GGI LECTURES ON SURJECTIVE SUBMERSIONS, BUNDLE GERBES, GROUP COHOMOLOGY AND APPLICATIONS

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Abstract. These are notes of a series of lectures presented at the GGI School on "Emergent Geometries from Strings and Quantum Fields", 12-17 June 2023. In these lectures I will give an introduction to various concepts from homological algebra, algebraic topology and differential geometry, with applications to T-duality. [Based on work with my students Gianni Gagliardo and Jaklyn Crilly.]

1. Surjective submersions

1.1. **Introduction.** Traditionally, we define a theory as given by an action $S[\Phi]$, depending on a set of fields Φ on a manifold M, given by a Lagrangian density $\mathcal{L}[\Phi]$ as follows

$$S[\Phi] = \int_M \mathcal{L}[\Phi]$$

However, this is not quite precise, as usually (when the manifold M has nontrivial topology) the fields are typically sections of bundles or connections on bundles which are only defined on M locally.

So, instead, we should find a cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M, such that on each U_i we have well-defined fields Φ and such that $\mathcal{L}_i(\Phi) = \mathcal{L}_j(\Phi)$ on overlaps $U_i \cap U_j$ by symmetries of the theory. Then we choose a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to the cover $\mathcal{U} = \{U_i\}_{i\in I}$. I.e. a set of functions $\rho_i : M \to \mathbb{R}$ such that $\operatorname{Supp} \rho_i \subset U_i$ and $\sum \rho_i = 1$, and define

$$S[\Phi] = \sum_i \int_M \rho_i \mathcal{L}_i[\Phi]$$

[Alternatively one can work with simplicial decompositions of M.]

More generally, we could allow for

$$\mathcal{L}_i[\Phi] = \mathcal{L}_j[\Phi] + d\Lambda_{ij}[\Phi]$$

on overlaps $U_i \cap U_j$, and include boundary terms in the action functional (cf. the definition of the holonomy of a gauge field or bundle gerbe connection).

Despite this adequate set-up it is usually not the way one proceeds in practise. For example, in the case of the simplest topologically nontrivial manifold, the circle (or the torus) one does not proceed by choosing an open cover, instead one uses periodic boundary conditions. How to interpret periodic boundary conditions in the context of surjective submersions is illustrated in the example below.

Example.

$$\begin{array}{ccc} y & & Y=\mathbb{R} \\ \big\downarrow & & \big\downarrow \pi \\ e^{2\pi i y} & & M = S^1 = \mathbb{R}/\mathbb{Z} \end{array}$$

A function $f : \mathbb{R} \to \mathbb{Z}$ (or $\mathbb{R}, \mathbb{C}, \mathsf{U}(1)$, etc) defines a function on S^1 iff $f(y_0) = f(y_1)$ for all (y_0, y_1) such that $y_0 - y_1 \in \mathbb{Z}$ (i.e. $\pi(y_0) = \pi(y_1)$).
Or equivalently, consider the fibered product $Y^{[2]} = Y \times_M Y = \{(y_0, y_1) \in Y \times Y \mid \pi(y_0) = \pi(y_1)\}$, then the condition on $f : Y \to \mathbb{Z}$ to define a function $f : M \to \mathbb{Z}$ is that $f(y_0) = f(y_1)$ for all $(y_0, y_1) \in Y^{[2]}$, i.e. we require the cocycle condition $(\delta f)(y_0, y_1) = f(y_1) - f(y_0) = 0$.

1.2. **Surjective submersions.**

Definition 1.1. (Surjective submersion) Let $\pi : Y \to M$ be a smooth map between smooth manifolds. If $d\pi_x : T_x Y \to T_{\pi(x)} M$ is surjective for all $x \in Y$, then π is called a submersion. If, in addition, π itself is surjective then it is called a surjective submersion.

Example.

- The canonical submersion $\pi : \mathbb{R}^k \to \mathbb{R}^\ell$, $k \geq \ell$, given by $\pi(x_1, \ldots, x_k) = (x_1, \ldots, x_\ell)$
- Our example $\pi : \mathbb{R} \to S^1$, or more generally $\pi : \mathbb{R}^n \to \mathbb{T}^n$.
- A principal G-bundle $\pi : P \to M$. [Similarly for a vector bundle $TM \to M$.]
- $Y = Y_{\mathcal{U}} = \amalg_i U_i$ for an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M.

Theorem 1.2. *(Local submersion theorem) Let $\pi : Y \to M$ be a submersion such that $\pi(y) = x$. Then there exist local coordinates around $y \in Y$ and $x \in M$ such that $\pi(x_1, \ldots, x_k) = (x_1, \ldots, x_\ell)$ locally.*

Theorem 1.3. *Let $\pi : Y \to M$ be a proper submersion and let M be connected, then π is surjective.*

[Recall that a map $\pi : Y \to M$ is proper if the inverse image $\pi^{-1}(K)$ of any compact subset $K \subseteq M$, is a compact subset of Y.]

A homomorphism between surjective submersions $\tilde{\pi} : \widetilde{Y} \to M$ and $\pi : Y \to M$ is a smooth map $s : \widetilde{Y} \to Y$ such that $\pi \circ s = \tilde{\pi}$, i.e. such that the following diagram is commutative

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\quad s \quad} & Y \\ & {\scriptstyle\tilde{\pi}} \searrow \quad \swarrow {\scriptstyle\pi} & \\ & M & \end{array}$$

Theorem 1.4. *(Properties of Surjective submersions) (cf.* [5, Thm 1.22]*)*

(i) *Diffeomorphisms are surjective submersions*
(ii) *If $\tilde{\pi} : \widetilde{Y} \to Y$ and $\pi : Y \to M$ are surjective submersions, then the composition $\pi \circ \tilde{\pi} : \widetilde{Y} \to M$ is a surjective submersion.*

(iii) *Let $\pi : Y \to N$ be a surjective submersion. If $f : M \to N$ is a smooth map, then*

$$Y_M \equiv Y \times_N M = \{(y, m) \in Y \times M \,:\, \pi(y) = f(m)\} \tag{1.1}$$

is a smooth submanifold of $Y \times M$, and the projection $\pi_2 : Y_M \to M$ is a surjective submersion.

$$\begin{array}{ccc} Y_M & \xrightarrow{\pi_1} & Y \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & N \end{array}$$

(iv) *For any surjective submersion $\pi : Y \to M$, there exists an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ and a smooth map $s : Y_{\mathcal{U}} \to Y$, where $Y_{\mathcal{U}} = \coprod_{i\in I} U_i$, such that $\pi \circ s = \pi_{\mathcal{U}}$.*

$$\begin{array}{ccc} Y_{\mathcal{U}} & \xrightarrow{\quad s \quad} & Y \\ & {\scriptstyle \pi_{\mathcal{U}}}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & M & \end{array}$$

(v) *If $\pi : Y \to M$ is a surjective submersion, then*
- (a) *A function $f : M \to \mathbb{R}$ is smooth $\iff$ The function $f \circ \pi : Y \to \mathbb{R}$ is smooth.*
- (b) *A map $f : M \to N$ is smooth $\iff$ $f \circ \pi : Y \to N$ is smooth.*
- (c) *A map $f : M \to N$ is a submersion $\iff$ $f \circ \pi : Y \to N$ is a submersion.*
- (d) *A map $f : M \to N$ is a surjective submersion $\iff$ $f \circ \pi : Y \to N$ is a surjective submersion.*
- (e) *A subset $U \subseteq M$ is closed (respectively open) $\iff$ $\pi^{-1}(U) \subseteq Y$ is closed (respectively open).*
- (f) *A subset $U \subseteq M$ is a smooth submanifold of M $\iff$ $\pi^{-1}(U) \subseteq Y$ is a smooth submanifold of Y*

Remark. The properties (i)-(iii) above show that surjective submersions define a Grothendieck topology on the category of smooth manifolds [2, 3], while properties (iv)-(v) essentially say that this topology is equivalent to the topology given by open sets.

There is a distinction between surjective submersions and locally trivial fibrations, which is often not fully appreciated. For completeness we give the definition of a locally trivial fibration and the main theorem linking the two.

Definition 1.5. (Locally trivial fibration). A locally trivial fibration $\pi : Y \to M$ is a smooth map such that for each $x \in M$, there exists a neighbourhood $U \subset M$ of x satisfying

(i) There exists a diffeomorphism $f : \pi^{-1}(U) \to U \times F$, where $F = \pi^{-1}(x)$ is the fiber over x.
(ii) The following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad f \quad} & U \times F \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi_U} & \\ & U & \end{array}$$

Theorem 1.6. *(Ehresmann theorem). If $\pi : Y \to M$ is a proper surjective submersion, then it is a locally trivial fibration.*

1.3. **Submersion cohomology.** We introduce the fibered product

$$Y^{[p]} = \underbrace{Y \times_M Y \times_M \ldots \times_M Y}_{p} = \{(y_1, \ldots, y_p) \in Y^p \mid \pi(y_1) = \ldots = \pi(y_p)\}$$

and cochains $C^p(Y, M; \mathcal{F}) = \{f : Y^{[p+1]} \to \mathcal{F}\}$, with values in an abelian group $\mathcal{F}$ (typically, we take $\mathcal{F}$ to be $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ or $\mathsf{U}(1)$), with the group law written additively). Then we have differentials $\delta : C^p(Y, M; \mathcal{F}) \to C^{p+1}(Y, M; \mathcal{F})$, defined by

$$(\delta f)(y_0, \ldots, y_{p+1}) = \prod_{i=0}^{p+1} (-1)^i g(y_0, \ldots, \widehat{y_i}, \ldots, y_{p+1})$$

satisfying $\delta^2 = 0$ (**EXERCISE**). The corresponding cohomology will be denoted by $H^p(Y, M; \mathcal{F})$, and we will refer to it as 'submersion cohomology'.

Lemma 1.7. *If there exists a global section $s : M \to Y$, then $H^p(Y, M; \mathcal{F}) = 0$ for all $p \geq 1$.*

Proof. Note that for $g \in C^p(Y, M; \mathcal{F})$ with $\delta g = 0$ we have

$$0 = (\delta g)(y_0, \ldots, y_{p+1}) = \sum_{i=0}^{p+1} (-1)^i g(y_0, \ldots, \widehat{y_i}, \ldots, y_{p+1})$$

Now let $x = \pi(y_0) = \ldots = \pi(y_{p+1})$, and define $h \in C^{p-1}(Y, M; \mathcal{F})$ by $h(y_1, \ldots, y_p) = g(s(x), y_1, \ldots, y_p)$, then

$$g(y_1, \ldots, y_{p+1}) = -\sum_{i=1}^{p+1} (-1)^i g(s(x), y_1, \ldots, \widehat{y_i}, \ldots, y_{p+1}) = (\delta h)(y_1, \ldots, y_{p+1})$$

□

Question: How does $H^p(Y, M; \mathcal{F})$ depend on the surjective submersion $\pi : Y \to M$, and how is $H^p(Y, M; \mathcal{F})$ related to the Čech-cohomology of M?
Suppose we have a morphism of surjective submersions, $s : \widetilde{Y} \to Y$, i.e. a commutative diagram

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\ s\ } & Y \\ & {\scriptstyle\tilde{\pi}}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & M & \end{array}$$

then we have an induced map $s^* : H^p(Y, M; \mathcal{F}) \to H^p(\widetilde{Y}, M; \mathcal{F})$. This map can fail to be either injective or surjective.

Theorem 1.8. *If we have two homomorphisms of surjective submersions $s, t : \widetilde{Y} \to Y$, then $s^* = t^*$*

Proof. The proof is modelled on [1, p.73], which discusses the special case of a refinement $\mathcal{V}$ of an open cover $\mathcal{U}$. We define the homotopy operator $h^p : C^p(Y) \to C^{p-1}(\widetilde{Y})$ by

$$(h^p g)(\tilde{y}_0, \ldots, \tilde{y}_{p-1}) = \sum_{i=0}^{p-1} (-1)^i g(s(\tilde{y}_0), \ldots, s(\tilde{y}_i), t(\tilde{y}_i), \ldots, t(\tilde{y}_{p-1}))$$

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & C^{p-1}(Y) & \xrightarrow{\ \delta\ } & C^p(Y) & \xrightarrow{\ \delta\ } & C^{p+1}(Y) & \longrightarrow & \ldots \\ & & & {\scriptstyle h^p}\swarrow & \downarrow\downarrow {\scriptstyle s^*,\, t^*} & \swarrow{\scriptstyle h^{p+1}} & & & \\ \ldots & \longrightarrow & C^{p-1}(\widetilde{Y}) & \xrightarrow[\tilde{\delta}]{} & C^p(\widetilde{Y}) & \xrightarrow[\tilde{\delta}]{} & C^{p+1}(\widetilde{Y}) & \longrightarrow & \ldots \end{array}$$

We claim that it satisfies

$$\tilde{\delta} h^p + h^{p+1}\delta = t^* - s^*$$

Then applying this to $g \in Z^p(Y)$, i.e. $\delta g = 0$, we have

$$t^* g - s^* g = \tilde{\delta}(h^p g)$$

which would prove the assertion.

Let's first prove the $p = 1$ case as an example

$$\begin{aligned}
\tilde{\delta}(h^1 g)(\tilde{y}_0, \tilde{y}_1) &= (h^1 g)(\tilde{y}_1) - (h^1 g)(\tilde{y}_0) \\
&= g(s(\tilde{y}_1), t(\tilde{y}_1)) - g(s(\tilde{y}_0), t(\tilde{y}_0)) \\
(h^2 \delta g)(\tilde{y}_0, \tilde{y}_1) &= (\delta g)(s(\tilde{y}_0), t(\tilde{y}_0), t(\tilde{y}_1)) - (\delta g)(s(\tilde{y}_0), s(\tilde{y}_1), t(\tilde{y}_1)) \\
&= (g(t(\tilde{y}_0), t(\tilde{y}_1)) - g(s(\tilde{y}_0), t(\tilde{y}_1)) + g(s(\tilde{y}_0), t(\tilde{y}_0))) \\
&\quad - (g(s(\tilde{y}_1), t(\tilde{y}_1)) - g(s(\tilde{y}_0), t(\tilde{y}_1)) + g(s(\tilde{y}_0), s(\tilde{y}_1))) \\
&= g(t(\tilde{y}_0), t(\tilde{y}_1)) + g(s(\tilde{y}_0), t(\tilde{y}_0)) - (s(\tilde{y}_1), t(\tilde{y}_1)) - g(s(\tilde{y}_0), s(\tilde{y}_1))
\end{aligned}$$

Adding the two equations, we have

$$(\tilde{\delta} h^1 g + h^2 \delta g)(\tilde{y}_0, \tilde{y}_1) = g(t(\tilde{y}_0), t(\tilde{y}_1)) - g(s(\tilde{y}_0), s(\tilde{y}_1)) = (t^* g - s^* g)(\tilde{y}_0, \tilde{y}_1)\,.$$

More generally (**EXERCISE**)

$$\begin{aligned}
(\tilde{\delta} h^p g)(\tilde{y}_0, \ldots, \tilde{y}_p) &= \sum_{i=0}^{p} (-1)^i (h^p g)(\tilde{y}_0, \ldots, \widehat{\tilde{y}_i}, \ldots, \tilde{y}_p) \\
&= \sum_{i=0}^{p} \sum_{j=0}^{i-1} (-1)^{i+j} g(s(\tilde{y}_0), \ldots, s(\tilde{y}_j), t(\tilde{y}_j), \ldots, \widehat{t(\tilde{y}_i)}, \ldots, t(\tilde{y}_p)) \\
&\quad + \sum_{i=0}^{p} \sum_{j=i+1}^{p} (-1)^{i+j+1} g(s(\tilde{y}_0), \ldots, \widehat{s(\tilde{y}_i)}, \ldots, s(\tilde{y}_j), t(\tilde{y}_j), \ldots, t(\tilde{y}_p)) \\
(h^{p+1} \delta g)(\tilde{y}_0, \ldots, \tilde{y}_p) &= \sum_{j=0}^{p} (-1)^j (\delta g)(s(\tilde{y}_0), \ldots, s(\tilde{y}_j), t(\tilde{y}_j), \ldots, t(\tilde{y}_p)) \\
&= \sum_{j=0}^{p} \sum_{i=0}^{j} (-1)^{i+j} g(s(\tilde{y}_0), \ldots, \widehat{s(\tilde{y}_i)}, \ldots, s(\tilde{y}_j), t(\tilde{y}_j), \ldots, t(\tilde{y}_p)) \\
&\quad + \sum_{j=0}^{p} \sum_{i=j}^{p} (-1)^{i+j+1} g)(s(\tilde{y}_0), \ldots, s(\tilde{y}_j), t(\tilde{y}_j), \ldots, \widehat{t(\tilde{y}_i)}, \ldots, t(\tilde{y}_p)) \\
&= \sum_{i=0}^{p} \sum_{j=i}^{p} (-1)^{i+j} g(s(\tilde{y}_0), \ldots, \widehat{s(\tilde{y}_i)}, \ldots, s(\tilde{y}_j), t(\tilde{y}_j), \ldots, t(\tilde{y}_p)) \\
&\quad + \sum_{i=0}^{p} \sum_{j=0}^{i} (-1)^{i+j+1} g(s(\tilde{y}_0), \ldots, s(\tilde{y}_j), t(\tilde{y}_j), \ldots, \widehat{t(\tilde{y}_i)}, \ldots, t(\tilde{y}_p))
\end{aligned}$$

Adding the two together, most of the terms cancel, except the ones for $j = i$, i.e.

$$\begin{aligned}
(\tilde{\delta} h^p g + h^{p+1} \delta g)(\tilde{y}_0, \ldots, \tilde{y}_p) &= \sum_{i=0}^{p} g(s(\tilde{y}_0), \ldots, \widehat{s(\tilde{y}_i)}, t(\tilde{y}_i), \ldots, t(\tilde{y}_p)) \\
&\quad - \sum_{i=0}^{p} g(s(\tilde{y}_0), \ldots, s(\tilde{y}_i), \widehat{t(\tilde{y}_i)}, \ldots, t(\tilde{y}_p))
\end{aligned}$$

The terms cancel pairwise, except for the very first and very last term, and we obtain

$$\begin{aligned}(\tilde{\delta} h^p g + h^{p+1}\delta g)(\tilde{y}_0, \ldots, \tilde{y}_p) &= g(t(\tilde{y}_0), \ldots, t(\tilde{y}_p)) - g(s(\tilde{y}_0), \ldots, s(\tilde{y}_p)) \\ &= (t^* g - s^* g)(\tilde{y}_0, \ldots, \tilde{y}_p)\end{aligned}$$

as asserted. □

In order to understand whether the map $s^* : H^p(Y, M; \mathcal{F}) \to H^p(\widetilde{Y}, M; \mathcal{F})$ is injective/surjective, we need to study the double complex

$$C^{p,q}(Y, \widetilde{Y}, M, \mathcal{F}) = \{f : Y^{[p+1]} \times_M \widetilde{Y}^{[q+1]} \to \mathcal{F}\}$$

We have differentials

$$\begin{aligned}\delta :& C^{p,q} \to C^{p+1,q} \\ \tilde{\delta} :& C^{p,q} \to C^{p,q+1} \\ D = \delta - (-1)^p \tilde{\delta} :& \bigoplus_{p+q=r} C^{p,q} \to \bigoplus_{p+q=r+1} C^{p,q}\end{aligned}$$

and we need to determine the conditions for zigzagging through this double complex to relate $C^{p,-1}$ and $C^{-1,p}$. [Challenging **EXERCISE**.]

1.4. **Cup product.** In case $\mathcal{F}$ has a multiplicative structure we can define a cup product $\cup : C^p(Y, M; \mathcal{F}) \times C^q(Y, M; \mathcal{F}) \to C^{p+q}(Y, M; \mathcal{F})$ in analogy with the cup product on Čech cochains

$$(f \cup g)(y_0, \ldots, y_{p+q}) = f(y_0, \ldots, y_p) g(y_p, \ldots, y_{p+q})$$

In fact, $\cup = \cup_0$, is part of a family of higher cup products

$$\cup_i : C^p(Y, M; \mathcal{F}) \times C^q(Y, M; \mathcal{F}) \to C^{p+q-i}(Y, M; \mathcal{F})$$

which are described combinatorially in [6]. Explicitly, $\cup_1$ is given by

$$(f \cup_1 g)(y_0, \ldots, y_{p+q-1}) = \sum_{j=0}^{p-1} (-1)^{(p-j)(q+1)} f(y_0, \ldots, y_j, y_{j+q}, \ldots, y_{p+q-1}) g(y_j, \ldots, y_{j+q})$$

For example, for $f, g \in C^1(Y, M; \mathcal{F})$,

$$(f \cup_1 g)(y_0, y_1) = f(y_0, y_1) g(y_0, y_1)$$

They satisfy the fundamental identity $(i \geq 1)$

$$f \cup_{i-1} g + (-1)^{pq-i} g \cup_{i-1} f = (-1)^{p+q-i} \Big(\delta(f \cup_i g) - \delta f \cup_i g - (-1)^p f \cup_i \delta g \Big)$$

This identity can be interpreted in various ways. Either the rhs is the correction to $\cup_{i-1}$ being graded commutative, or the lhs is the correction to δ being a graded derivation of $\cup_i$.

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