1]\rangle + \langle B_{S^1}, [A_1, A_{d\theta}]\rangle$ (gen. fun. for cup-product on $H^\bullet(S^1)$) $- i\hbar \log\det G(\mathrm{ad}_{A_{d\theta}})$

gh: 1, 0, −2, −1

$1, d\theta$ - basis in $H^\bullet(S^1)$

pt, S^1 - basis in $H_\bullet(S^1)$

more generally, $S_{H^\bullet(M)}$ is a gen. fun. for "q L_∞ structure" on $H^\bullet(M, \mathfrak{g})$

- Massey operations (encode RHT of M if $\pi_1(M)=0$) + "quantum Massey operations" (related to R-torsion)

Rem: generally, a power series solution of CME $S(\varphi) = \sum_{k\geq 1} \frac{1}{(k+1)!}(\varphi, l_k(\varphi, \ldots, \varphi))$ on $\mathcal{F} = V[1]$ (gr. v.sp. with (,) inner product of deg = −3)
corresponds to a cyclic L_∞ algebra structure on V, given by (,), $\{l_k\}_{k\geq 1}$.

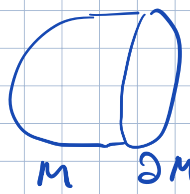
BV pushforward along $\mathcal{F} \longrightarrow \mathcal{F}'$ ($\mathcal{F} = V[1]$, $\mathcal{F}' = V'[1]$) corresponds (at 0-loop level) to a homotopy transfer of cyclic L_∞ algebras.

Ex: CS $\leftrightarrow$ cyclic dgLa $\Omega^\bullet(M, \mathfrak{g}), d, [\,,]$, $(\alpha, \beta) = \int_M \mathrm{tr}\, \alpha \wedge \beta$

effective action on $H^\bullet(M, \mathfrak{g})$ - see A. Cattaneo, P.M. "Remarks on CS invariants", 2008

Field theory on manifolds with boundary

non-gauge theory on M with bdry



F_M

$\downarrow \pi^{pre}$ ← taking normal jet of a field at ∂M

$\Phi^{pre}_{\partial M} = \{\text{normal } \infty\text{-jets of fields at } \partial M\}$, $\alpha^{pre}_{\partial M} \in \Omega^1(\Phi^{pre}_{\partial M})$, $\omega^{pre}_\partial = \delta\alpha^{pre}_\partial$

boundary pre-phase space

($\alpha^{pre}_{\partial M}$ ↑ boundary term in $\delta S_M = \int_M \underbrace{(\ldots)}_{EL}\delta\varphi + \underbrace{\int_{\partial M} \ldots}_{\alpha^{pre}_{\partial M}}$)

$\downarrow$ presymplectic reduction (π: $F_M \to \Phi_{\partial M}$)

$\Phi_{\partial M} = \Phi^{pre}_{\partial M} / \ker \omega^{pre}_{\partial M}$, $\omega_{\partial M}$

– a symplectic mfd ("phase space") (smooth in nice cases)

$EL_M \subset F_M$ – space of solutions of EL eq.

$L_{M,\partial M} = \pi(EL_M) \subset \Phi_{\partial M}$ – in good cases Lagrangian submanifold

Ex: scalar field (M,g) Riemannian n-mfd with bdry

$F_M = C^\infty(M)$, $S_M = \int_M \frac{1}{2} d\varphi \wedge * d\varphi$

$$\delta S_M = \int_M (-1)^{n+1} \underbrace{d\delta\varphi \wedge * d\varphi}_{d(\delta\varphi\wedge * d\varphi) + \delta\varphi\, \underbrace{d*d\varphi}_{-*d^*d\varphi = -*\Delta\varphi}} = \int_M \Delta\varphi\, \delta\varphi + \overbrace{\int_{\partial M} \underbrace{(*d\varphi)|_{\partial M}}_{*_\partial(\partial_n\varphi)} \delta\varphi}^{\alpha^{pre}_{\partial M}}$$

($\partial_n\varphi$ ↑ normal derivative at bdry)

$$\omega^{pre}_{\partial M} = \int_{\partial M} d\mathrm{vol}_{\partial M}\, \delta(\partial_n\varphi) \wedge \delta\varphi = (*)$$

$\Phi_{\partial M} = \{1^{st} \text{ normal jets of } \varphi \text{ on } \partial M\} = C^\infty(\partial M) \oplus C^\infty(\partial M) = \{(\varphi_\partial, \partial_n\varphi_\partial)\}$,

$\omega_{\partial M} = (*)$

· if $M = \underset{(n-1)\text{-mfd}}{\Sigma} \times [t_0, t_1]$, then

$$L_{M,\partial M} \overset{Lagr}{\subset} \overline{\Phi_{\Sigma\times\{t_0\}}} \times \Phi_{\Sigma\times\{t_1\}}$$

$$\| \quad \text{graph}(\text{Flow}_{t_1 - t_0} X_H)$$

X_H = Ham. v. field on Φ_Σ

for $H = \frac{(\partial_n\varphi_\partial)^2}{2} - \frac{d\varphi_\partial \wedge *_\partial d\varphi_\partial}{2\,d\mathrm{vol}_\Sigma}$

(BV) gauge theory on M with bdry

bulk data: $(\mathcal{F}_M, S_M, Q_M, \omega_M)$ — S_M: action, gh 0; Q_M: coh. v. f., gh +1; ω_M: (-1)-symp. form, gh -1

where $\delta S_M = \iota_{Q_M}\omega_M + \pi^* \alpha_{\partial M}$

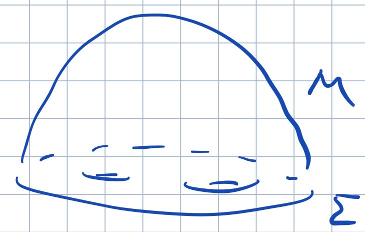
π (from $\mathcal{F}_M$ down to $\Phi_{\partial M}$); $\pi_* = d\pi$ (from Q_M down to $Q_{\partial M}$)

boundary data: $(\Phi_{\partial M}, S_{\partial M}, Q_{\partial M}, \omega_{\partial M} = \delta\alpha_{\partial M})$ — $S_{\partial M}$: gh +1, "BFV action"; $Q_{\partial M}$: gh +1; $\omega_{\partial M}$: gh 0, exact

dg Hamiltonian mfd

$\delta S_{\partial M} = \iota_{Q_{\partial M}}\omega_{\partial M}$ i.e., $S_{\partial M}$ is a Hamiltonian for $Q_{\partial M}$

Example: Chern-Simons. M - oriented 3-mfd w/ bdry Σ, cpt



bulk data

$\mathcal{F}_M = \Omega^\bullet(M, \mathfrak{g})[1] \ni \mathcal{A}$, $\quad S_M = \int_M \frac{1}{2}\langle \mathcal{A}, d\mathcal{A}\rangle + \frac{1}{6}\langle \mathcal{A}, [\mathcal{A},\mathcal{A}]\rangle$

$$Q_M = \int_M \langle d\mathcal{A} + \frac{1}{2}[\mathcal{A},\mathcal{A}], \frac{\delta}{\delta \mathcal{A}}\rangle$$

$$\omega_M = \int_M \frac{1}{2}\langle \delta\mathcal{A}, \delta\mathcal{A}\rangle$$

bdry data $\mathcal{F}_\Sigma = \Omega^\bullet(\Sigma, \mathfrak{g})$, $\quad S_\Sigma = \int_\Sigma \frac{1}{2}\langle \mathcal{A}_\Sigma, d\mathcal{A}_\Sigma\rangle + \frac{1}{6}\langle \mathcal{A}_\Sigma, [\mathcal{A}_\Sigma, \mathcal{A}_\Sigma]\rangle$

$$= \int_\Sigma \langle c, \underbrace{dA + \tfrac{1}{2}[A,A]}_{F_A}\rangle + \langle A^+, \tfrac{1}{2}[c,c]\rangle$$

where $\mathcal{A}_\Sigma = c_\Sigma + A_\Sigma + A^+_\Sigma$

$Q_{\partial M}$ = same fla as in the bulk,

$$c \mapsto \tfrac{1}{2}[c,c]$$
$$A \mapsto d_A c$$
$$A^+ \mapsto F_A + [c, A^+]$$

Rem BFV formalism (for cohom. resolution of a coisotropic reduction):

$C \subset (\Phi, \omega)$ — C coiso submfd, (Φ,ω) symp mfd. Want to describe the reduction $\underline{C} = C/_{\ker \omega|_C}$.

One looks for a dg Ham. mfd $(\Phi^{BFV}, S^{BFV}, Q^{BFV}, \omega^{BFV})$ (S^{BFV}: Ham. for Q; Q^{BFV}: coh. v.f.; ω^{BFV}: symp) s.t. $\Phi^{gh=0} = \Phi$, $\mathrm{zero}(Q) \cap \Phi = C$

then: char. distribution $\ker \omega|_C$ at $x \in C$ $= \mathrm{im}(Q)$

$\rightsquigarrow C^\infty(\underline{C}) = H^0_Q(C^\infty(\Phi))$ if $\underline{C}$ is smooth.

> rem: for (Φ, Q) a dg mfd, $x \in \mathrm{zero}(Q)$, $\hat{Q}_x \in \mathrm{End}(T_x\Phi)$ (linearization of Q) satisfies $\hat{Q}_x^2 = 0$
> $\rightsquigarrow \underbrace{\mathrm{im}\,\hat{Q}_x}_{\text{integrable distribution on } \mathrm{zero}(Q)} \subset \ker \hat{Q}_x = T_x\,\mathrm{zero}(Q)$

Ex: n CS bdry data, $C = \mathrm{FlatConn}_{\Sigma,G} \subset \Phi = \mathrm{Conn}_{\Sigma,G}$

$= \mu^{-1}(0)$, $\mu: {}^{\mathcal{G}_\Sigma} \circlearrowright \mathrm{Conn}_{\Sigma,G} \to \Omega^2(\Sigma, \mathfrak{g}) \simeq \mathfrak{g}_\Sigma^*$, $A \mapsto F_A$ — Atiyah-Bott moment map

$\underline{C} = \mathcal{M}_{flat}(\Sigma, G) \simeq \mathrm{Hom}(\pi_1\Sigma, G)/G = \mathrm{Conn}_\Sigma /\!\!/ \mathcal{G}_\Sigma$ (Marsden-Weinstein reduction)

moduli space of flat connections

note: $S^{BFV} = \int_\Sigma \langle c, \mu(A)\rangle + \ldots$ (μ: moment map)

Induced data, reductions/moduli spaces

$\mathrm{zero}(Q_M) = \mathcal{EL}_M \subset \mathcal{F}_M$ ($\mathcal{EL}_M$: sols of EL eqs)

$\pi: \mathcal{EL}_M \to L_{M,\partial M}$, $\pi: \mathcal{F}_M \to \Phi_{\partial M}$

$L_{M,\partial M} := \pi(\mathcal{EL}_M) \subset C_{\partial M} \subset \Phi_{\partial M}$ ($L_{M,\partial M}$ Lagr in $\Phi_{\partial M}$; $C_{\partial M}$ coiso)

$C_{\partial M} = \mathrm{zero}(Q_{\partial M})$ = "constraint surface"

Q-moduli spaces

$\mathcal{M}_M := \mathrm{zero}(Q)/\mathrm{im}(Q)$, Π — $(+1)$-Poisson structure inherited from ω

$\downarrow \pi_*$

$\mathcal{M}_{\partial M} := \mathrm{zero}(Q_\partial)/\mathrm{im}(Q_\partial)$, $\omega_{\partial M}$ — (reduced) 0-symp. str.

$\mathrm{im}\,\pi_* = \underline{L}_{M,\partial M}$ - Lagrangian in $\mathcal{M}_{\partial M}$

symp. leaves of Π = fibers of π_*.

Ex: Chern-Simons,
restrict to gh=0 part:

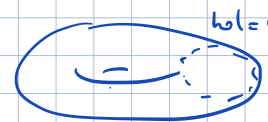
$$\mathrm{FlatConn}_M \subset \mathrm{Conn}_M$$
$$\quad\ \ \| \qquad\qquad \|$$
$$\quad\ \ EL \qquad\qquad F$$

$$L_{M,\Sigma} \subset \mathrm{FlatConn}_\Sigma \subset \mathrm{Conn}_\Sigma$$
$$\qquad\qquad\qquad \| \qquad\qquad\quad \|$$
$$\qquad\qquad\qquad C \qquad\qquad\quad \Phi$$

$L_{M,\Sigma}$ " $= \{A_\Sigma$ that can be extended as flat conn. over $M\}$"

$\rightsquigarrow$ reduction by gauge symmetry

$$\mathcal{M}_{flat}(M)$$
$$\downarrow \pi_* = i^* \qquad i: \Sigma \hookrightarrow M$$
$$L_{M,\Sigma} \underset{\text{Lagr.}}{\subset} \mathcal{M}_{flat}(\Sigma),\ \omega_{AB}$$

e.g.: $M =$  Solid torus

$L = \{$gauge classes of flat conn. on $\Sigma = S^1 \times S^1$ with trivial holonomy along the contractible $S^1\}$

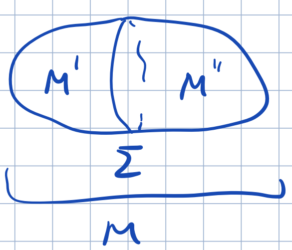
Ex: abelian ($G = \mathbb{R}$) CS (in all ghost degrees)

$$\Omega_{cl}(M)[1] \subset \Omega^\bullet(M)[1]$$
$$\downarrow i^*$$
$$\{\alpha \in \Omega(\Sigma) \text{ s.t. } \exists \tilde{\alpha} \in \Omega_{cl}(M) \text{ with } \tilde{\alpha}|_\Sigma = \alpha_\Sigma\} \subset \Omega_{cl}(\Sigma)[1] \subset \Omega^\bullet(\Sigma)[1]$$

$$\Big\|$$

$H^\bullet(M)[1]$ de Rham coh.
$\downarrow i^*$
$\Pi \in H^\bullet(M) \otimes H^\bullet(M)$ dual of Lefschetz pairing
$H^\bullet(\Sigma)[1]$ symplectic (w.r.t. Poincaré duality)

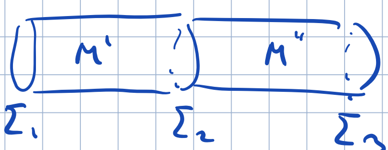
Gluing (of field theories along boundary)



gluing of manifolds $\longrightarrow$ fiber product of bulk data over bdry data

$$F_M = F_{M'} \underset{\Phi_\Sigma}{\times} F_{M''}$$

induced picture:



canonical relation / (= Lagr. correspondence)

$$\Phi_{\Sigma_1} \xrightarrow{L_{M'}} \Phi_{\Sigma_2} \xrightarrow{L_{M''}} \Phi_{\Sigma_3}$$

L_M – composition of relations

Rem: Corner ("BF^{k}V") structures

codim $= k$ stratum γ of M (corner) $\longrightarrow$ $(\Phi_\gamma, S_\gamma, Q_\gamma, \omega_\gamma = \delta\alpha_\gamma)$ – exact $(k-1)$-Ham. dg mfd

gh $=$ k (for S_γ), $+1$ (for Q_γ, coh. vf.), $k-1$

str. rel. $\delta S_\gamma^{(k)} = \iota_{Q_\gamma}\omega_\gamma^{(k)} + \pi^* \alpha_{\partial\gamma}^{(k+1)}$ ← codim. ; $\partial\gamma$ ← union of codim $= k+1$ strata

Ex: AKSZ theories

$\mathcal{F}_M = \mathrm{Map}(T[1]M, \mathcal{N})$

stratum γ $\rightsquigarrow$ $\mathcal{F}_\gamma = \mathrm{Map}(T[1]\gamma, \mathcal{N})$

$$S_\gamma = \int_\gamma \alpha_i(X)\, dX^i + \Theta(X)$$, X^i - superfield on γ

$$\omega_\gamma = \int_\gamma \tfrac{1}{2}\omega_{ij}(X)\, \delta X^i \wedge \delta X^j$$

So: densities look the same in all codimensions, but integration $\int_\gamma$ selects a particular degree.

Ex: electromagnetism (1st order formalism)

$$S_M = \int_M B \wedge dA + \tfrac{1}{2} B \wedge * B + A^+ dc$$

bulk: (BV)

$$\mathcal{F}_M = \underset{c}{\Omega^0} \xrightarrow{d} \underset{A}{\Omega^1} \xrightarrow{d} \underset{B^+}{\Omega^2}$$
$$\underset{B}{\Omega^{n-2}} \xrightarrow{d} \underset{A^+}{\Omega^{n-1}} \xrightarrow{d} \underset{c^+}{\Omega^n}$$

Q: $c \to 0$, $A \to dc$, $B^+ \to dA + *B$, $B \to 0$, $A^+ \to dB$, $c^+ \to dA^+$

bdry: (BFV)

$$\Phi_\Sigma = \underset{c}{\Omega^0} \to \underset{A}{\Omega^1}$$
$$\underset{B}{\Omega^{n-2}} \to \underset{A^+}{\Omega^{n-1}}$$

Q_Σ: $c \to 0$, $A \to dc$, $B \to 0$, $A^+ \to dB$

$$\alpha_\Sigma = \int_\Sigma B\,\delta A + A^+ \delta c$$

$$S_\Sigma = \int_\Sigma B\, dc$$

$C \subset \Phi_\Sigma$

$\{(A,B) \mid dB = 0\}$ = C, Φ_Σ = $\Omega^1 \oplus \Omega^{n-2}$

$dB=0$ ← "Gauss law"

codim = 2 structure ("BFFV")

$$\Phi_\gamma = \underset{c}{\Omega^0} \oplus \underset{B}{\Omega^{n-2}}$$

$Q_\gamma = 0$

$$\alpha = \int_\gamma B\,\delta c$$

$S_\gamma = 0$

Ex: Yang-Mills: (1st order)

bulk $$S_M = \int_M \mathrm{tr}\left(B \wedge F_A + \tfrac{1}{2} B \wedge *B + A^+ d_A c + \tfrac{1}{2} c^+ [c,c] + B^+ [B,c]\right)$$

bdry $$S_\Sigma = \int_\Sigma \mathrm{tr}\, B\, d_A c + \tfrac{1}{2} A^+ [c,c]$$

codim=2 $$S_\gamma = \int_\gamma \mathrm{tr}\, \tfrac{1}{2} B [c,c]$$

Q:
- $c \to \tfrac{1}{2}[c,c]$
- $A \to d_A c$
- $B \to [B,c]$
- $A^+ \to d_A B + [A^+, c]$
- $B^+ \to F_A + *B + [B^+, c]$
- $c^+ \to d_A A^+ + [c, c^+] + [B, B^+]$

Aside: formalism of densities / variational bicomplex $\quad \Omega^{\bullet,\bullet}_{loc}(M\times \mathcal{F}_M) \circlearrowleft Q$

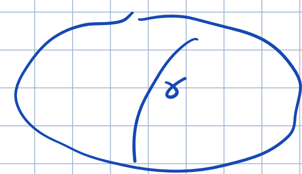
$$S_M = \int_M L^{(n)}, \quad S_\gamma = \int_\gamma L^{(\dim\gamma)} \qquad \omega_\gamma = \int_\gamma \underline{\omega}^{(\dim\gamma)}, \quad \alpha_\gamma = \int_\gamma \underline{\alpha}^{(\dim\gamma)}$$

(γ ↑ stratum in M)

data: $L^\bullet \in \bigoplus_k \Omega^{n-k,0}_{loc}(M\times\mathcal{F}_M)_k$ (↖ gh), $\quad \underline{\alpha}^\bullet \in \bigoplus_k (\Omega^{n-k,1}_{loc})_{k-1}, \quad \underline{\omega}^\bullet \in \bigoplus_k (\Omega^{n-k,2}_{loc})_{k-1}$

structure eqs:

$$Q^2 = 0$$
$$\underline{\omega}^\bullet = \delta \underline{\alpha}^\bullet$$
$$\boxed{\delta L^\bullet = \iota_Q \underline{\omega}^\bullet + d\underline{\alpha}^{\bullet-1}}$$



Ex: in CS, $\quad L^\bullet = \frac{1}{2}\langle \mathcal{A}, d\mathcal{A}\rangle + \frac{1}{6}\langle \mathcal{A}, [\mathcal{A},\mathcal{A}]\rangle$

$\qquad\qquad\qquad \alpha^\bullet = \frac{1}{2}\langle \mathcal{A}, \delta\mathcal{A}\rangle$

Rem: "f-transformations" of BV-BFV data:

$f_\Sigma \in C^\infty(\Phi_\Sigma)$

$$S_M \to S_M + \pi^* f_\Sigma$$
$$\alpha_\Sigma \longrightarrow \alpha_\Sigma + \delta f_\Sigma$$

) gives a way to change polarization in BV-BFV quantization!

in "densities" formalism: choose $\underline{f}^\bullet \in \bigoplus \Omega^{n-k,0}_{loc}(M\times\mathcal{F}_M)_{k-1}$

f-transform:

$$L^\bullet \to L^\bullet + d\underline{f}^{\bullet-1}$$
$$\underline{\alpha}^\bullet \mapsto \underline{\alpha}^\bullet + \delta \underline{f}^\bullet$$

"CME" with boundary

ver. 1: $\quad L_{Q_M} S_M = \pi^*(2S_\Sigma - \iota_{Q_\Sigma}\alpha_\Sigma)$

ver. 2: $\quad \frac{1}{2}\iota_{Q_M}\iota_{Q_M}\omega_M = \pi^* S_\Sigma$

via "densities": $\quad \frac{1}{2}\iota_Q\iota_Q\underline{\omega}^\bullet = dL^{\bullet-1}$

Rem "(extension) depth" of a BV-BFV theory.

a field theory has "depth D" if $\forall$ stratum γ of codim $\leq D-1$ $\mathcal{M}_\gamma \to \mathcal{M}_{\partial\gamma}$ has finite-dimensional fibers

(idea: degrees of freedom are supported on codim $= D$ submanifolds)

e.g.:

theory	D
scalar field	1
n-dim AKSZ thy	n
Yang-Mills	2

case $D = n$ - topological theory (only global d.o.f.)

[? example: fibers of maps btw moduli spaces for EM]

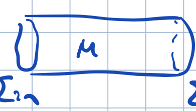
Intermezzo / reminder: Atiyah-Segal axioms of QFT

n-dim QFT is a functor of sym. monoidal categories

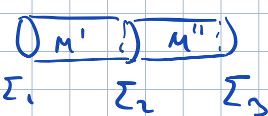
$$\mathrm{Cob}_n^{\mathrm{Geom}} \xrightarrow{(\mathcal{H}, Z)} \mathrm{Vect}$$

Ob: closed $(n-1)$-mflds Σ $\longmapsto$ vect. spaces $/\mathbb{C}$ $\mathcal{H}_\Sigma$ (space of states)

Mor: n-mfld with bdry $\longmapsto$ linear map $Z_M : \mathcal{H}_{\Sigma_{in}} \to \mathcal{H}_{\Sigma_{out}}$



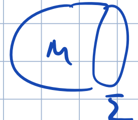
functoriality: gluing $\longmapsto$ composition $\mathcal{H}_{\Sigma_1} \xrightarrow{Z_{M'}} \mathcal{H}_{\Sigma_2} \xrightarrow{Z_{M''}} \mathcal{H}_{\Sigma_3}$



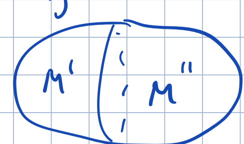
multiplicativity: $\sqcup \longmapsto \otimes$

$\emptyset \in \mathrm{Ob} \longmapsto \mathbb{C}$

Another description:

$Z($  $) \in \mathcal{H}_\Sigma$ - vector in the bdry space of states

gluing:

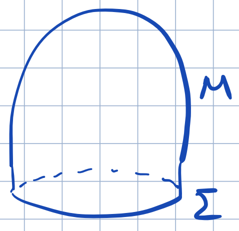


$\underbrace{\quad\Sigma\quad}_{M}$

$$Z(M) = \langle Z(M'), Z(M'') \rangle_{\mathcal{H}_\Sigma}$$

(pairing in the Hilbert space)

Quantum BV-BFV Formalism



$M \longrightarrow$ (i) $\mathcal{V}_M^{(r)}, \omega_M$ — odd-symp. graded vector space ($gh = -1$) — "residual fields"

(ii) $Z_M^{(r)} \in \mathcal{H}_\Sigma \otimes \mathrm{Dens}^{1/2}(\mathcal{V}_M)$

satisfying the modified QME:

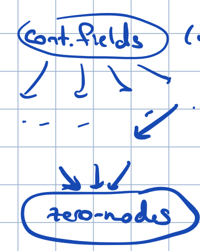
$$\left(\frac{i}{\hbar}\Omega_\Sigma - i\hbar\Delta_{\mathcal{V}_M}\right) Z_M = 0$$

$\Sigma \longrightarrow (\mathcal{H}_\Sigma^\bullet, \Omega_\Sigma)$ chain complex

(Ω_Σ: quantum BFV operator)

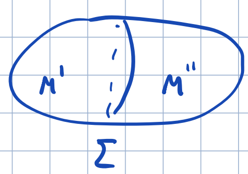
Rem.: one has a poset $\mathcal{R}$ of possible realizations r of residual fields $\mathcal{V}^r$;

for $r_1 > r_2$, $Z^{r_2} = P_* Z^{r_1}$ (P_*: BV pushforward)



- Gluing

$$Z\left(\quad \right) = P_* \langle Z_{M'}, Z_{M''} \rangle_{\mathcal{H}_\Sigma}$$



(P_*: BV pushforward along $\mathcal{V}_{M'} \times \mathcal{V}_{M''} \longrightarrow \mathcal{V}_M$)

Idea of quantization (cl. BV-BFV $\longrightarrow$ q. BV-BFV)

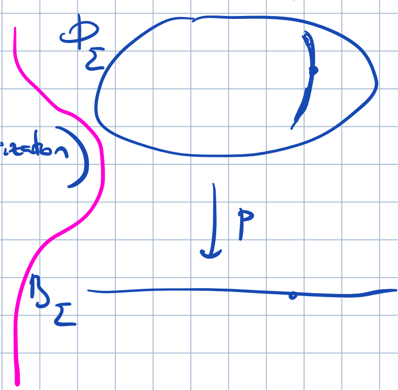
boundary quantization: $(\Phi_\Sigma, S_\Sigma, Q_\Sigma, \omega_\Sigma = \delta\alpha_\Sigma)$

choose a fibrating Lagrangian polarization $\Phi_\Sigma \xrightarrow{p} B_\Sigma$

Then: $\mathcal{H}_\Sigma := \mathrm{GeomQ}(\Phi_\Sigma, \omega_\Sigma, \Phi_\Sigma \times \mathbb{C}^*, \alpha_\Sigma, \text{polarization } p)$

($\Phi_\Sigma \times \mathbb{C}^* \to \Phi_\Sigma$: prequantum line bundle)

$= \mathrm{Dens}^{1/2}(B_\Sigma)$



fibers of p = Lag. submanifolds

assumption: $\alpha_\Sigma \big|_{\text{fibers of } p} = 0$

$\Omega_\Sigma :=$ geometric quantization of S_Σ

(ordering ambiguity; want $\Omega_\Sigma^2 = 0$)

bulk quantization

$\mathcal{F}_M \supset \mathcal{F}^b$ - fields subject to b.c. b.

$\pi \downarrow$

$\Phi_\Sigma \overset{\text{Lagr.}}{\supset} p^{-1}(b)$ - $\overset{\text{Lagr.}}{\text{boundary}}$ condition on fields

$p \downarrow$

$\mathcal{B}_\Sigma \ni b$

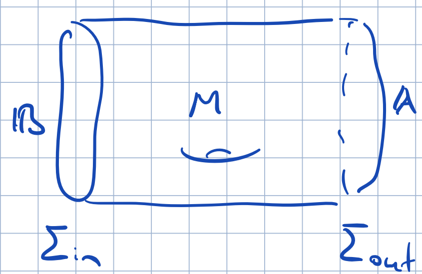
0^{-th} approximation:
$$Z(b) = \int_{\mathcal{L} \underset{\text{Lagr.}}{\subset} \mathcal{Y}} e^{\frac{i}{\hbar} S(b+\varphi)} D\varphi$$
- we split $\mathcal{F} \simeq \mathcal{B} \times \mathcal{Y}$, $\mathcal{Y} \supset \mathcal{L}$

this integral is usually obstructed (as a perturbed Gaussian integral) by zero-modes $\rightsquigarrow$ $\mathcal{Y} = \mathcal{V} \times \mathcal{Y}''$

($\mathcal{V}$: res. fields / zero-modes)

1^{st} approx
$$Z(b, \varphi_{res}) = \int_{\mathcal{L}'' \underset{\text{Lagr}}{\subset} \mathcal{Y}''} e^{\frac{i}{\hbar} S(b + \varphi_{res} + \varphi'')} D\varphi'' \rightsquigarrow Z \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B} \times \mathcal{Y}) \simeq \mathcal{H}_\Sigma \otimes \mathrm{Dens}^{\frac{1}{2}}(\mathcal{Y})$$

Example: abelian BF



$$\mathcal{F} = \overset{\mathcal{A}}{\Omega^\bullet(M)[1]} \oplus \overset{\mathcal{B}}{\Omega^\bullet(M)[n-2]}$$

$\pi \downarrow$

$\Phi_\partial = \Phi_{out} \oplus \Phi_{in}$, with $\Phi_{out} = \Omega(\Sigma_{out}) \oplus \Omega(\Sigma_{out})$, $\Phi_{in} = \Omega(\Sigma_{in}) \oplus \Omega(\Sigma_{in})$

$p \downarrow$

$\mathcal{B} = \mathcal{B}_{out} \oplus \mathcal{B}_{in}$, with $\mathcal{B}_{out} = \underset{\mathbb{A}_{out}}{\Omega(\Sigma_{out})[1]}$, $\mathcal{B}_{in} = \underset{\mathbb{B}_{in}}{\Omega(\Sigma_{in})[n-2]}$

$\mathcal{Y}$ = fiber of $p \circ \pi$ = $\Omega^\bullet(M, \Sigma_{out})[1] \oplus \Omega^\bullet(M, \Sigma_{in})[n-2]$

$\wr$ def. retraction

$\underset{\text{zero-modes}}{\mathcal{V}} = H^\bullet(\mathcal{Y}) = H^\bullet(M, \Sigma_{out})[1] \oplus H^\bullet(M, \Sigma_{in})[n-2]$, $\omega_\mathcal{V}$ = Poincaré-Lefschetz pairing

gauge-fixing: choose reps χ_a of cohom. classes in $H(M, \Sigma_{out})$; χ^a - dual reps of $H(M, \Sigma_{in})$

contraction (chain homotopy) $K: \Omega^\bullet(M, \Sigma_{out}) \to \Omega^{\bullet-1}(M, \Sigma_{out})$

given by integral kernel $\underset{\text{propagator}}{\eta} \in \Omega^{n-1}(C_2(M))$ s.t.
- $\eta(x,y) = 0$ if $x \in \Sigma_{out}$ or $y \in \Sigma_{in}$
- $d\eta = \sum_a \chi_a \otimes \chi^a$
- $\int_{S_\varepsilon^{n-1}(y) \ni x} \eta(x,y) = 1$

Space of states: (quantization of boundary)

$\Phi, S^{BFV} = \int_{\partial M} \langle B, d\alpha + \frac{1}{2}[\alpha,\alpha]\rangle \rightsquigarrow \mathcal{H} = \underbrace{\mathcal{H}_{out}}_{Fun(\mathbb{A}_{out})} \otimes \underbrace{\mathcal{H}_{in}}_{Fun(\mathbb{B}_{in})}$, $\Omega = \Omega_{out} \otimes id + id \otimes \Omega_{in}$

$Fun(\mathbb{A}_{out}) = \left\{ \sum_{k \geq 0} \int_{Conf_k(\Sigma_{out})} \psi_k \, \pi_1^* \mathbb{A}_{out} \cdots \pi_k^* \mathbb{A}_{out} \right\}$

↑ k-point wavefunction, $\psi_k \in C^\infty(Conf_k(\Sigma_{out}))$

$$\Omega_{out} = \int_{\Sigma_{out}} d\mathbb{A} \frac{\delta}{\delta \mathbb{A}}$$

$$\Omega_{in} = \int_{\Sigma_{in}} d\mathbb{B} \frac{\delta}{\delta \mathbb{B}}$$

$\int_M \mathbb{B} d\alpha + \int_{\Sigma_{in}} \mathbb{B} d\mathbb{A}$ – to account for the desired $\mathbb{B}$-polarization on Σ_{in}: $\alpha_{new} = \int_{\Sigma_{out}} \mathbb{B}\,\delta\mathbb{A} + \int_{\Sigma_{in}} \mathbb{A}\,\delta\mathbb{B}$

<u>bulk quantization</u>

"fluctuations of α, $\mathbb{B}$ into the bulk"; discontinuous extension

$$Z(\mathbb{A}, \mathbb{B}; a, b) = \int_{\mathcal{L} \subset \mathcal{Y}''} D\alpha D\beta \, e^{\frac{i}{\hbar} S(\alpha = \tilde{\mathbb{A}} + i(a) + \alpha,\ \mathbb{B} = \tilde{\mathbb{B}} + i(b) + \beta)} =$$

$S(\ldots) = \int_{\Sigma_{out}} \mathbb{A}(b+\beta) + \int_{\Sigma_{in}} \mathbb{B}(a+\alpha) + \int_M \beta\, d\alpha$

$(a,b) \in \mathcal{V}$ – rel. cohomology classes on M

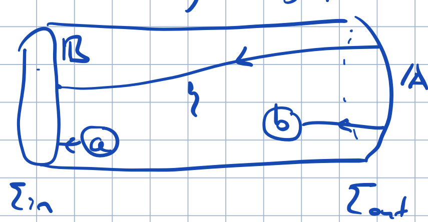
$\mathcal{L} \subset \mathcal{Y}''$: $im(K) \oplus im(K^\vee)$

$$= \tau(M, \Sigma_{out})\, e^{\frac{i}{\hbar}\left(\int_{\Sigma_{out}} b\,\mathbb{A} + \int_{\Sigma_{in}} a\,\mathbb{B} + \int_{\Sigma_{in} \times \Sigma_{out}} \mathbb{B}(x)\,\eta(x,y)\,\mathbb{A}(y)\right)}$$

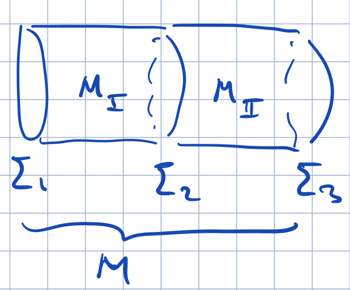
$\tau(M,\Sigma_{out}) \in Det\, H^\bullet(M, \Sigma_{out})$

↑ analytic torsion

Feynman graphs:

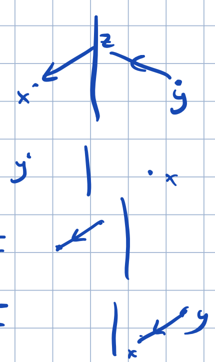


<u>Gluing of propagators</u>



$$\eta^{glued}(x,y) = \begin{cases} \int_{z \in \Sigma_2} \eta_I(x,z)\,\eta_{II}(z,y) & \text{if } x \in M_I,\ y \in M_{II} \\ 0 & \text{if } x \in M_{II},\ y \in M_I \\ \eta_I & \text{if } x,y \in M_I \\ \eta_{II} & \text{if } x,y \in M_{II} \end{cases}$$

↑ for non-minimal res. fields $\mathcal{V}_I \oplus \mathcal{V}_{II}$ on M

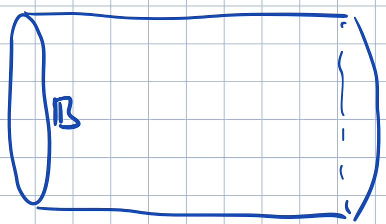


Class of examples: "BF-like" AKSZ theories

$\mathcal{F} = \Omega^\bullet(M)\otimes V \oplus \Omega^\bullet(M)\otimes V^*[n-1]$, $\Theta \in C^\infty(\overbrace{V\oplus V^*[n-1]}^{\mathcal{N}})$ polynomial potential satisfying $\{\Theta,\Theta\}=0$

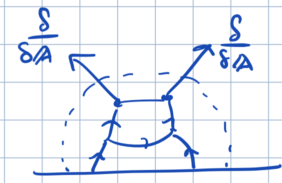
(first summand: $\mathcal{A}$, with V a gr. v. sp; second summand: $\mathcal{B}$)

$$S = \int_M \mathcal{B}\, d\mathcal{A} + \Theta(\mathcal{A},\mathcal{B})$$



$\mathcal{H}_{out} = \mathrm{Fun}(\mathbb{A})$

$\Omega_{out} = \int_{\Sigma_{out}} d\mathbb{A}\frac{\delta}{\delta \mathbb{A}} +$

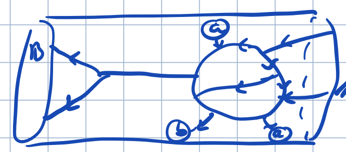


conf. space integral (configuration in half-space $\mathbb{R}^n_+ = (T_x M)_+$)

$\mathcal{H}_{in}$, Ω_{in} - similar

partition function:

$$Z(\mathbb{A},\mathbb{B}; a, b) = \tau(M,\Sigma_{out})\cdot \exp \sum_{\text{conn. graphs}}$$



Feynman rules

contrib. of a graph Γ:

$$\frac{\hbar^{-\chi(\Gamma)}}{|\mathrm{Aut}\,\Gamma|}\int_{\mathrm{Conf}_\Gamma(M)} \omega_\Gamma$$

(ω_Γ: form on conf. space)

$$\int_{\Sigma_{out}^{\times k}\times\Sigma_{in}^{\times l}} \mathbb{A}(x_1)\ldots\mathbb{A}(x_k)\,\mathbb{B}(y_1)\ldots\mathbb{B}(y_l)\; a_{a_1}\ldots a_{a_r} b^{b_1}\ldots b^{b_s}\cdot\int_{M^{\times m}} \prod_{\text{edges}}\eta \prod_{\text{vertices}} \text{coeff of } P$$

ω_Γ:

- bdry vertices: $\leftarrow\!\!|\,\mathbb{A}$, $\;|\!\!\rightarrow \mathbb{B}$
- bulk vertices: (star of arrows) - coeff of P
- "leaves": $\leftarrow$ ⓐ, ⓑ $\leftarrow$
- propagators: $\longleftarrow$ labelled η

Thm (CMR)

(i) $\Omega^2 = 0$

(ii) $\left(\frac{i}{\hbar}\Omega - i\hbar\Delta_V\right)Z = 0$ — mQME

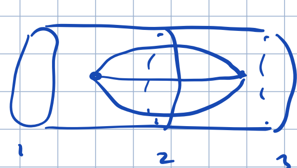
(iii) changing the gauge-fixing ϕ changes $Z \to Z + \left(\frac{i}{\hbar}\Omega - i\hbar\Delta_V\right)(\cdots)$

} from Stokes' thm for conf. space integrals

(iv) Z satisfies gluing f-la $\leftarrow$ from gluing f-la for propagators:

$$Z_M\left(\mathbb{A}_3,\mathbb{B}_1\,\middle|\,{}^{a_I, b_I}_{a_{II}, b_{II}}\right) = \int D\mathbb{A}_2 D\mathbb{B}_2\, Z_{M_{II}}(\mathbb{A}_3,\mathbb{B}_2; a_{II}, b_{II})\, e^{\frac{i}{\hbar}\int_{\Sigma_2}\mathbb{B}_2\mathbb{A}_2}\, Z_{M_I}(\mathbb{A}_2,\mathbb{B}_1; a_I, b_I)$$

Feynman graphs on M are cut into graphs on M_I, M_{II}



Idea of proof of mQME:

$$\sum_\Gamma \int_{\mathrm{Conf}_\Gamma} d\omega_\Gamma = \sum_\Gamma \int_{\partial \mathrm{Conf}_\Gamma} \omega_\Gamma$$

$\parallel$

$\frac{i}{\hbar}\Omega_0 Z - i\hbar \Delta_\nu Z$

(Ω_0: d acts on $\mathbb{A},\mathbb{B}$; Δ_ν: d acts on a propagator)

$\longrightarrow$ (1) two vertices collide in the bulk
- cancels out in $\sum_\Gamma$ due to CME for S

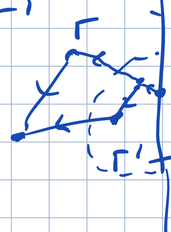
(2) ≥ 3 vertices collide in the bulk ("hidden faces")
$\to$ Kontsevich's vanishing lemmas

(3) several bulk/bdry vertices collapse at a pt. of bdry

$$\int_{\partial_{\Gamma'} \mathrm{Conf}_\Gamma} \omega_\Gamma = \int_{\mathrm{Conf}_{\Gamma'}} \int_{\mathrm{Conf}_{\Gamma/\Gamma'}} \omega_\Gamma = \frac{i}{\hbar}\Omega_{\Gamma'} Z_{\Gamma/\Gamma'}$$

$\partial_{\Gamma'}$: this stratum (subgraph Γ' collapses)

$\mathrm{Conf}_{\Gamma/\Gamma'}$: vertices & edges of Γ' collapsed



collapsing subgraph

<u>Ex:</u> non-abelian BF:

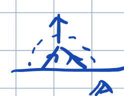
Ω_Σ = Stand. quantization of $\underbrace{S_\Sigma^{BFV}}_{\int_\Sigma \langle \mathbb{B}, d\mathbb{A} + \frac{1}{2}[\mathbb{A},\mathbb{A}]\rangle} - i\hbar \sum_{j=0}^{[\frac{n-3}{4}]} \int_\Sigma \gamma_j \, \mathrm{tr}\, \mathrm{ad}_{\mathbb{A}}^{n-4j}$

$\mathbb{A} \to \mathbb{A}\cdot$, $\mathbb{B} \to i\hbar \frac{\delta}{\delta \mathbb{A}}$ on $\mathbb{A}$-bdry

$\mathbb{A} \to i\hbar \frac{\delta}{\delta \mathbb{B}}$, $\mathbb{B} \to \mathbb{B}\cdot$ of $\mathbb{B}$-bdry

or


$\langle \mathbb{B}, d\mathbb{A} + \frac{1}{2}[\mathbb{A},\mathbb{A}]\rangle$ from



or


$\gamma_j \in \Omega^{4j}(\Sigma)$ - inv. polynomial, with univ. coeffs, of the curvature of the connection used in construction of the propagator (Pontryagin class)

<u>Ex:</u> Poisson σ-model

$$S = \int_M \eta_i dX^i + \frac{1}{2}\pi^{ij}(X)\eta_i \eta_j$$

$X = \mathbb{B}$
$\eta = \mathbb{A}$

vertices $\left\{ \cdots \right\}$ $k \geq 0$ out, 2 in

(A-) on η-boundary:

Ω_Σ = stand. quant. of $\int_\Sigma \eta_i dX^i + \frac{1}{2}\Pi^{ij}(X)\eta_i\eta_j$, ($\Sigma$: curve)

$$\Pi^{ij} = \frac{x^i \star x^j - x^j \star x^i}{-i\hbar} = \pi^{ij} + O(\hbar)$$