EW SMEFT – Exercise sheet 1

Invariance under basis change

A note on notation & conventions. In this sheet we will always use Q_i, C_i to indicate Warsaw basis operators and Wilson coefficients, and $\mathcal{O}_i, f_i$ to indicate HISZ basis operators and their coefficients. We will also work with a $U(3)^5$ flavor symmetry for practicity, so all flavor indices are implicitly contracted and we use $\{m_W, m_Z, G_F\}$ as EW inputs.
A list of Warsaw basis operators relevant to this exercise is provided in the appendix.

1. The following operators are present in the HISZ basis but not in the Warsaw one:

$$\mathcal{O}_B = iB_{\mu\nu}D^\mu\Phi^\dagger D^\nu\Phi \qquad\qquad \mathcal{O}_W = iW^I_{\mu\nu}D^\mu\Phi^\dagger\sigma^I D^\nu\Phi \tag{1}$$

   (a) Convince yourself that the i factor is required for the operators to be Hermitian.

   (b) Translate them to the Warsaw basis:

   Step 1. Start from the operators

$$A_W = (D^\mu W^I_{\mu\nu})(i\Phi^\dagger \overleftrightarrow{D}^{I\nu}\Phi) \qquad\qquad A_B = (\partial^\mu B_{\mu\nu})(i\Phi^\dagger\overleftrightarrow{D}^\nu\Phi) \tag{2}$$

   and rewrite them in terms of $\mathcal{O}_W$, $\mathcal{O}_B$ and Warsaw basis operators. Use:

$$i\Phi^\dagger\overleftrightarrow{D}^\nu\Phi = i\Phi^\dagger(D^\nu\Phi) - i(D^\nu\Phi^\dagger)\Phi \tag{3}$$

$$i\Phi^\dagger\overleftrightarrow{D}^{I\nu}\Phi = i\Phi^\dagger\sigma^I(D^\nu\Phi) - i(D^\nu\Phi^\dagger)\sigma^I\Phi \tag{4}$$

$$[D_\mu, D_\nu]\Phi = \left[\frac{ig}{2}W^I_{\mu\nu}\sigma^I + \frac{ig'}{2}B_{\mu\nu}\right]\Phi \tag{5}$$

   Step 2. Now write A_W, A_B as a function of Warsaw basis operators only. Use the Equations of motion:

$$D^\mu W^I_{\mu\nu} = \frac{g}{2}(\Phi^\dagger i\overleftrightarrow{D}^I_\nu\Phi) + g\left(\bar{q}\frac{\sigma^I}{2}\gamma_\nu q + \bar{l}\frac{\sigma^I}{2}\gamma_\nu l\right) \tag{6}$$

$$\partial^\mu B^I_{\mu\nu} = \frac{g'}{2}(\Phi^\dagger i\overleftrightarrow{D}_\nu\Phi) + g'\left(\frac{1}{6}\bar{q}\gamma_\nu q + \frac{2}{3}\bar{u}\gamma_\nu u - \frac{1}{3}\bar{d}\gamma_\nu d - \frac{1}{2}\bar{l}\gamma_\nu l - \bar{e}\gamma_\nu e\right) \tag{7}$$

   You will also need the relations [*Bonus*: prove them!]

$$(\Phi^\dagger i\overleftrightarrow{D}_\nu\Phi)^2 = 4Q_{HD} + Q_{H\Box} \tag{8}$$

$$(D_\mu\Phi^\dagger D^\mu\Phi)\Phi^\dagger\Phi = \frac{Q_{H\Box}}{2} - m_h^2(\Phi^\dagger\Phi)^2 + \lambda Q_H + (Q_{eH} + Q_{uH} + Q_{dH} + \text{h.c.}) \tag{9}$$

   (the term $(\Phi^\dagger\Phi)^2$ appears at $d = 4$ and remains explicitly in the solutions)

Step 3. Put everything together equating the two expressions for A_W, A_B that you got at steps 1 and 2. Solve for $\mathcal{O}_W$ and $\mathcal{O}_B$.

(c) Determine the Wilson coefficients of the Warsaw basis operators as a function of those in the HISZ basis (i.e. $C_i(f_W, f_B)$).

2. Verify the invariance under basis change for the process $e^+e^- \to W^+W^-$. For this exercise and the next we will only look at $\mathcal{O}_B$, as the case of $\mathcal{O}_W$ is very similar.

   For this first example it is sufficient to look at the Feynman rules stemming from the operators in the two bases to verify that the basis change derived above leads exactly to the same scattering amplitude.

   (a) Consider first the Feynman rules in the Warsaw basis. The relevant ones are: (only operators relevant for the basis conversion are retained, all momenta incoming)

$$\begin{aligned}
\gamma_\rho W^+_\mu W^-_\nu \quad & ie\left[\eta^{\mu\nu}(p_{W+}-p_{W-})^\rho + \eta^{\mu\rho}(p_\gamma - p_{W+})^\nu - \eta^{\nu\rho}(p_\gamma - p_{W-})^\mu\right]\left[1-\left(\frac{\bar{C}_{HD}}{4t_\theta^2}+\frac{\bar{C}_{HWB}}{t_\theta}\right)\right] \\
& + ie\left[\eta^{\mu\rho}p_\gamma^\nu - \eta^{\nu\rho}p_\gamma^\mu\right]\frac{\bar{C}_{HWB}}{t_\theta} \\
Z_\rho W^+_\mu W^-_\nu \quad & igc_\theta\left[\eta^{\mu\nu}(p_{W+}-p_{W-})^\rho + \eta^{\mu\rho}(p_Z - p_{W+})^\nu - \eta^{\nu\rho}(p_Z - p_{W-})^\mu\right]\left[1+\frac{\bar{C}_{HD}}{4}+t_\theta \bar{C}_{HWB}\right] \\
& - ie\left[\eta^{\mu\rho}p_Z^\nu - \eta^{\nu\rho}p_Z^\mu\right]\bar{C}_{HWB} \\
\bar{e}e\gamma_\mu \quad & ie\gamma^\mu\left[1-\left(\frac{\bar{C}_{HD}}{4t_\theta^2}+\frac{\bar{C}_{HWB}}{t_\theta}\right)\right] \\
\bar{e}eZ_\mu \quad & i\frac{g}{c_\theta}\frac{c_{2\theta}}{2}(\gamma^\mu P_L)\left[1+\frac{1}{c_{2\theta}}\left(\bar{C}^{(1)}_{Hl}+\frac{\bar{C}_{HD}}{4}(3-2s_\theta^2)+s_{2\theta}\bar{C}_{HWB}\right)\right] \\
& - i\frac{g}{c_\theta}s_\theta^2(\gamma^\mu P_R)\left[1-\frac{1}{2s_\theta^2}\left(\bar{C}_{He}+\frac{\bar{C}_{HD}}{2}(2-s_\theta^2)+s_{2\theta}\bar{C}_{HWB}\right)\right]
\end{aligned}$$

   Rewrite them replacing the C_i with the functions of f_B determined at point (c) above.

   (b) Now consider the rules for the operator $\mathcal{O}_B$:

$$\begin{aligned}
\gamma_\rho W^+_\mu W^-_\nu \quad & i\frac{g^2 c_\theta}{4}\bar{f}_B(\eta^{\mu\rho}p_\gamma^\nu - \eta^{\nu\rho}p_\gamma^\mu) \\
Z_\rho W^+_\mu W^-_\nu \quad & -i\frac{g^2 s_\theta}{4}\bar{f}_B(\eta^{\mu\rho}p_Z^\nu - \eta^{\nu\rho}p_Z^\mu)
\end{aligned}$$

   Verify that they coincide with those obtained at the previous point, starting from Warsaw basis operators. This directly proves that the two bases give identical predictions at amplitude level.

   (c) Convince yourself that the equivalence holds for the γ and Z diagrams individually and also independently for each chirality in initial state.

3. Verify the invariance under basis change in $\bar{u}u \to Zh$.
   This case is less trivial! We will actually need to compute the amplitudes.

   (a) Compute first the scattering amplitude with one $\mathcal{O}_B$ insertion. The relevant Feynman rules are:

$$Z_\mu Z_\nu h \qquad \frac{ie}{2vc_\theta}\bar{f}_B\left[\eta^{\mu\nu}p_h\cdot(p_{Z1}+p_{Z2})-p_h^\nu p_{Z2}^\mu-p_h^\mu p_{Z1}^\nu\right]$$

$$\gamma_\mu Z_\nu h \qquad -\frac{ig}{2v}\bar{f}_B\left[\eta^{\mu\nu}p_h\cdot p_\gamma-p_\gamma^\nu p_h^\mu\right]$$

⚠ The rules are given for all momenta *incoming*. Change signs appropriately when inserting in the diagrams.

(b) Now consider the FR for the Warsaw basis and convert them to f_B:

$$Z_\mu Z_\nu h \qquad \frac{ig^2v}{2c_\theta^2}\eta^{\mu\nu}\left[1+\left(\bar{C}_{H\Box}+\frac{\bar{C}_{HD}}{4}\right)\right]+\frac{4is_\theta}{v}\left[p_{Z1}^\nu p_{Z2}^\mu-p_{Z1}\cdot p_{Z2}\eta^{\mu\nu}\right]\left(\bar{C}_{HWB}c_\theta+\bar{C}_{HB}s_\theta\right)$$

$$\gamma_\mu Z_\nu h \qquad -\frac{2i}{v}\left[p_\gamma^\nu p_Z^\mu-p_\gamma\cdot p_Z\eta^{\mu\nu}\right]\left(c_{2\theta}\bar{C}_{HWB}+s_{2\theta}\bar{C}_{HB}\right)$$

$$\bar{u}uZ_\mu \qquad -i\frac{g}{c_\theta}\left(\frac{1}{2}-\frac{2s_\theta^2}{3}\right)(\gamma^\mu P_L)\left[1-\frac{3}{3-4s_\theta^2}\left(\bar{C}_{Hq}^{(1)}+\bar{C}_{HD}\frac{-5+4s_\theta^2}{12}-\frac{2s_{2\theta}}{3}\bar{C}_{HWB}\right)\right]$$
$$+i\frac{g}{c_\theta}\frac{2}{3}s_\theta^2(\gamma^\mu P_R)\left[1+\frac{3}{4s_\theta^2}\left(\bar{C}_{Hu}+\bar{C}_{HD}\frac{-2+s_\theta^2}{3}-\frac{2s_{2\theta}}{3}\bar{C}_{HWB}\right)\right]$$

$$\bar{u}uZ_\mu H \qquad \frac{ig}{vc_\theta}\left[\bar{C}_{Hq}^{(1)}(\gamma^\mu P_L)+\bar{C}_{Hu}(\gamma^\mu P_R)\right]$$

(c) Using the Warsaw basis FR simplified at the previous point, compute the scattering amplitude linear in f_B.

⚠ There are 3 diagrams here: with a Z, with a γ and with a contact term.

(d) Verify that the total amplitudes computed directly with $\mathcal{O}_B$ and through the translation from the Warsaw basis are identical. This should be true independently for each chirality in initial state.

Unlike in the previous example, here the equivalence does not occur diagram by diagram or vertex by vertex, but requires to sum all contributions!

A List of relevant Warsaw basis operators

$$Q_{HW}=W^I_{\mu\nu}W^{I\mu\nu}\Phi^\dagger\Phi \qquad Q_{HD}=(\Phi^\dagger D_\mu\Phi)(D^\mu\Phi^\dagger\Phi)$$
$$Q_{HB}=B_{\mu\nu}B^{\mu\nu}\Phi^\dagger\Phi \qquad Q_{H\Box}=\Box(\Phi^\dagger\Phi)(\Phi^\dagger\Phi) \qquad Q_H=(\Phi^\dagger\Phi)^3$$
$$Q_{HWB}=B_{\mu\nu}W^{I\mu\nu}\Phi^\dagger\sigma^I\Phi$$

$$Q_{Hl}^{(3)}=(i\Phi^\dagger\overleftrightarrow{D}^{I\nu}\Phi)(\bar{l}\gamma_\nu\sigma^I l) \qquad Q_{Hq}^{(3)}=(i\Phi^\dagger\overleftrightarrow{D}^{I\nu}\Phi)(\bar{q}\gamma_\nu\sigma^I q)$$
$$Q_{Hl}^{(1)}=(i\Phi^\dagger\overleftrightarrow{D}^{\nu}\Phi)(\bar{l}\gamma_\nu l) \qquad Q_{Hq}^{(1)}=(i\Phi^\dagger\overleftrightarrow{D}^{\nu}\Phi)(\bar{q}\gamma_\nu q)$$
$$Q_{Hu}=(i\Phi^\dagger\overleftrightarrow{D}^{\nu}\Phi)(\bar{u}\gamma_\nu u) \qquad Q_{Hd}=(i\Phi^\dagger\overleftrightarrow{D}^{\nu}\Phi)(\bar{d}\gamma_\nu d)$$
$$Q_{He}=(i\Phi^\dagger\overleftrightarrow{D}^{\nu}\Phi)(\bar{e}\gamma_\nu e)$$

$$Q_{uH}=(\bar{q}\tilde{\Phi}Y_u^\dagger u)(\Phi^\dagger\Phi) \qquad Q_{dH}=(\bar{q}\Phi Y_d^\dagger d)(\Phi^\dagger\Phi)$$
$$Q_{eH}=(\bar{l}\Phi Y_e^\dagger e)(\Phi^\dagger\Phi)$$

EW SMEFT – Exercise sheet 1 – Solutions

Invariance under basis change

1. (a)

$$\mathcal{O}_B^\dagger = (-i)B_{\mu\nu}(D^\mu\Phi)(D^\nu\Phi^\dagger) = iB_{\mu\nu}(D^\nu\Phi^\dagger)(D^\mu\Phi) = \mathcal{O}_B \tag{1}$$

and the same for $\mathcal{O}_W$. If the i were not there, there would be an overall minus sign to the right hand side.

(b) **Step 1.**

$$\begin{aligned}
A_W &= -W^I_{\mu\nu}D^\mu\left(\Phi^\dagger i\overleftrightarrow{D}^{I\nu}\Phi\right) && (2)\\
&= -2iW^I_{\mu\nu}D^\mu\Phi^\dagger\sigma^I D^\nu\Phi + \frac{g}{2}W^I_{\mu\nu}W^{I\mu\nu}\Phi^\dagger\Phi + \frac{g'}{2}W^I_{\mu\nu}B^{\mu\nu}\Phi^\dagger\sigma^I\Phi && (3)\\
&= -2\mathcal{O}_W + \frac{g}{2}Q_{HW} + \frac{g}{2}Q_{HWB}\,, && (4)\\
A_B &= -B_{\mu\nu}D^\mu\left(i\Phi^\dagger\overleftrightarrow{D}^\nu\Phi\right) && (5)\\
&= -2iB_{\mu\nu}D^\mu\Phi^\dagger D^\nu\Phi + \frac{g'}{2}B_{\mu\nu}B^{\mu\nu}\Phi^\dagger\Phi + \frac{g}{2}W^I_{\mu\nu}B^{\mu\nu}\Phi^\dagger\sigma^I\Phi && (6)\\
&= -2\mathcal{O}_B + \frac{g'}{2}Q_{HB} + \frac{g}{2}Q_{HWB}. && (7)
\end{aligned}$$

Step 2.

$$A_W = g\left(2\Phi^\dagger\Phi\left(D_\mu\Phi^\dagger D^\mu\Phi\right) + \frac{Q_{H\Box}}{2} + \frac{1}{2}\left(Q^{(3)}_{Hl} + Q^{(3)}_{Hq}\right)\right), \tag{8}$$

$$= g\left[\frac{3}{2}Q_{H\Box} - 2m_h^2(\Phi^\dagger\Phi)^2 + 2\lambda Q_H + 2\left(Q_{eH} + Q_{uH} + Q_{dH} + \text{h.c.}\right) + \frac{1}{2}\left(Q^{(3)}_{Hl} + Q^{(3)}_{Hq}\right)\right] \tag{9}$$

$$A_B = g'\left(\frac{1}{6}Q^{(1)}_{Hq} + \frac{2}{3}Q_{Hu} - \frac{1}{3}Q_{Hd} - \frac{1}{2}Q^{(1)}_{Hl} - Q_{He} + \frac{Q_{H\Box}}{2} + 2Q_{HD}\right). \tag{10}$$

Step 3.

$$\mathcal{O}_W = \frac{1}{2}\Bigg[\frac{g}{2}Q_{HW} + \frac{g'}{2}Q_{HWB} - \frac{3g}{2}Q_{H\Box} + 2gm^2\left(\Phi^\dagger\Phi\right)^2 - 2g\lambda Q_H$$
$$- g\left(Q_{eH} + Q_{uH} + Q_{dH} + \text{h.c.}\right) - g\left(\frac{1}{2}Q_{Hl}^{(3)} + \frac{1}{2}Q_{Hq}^{(3)}\right)\Bigg], \tag{11}$$

$$\mathcal{O}_B = \frac{1}{2}\Bigg[\frac{g'}{2}Q_{HB} + \frac{g}{2}Q_{HWB} - \frac{g'}{2}Q_{H\Box} - 2g'Q_{HD}$$
$$- g'\left(\frac{1}{6}Q_{Hq}^{(1)} + \frac{2}{3}Q_{Hu} - \frac{1}{3}Q_{Hd} - \frac{1}{2}Q_{Hl}^{(1)} - Q_{He}\right)\Bigg]. \tag{12}$$

(c) Conversion of the Wilson coefficients:

$$\begin{aligned} &\mathcal{O} = RQ \\ &f\mathcal{O} = CQ = CR^{-1}\mathcal{O} \qquad\qquad \to f = CR^{-1}, \quad C = fR \end{aligned} \tag{13}$$

where R is the basis rotation matrix between operators. Since we do not have the complete HISZ basis, we can only convert C into f and not vice versa.
In order to translate from Warsaw to $\mathcal{O}_W$ one has to map simultaneously

$$\begin{aligned} &C_{HW} \to \frac{g}{4}f_W \qquad C_{HWB} \to \frac{g'}{4}f_W \qquad C_{H\Box} \to -\frac{3g}{4}f_W \qquad C_H \to -g\lambda f_W \\ &C_{Hq}^{(3)} \to -\frac{g}{4}f_W \qquad C_{Hl}^{(3)} \to \frac{g}{4}f_W \\ &C_{uH} \to -\frac{g}{2}f_W \qquad C_{dH} \to -\frac{g}{2}f_W \qquad C_{eH} \to -\frac{g}{2}f_W \end{aligned} \tag{14}$$

(and add the quartic contribution to the potential).
In order to translate from Warsaw to $\mathcal{O}_B$ one has to map simultaneously

$$\begin{aligned} &C_{HB} \to \frac{g'}{4}f_B \qquad C_{HWB} \to \frac{g}{4}f_B \qquad C_{H\Box} \to -\frac{g'}{4}f_B \quad C_{HD} \to -g'f_B \\ &C_{Hq}^{(1)} \to -\frac{g'}{12}f_B \qquad C_{Hu} \to -\frac{g'}{3}f_B \quad C_{Hd} \to \frac{g'}{6}f_B \qquad C_{Hl}^{(1)} \to \frac{g'}{4}f_B \qquad C_{He} \to \frac{g'}{2}f_B \end{aligned} \tag{15}$$

2. (a) The shifts appearing in the FR map to

$$\frac{\bar{C}_{HD}}{4t_\theta^2} + \frac{\bar{C}_{HWB}}{t_\theta} \to 0 \tag{16}$$

$$\bar{C}_{HWB} \to \frac{g}{4}\bar{f}_B \tag{17}$$

$$\bar{C}_{Hl}^{(1)} + \frac{\bar{C}_{HD}}{4}(3 - 2s_\theta^2) + s_{2\theta}\bar{C}_{HWB} \to 0 \tag{18}$$

$$\bar{C}_{He} + \frac{\bar{C}_{HD}}{2}(2 - s_\theta^2) + s_{2\theta}\bar{C}_{HWB} \to 0 \tag{19}$$

so, of all contributions, only the corrections to $WW\gamma$ and WWZ survive and they become

$$\gamma_\rho W^+_\mu W^-_\nu \quad ie\left[\eta^{\mu\rho}p^\nu_\gamma - \eta^{\nu\rho}p^\mu_\gamma\right]\frac{g}{4t_\theta}\bar{f}_B$$

$$Z_\rho W^+_\mu W^-_\nu \quad -ie\left[\eta^{\mu\rho}p^\nu_Z - \eta^{\nu\rho}p^\mu_Z\right]\frac{g}{4}\bar{f}_B$$

(b) the rules for the operator $\mathcal{O}_B$ are indeed identical to those derived at the previous point.

(c) The equivalence holds for the γ and Z diagrams individually because it happens separately for the $WW\gamma$ and WWZ vertices. It also holds independently for each chirality in initial state, because the fermionic current is unaffected by the operators in any case.

3. (a) There are two diagrams, one with a Z and one with a γ in s channel. Let's compute them separately. We will use $q = p_Z + k$ for the s channel momentum, p_Z for the momentum of the outgoing Z and k for the momentum of the outgoing Higgs. We also shorten

$$J^\rho_{uuZ,SM} = \bar{u}\gamma^\rho(g^u_L P_L + g^u_R P_R)v \qquad g^u_L = \frac{1}{2} - \frac{2}{3}s^2_\theta \qquad g^u_R = -\frac{2}{3}s^2_\theta \tag{20}$$

$$J^\rho_{uu,\gamma} = \frac{2}{3}\bar{u}\gamma^\rho v \tag{21}$$

$$\begin{aligned}
A^{\mathcal{O}_B}_Z &= -\frac{ig}{c_\theta}J^\rho_{uuZ,SM}\frac{-i}{q^2-m_Z^2}\left(\eta^{\mu\rho} - \frac{q^\rho q^\mu}{m_Z^2}\right)\frac{ie}{vc_\theta}\bar{f}_B\left[-\eta^{\mu\nu}\frac{(q-p_Z)^2}{2} - q^\nu p^\mu_Z + \frac{1}{2}q^\mu q^\nu\right]\varepsilon^*_\nu(p_Z)\\
&= -\frac{ige}{vc^2_\theta}\bar{f}_B\frac{J^\rho_{uuZ,SM}}{q^2-m_Z^2}\left[-\eta^{\rho\nu}\frac{k^2}{2} - q^\nu p^\rho_Z + \frac{1}{2}q^\nu q^\rho + \frac{q^\rho q^\nu}{m_Z^2}\frac{(q-p_Z)^2}{2} + \frac{q^\rho q^\nu}{m_Z^2}q\cdot p_Z - \frac{1}{2}q^\rho q^\nu\frac{q^2}{m_Z^2}\right]\varepsilon^*_\nu(p_Z)\\
&= -\frac{ige}{vc^2_\theta}\bar{f}_B\frac{J^\rho_{uuZ,SM}}{q^2-m_Z^2}\left[-\frac{m_h^2}{2}\eta^{\rho\nu} - q^\nu p^\rho_Z + \frac{q^\rho q^\nu}{m_Z^2}\left(\frac{1}{2}m_Z^2 + \frac{1}{2}q^2 + \frac{1}{2}m_Z^2 - q\cdot p_Z + q\cdot p_Z - \frac{1}{2}q^2\right)\right]\varepsilon^*_\nu(p_Z)\\
&= -\frac{ige}{vc^2_\theta}\bar{f}_B\frac{J^\rho_{uuZ,SM}}{q^2-m_Z^2}\left[-\frac{m_h^2}{2}\eta^{\rho\nu} + k^\rho q^\nu\right]\varepsilon^*_\nu(p_Z)
\end{aligned}\tag{22}$$

and

$$\begin{aligned}
A^{\mathcal{O}_B}_\gamma &= -ieJ^\rho_{uu\gamma,SM}\frac{-i\eta^{\mu\rho}}{q^2}\frac{-ig}{2v}\bar{f}_B\left[-\eta^{\mu\nu}q\cdot k + q^\nu k^\mu\right]\varepsilon^*_\nu(p_Z)\\
&= i\frac{eg}{2v}\bar{f}_B\frac{J^\rho_{uu\gamma,SM}}{q^2}\left[-\eta^{\rho\nu}\frac{q^2+m_h^2-m_Z^2}{2} + q^\nu k^\rho\right]\varepsilon^*_\nu(p_Z)
\end{aligned}\tag{23}$$

(b) The converted FR are

$$Z_\mu Z_\nu h \quad \frac{ig^2v}{2c^2_\theta}\eta^{\mu\nu}\left[1 - \frac{g'}{2}\bar{f}_B\right] + \frac{4is_\theta}{v}\left[p^\nu_{Z1}p^\mu_{Z2} - p_{Z1}\cdot p_{Z2}\eta^{\mu\nu}\right]\frac{g}{4c_\theta}\bar{f}_B$$

$$\gamma_\mu Z_\nu h \quad -\frac{2i}{v}\left[p^\nu_\gamma p^\mu_Z - p_\gamma\cdot p_Z\eta^{\mu\nu}\right]\frac{g}{4}\bar{f}_B$$

$$\bar{u}uZ_\mu \quad -i\frac{g}{c_\theta}\left(\frac{1}{2} - \frac{2s^2_\theta}{3}\right)(\gamma^\mu P_L) + i\frac{g}{c_\theta}\frac{2}{3}s^2_\theta(\gamma^\mu P_R)$$

$$\bar{u}uZ_\mu H \quad -\frac{igs_\theta}{3vc^2_\theta}\bar{f}_B\left[\frac{1}{4}(\gamma^\mu P_L) + (\gamma^\mu P_R)\right]$$

(c)

$$
\begin{aligned}
A_Z^{\text{Warsaw}\to \text{f}_\text{B}} &= -\frac{ig}{c_\theta} J^\rho_{uuZ,SM} \frac{-i}{q^2-m_Z^2}\left(\eta^{\mu\rho} - \frac{q^\rho q^\mu}{m_Z^2}\right) \frac{ie}{vc_\theta} \bar{f}_B \left[-(m_Z^2 - p_Z\cdot q)\eta^{\mu\nu} - q^\nu p_Z^\mu\right] \varepsilon^*_\nu(p_Z) \\
&= -\frac{ige}{vc_\theta^2}\bar{f}_B \frac{J^\rho_{uuZ,SM}}{q^2-m_Z^2}\left[\eta^{\rho\nu} p_Z\cdot k - q^\nu p_Z^\rho - \frac{q^\rho q^\nu}{m_Z^2}(p_Z\cdot k) + \frac{q^\rho q^\nu}{m_Z^2} q\cdot p_Z\right]\varepsilon^*_\nu(p_Z) \\
&= -\frac{ige}{vc_\theta^2}\bar{f}_B \frac{J^\rho_{uuZ,SM}}{q^2-m_Z^2}\left[\eta^{\rho\nu} p_Z\cdot k - q^\nu p_Z^\rho + q^\rho q^\nu\right]\varepsilon^*_\nu(p_Z) \\
&= -\frac{ige}{vc_\theta^2}\bar{f}_B \frac{J^\rho_{uuZ,SM}}{q^2-m_Z^2}\left[\eta^{\rho\nu} \frac{q^2 - m_Z^2 - m_h^2}{2} + k^\rho q^\nu\right]\varepsilon^*_\nu(p_Z) \\
&= -\frac{ige}{vc_\theta^2}\bar{f}_B \left[\bar{u}\gamma^\rho(g_L^u P_L + g_R^u P_R)v\right]\left[\frac{1}{2}\eta^{\rho\nu}\right]\varepsilon^*_\nu(p_Z) + A_Z^{\mathcal{O}_B} \qquad (24)
\end{aligned}
$$

Note that in the last line, in subtracting $A_Z^{\mathcal{O}_B}$, we remained with a factor $(q^2 - m_Z^2)$ in the numerator that canceled out with the propagator dependence in the denominator. For the photon we have something similar:

$$
\begin{aligned}
A_\gamma^{\text{Warsaw}\to \text{f}_\text{B}} &= -ieJ^\rho_{uu\gamma,SM}\frac{-i\eta^{\mu\rho}}{q^2}\frac{-ig}{2v}\bar{f}_B\left[\eta^{\mu\nu} q\cdot p_Z - q^\nu p_Z^\mu\right]\varepsilon^*_\nu(p_Z) \\
&= i\frac{eg}{2v}\bar{f}_B\frac{J^\rho_{uu\gamma,SM}}{q^2}\left[\eta^{\nu\rho}\frac{q^2+m_Z^2-m_h^2}{2} - q^\nu p_Z^\rho\right]\varepsilon^*_\nu(p_Z) \qquad (25) \\
&= i\frac{eg}{2v}\bar{f}_B J^\rho_{uu\gamma,SM}\left[\eta^{\nu\rho} - \frac{q^\nu q^\rho}{q^2}\right] + A_\gamma^{\mathcal{O}_B} \qquad (26)
\end{aligned}
$$

In the last row, the term in $q^\nu q^\rho$ can be removed because $J^\rho_{uu\gamma} q_\rho = 0$.
Now the contact terms: for these we need to split LH and RH up quarks:

$$
A_{uuZH,L}^{\text{Warsaw}\to \text{f}_\text{B}} = -\frac{ieg}{12vc_\theta^2}\bar{f}_B(\bar{u}\gamma^\nu P_L v)\varepsilon^*_\nu(p_Z) \qquad (27)
$$

$$
A_{uuZH,R}^{\text{Warsaw}\to \text{f}_\text{B}} = -\frac{ieg}{3vc_\theta^2}\bar{f}_B(\bar{u}\gamma^\nu P_R v)\varepsilon^*_\nu(p_Z) \qquad (28)
$$

(d) Let's put everything together, for LH and RH up quarks separately:

$$
\begin{aligned}
&A_{Z,L}^{\text{Warsaw}\to \text{f}_\text{B}} + A_{\gamma,L}^{\text{Warsaw}\to \text{f}_\text{B}} + A_{uuZH,L}^{\text{Warsaw}\to \text{f}_\text{B}} \qquad (29) \\
&= A_{Z,L}^{\mathcal{O}_B} + A_{\gamma,L}^{\mathcal{O}_B} + \frac{ieg}{vc_\theta^2}\bar{f}_B\varepsilon^*_\nu(p_Z)(\bar{u}\gamma^\nu P_L v)\left[-g_L^u\frac{1}{2} + \frac{c_\theta^2}{2}\frac{2}{3} - \frac{1}{12}\right] \qquad (30) \\
&= A_{Z,L}^{\mathcal{O}_B} + A_{\gamma,L}^{\mathcal{O}_B} \qquad (31)
\end{aligned}
$$

and

$$
\begin{aligned}
&A_{Z,R}^{\text{Warsaw}\to \text{f}_\text{B}} + A_{\gamma,R}^{\text{Warsaw}\to \text{f}_\text{B}} + A_{uuZH,R}^{\text{Warsaw}\to \text{f}_\text{B}} \qquad (32) \\
&= A_{Z,R}^{\mathcal{O}_B} + A_{\gamma,R}^{\mathcal{O}_B} + \frac{ieg}{vc_\theta^2}\bar{f}_B\varepsilon^*_\nu(p_Z)(\bar{u}\gamma^\nu P_R v)\left[-g_R^u\frac{1}{2} + \frac{c_\theta^2}{2}\frac{2}{3} - \frac{1}{3}\right] \qquad (33) \\
&= A_{Z,L}^{\mathcal{O}_B} + A_{\gamma,L}^{\mathcal{O}_B} \qquad (34)
\end{aligned}
$$

as

$$-\frac{g_L^u}{2}+\frac{c_\theta^2}{3}-\frac{1}{12}=-\frac{1}{4}+\frac{1}{3}s_\theta^2+\frac{c_\theta^2}{3}-\frac{1}{12}=0 \tag{35}$$

$$-\frac{g_R^u}{2}+\frac{c_\theta^2}{3}-\frac{1}{3}=\frac{1}{3}s_\theta^2+\frac{c_\theta^2}{3}-\frac{1}{3}=0 \tag{36}$$

It's interesting to note that really all diagrams are needed in order for the contributions to cancel.

The messages of these exercises are:

- Naive operator classifications into "bosonic operators" vs "fermionic operators" or "operators affecting TGC" etc are basis dependent statements.
- When computing a SMEFT observable, in order to have a physical, basis independent result, we need to retain contributions from *all* operators in the basis. In the example above, if we had neglected one or more Warsaw basis operators, we wouldn't have been able to reconstruct properly the effect of $\mathcal{O}_B$.

EW SMEFT – Exercise sheet 2

The W mass in SMEFT

1. Compute the correction to m_W^2 in the Warsaw basis, at tree level and up to Λ^{-4} corrections.

   (a) Remember that, at LO, the W pole mass in the SMEFT Lagrangian is just $m_W^2 = g^2 v_T^2/4$, and leave the result written in terms of $\delta g/g$ etc.

   (b) Using the shift formulas in the appendix, specialize the result to the input parameters set $\{\alpha, G_F, m_Z\}$ and write the correction to m_W in terms of the Wilson coefficients.

   (c) Do the same for the input parameters set $\{m_W, G_F, m_Z\}$. Verify that the correction to m_W vanishes in this case, as expected for any input quantities,

2. Now we will relate the m_W correction to the ρ parameter.

   (a) Compute the relative correction to $\cos^2\theta$ (i.e. $\delta c_\theta^2/c_\theta^2$) in the Warsaw basis, again at LO and up to Λ^{-4} corrections.
   Remember that the angle is defined by

   $$\theta = \arctan\left[\frac{g'}{g} + \frac{1}{2}\frac{gg'}{g^2+(g')^2}\bar{C}_{HWB}\right] \tag{1}$$

   As above, leave the result in the inputs-independent form, as a function of $\delta g/g$ etc.

   (b) Write $\delta c_\theta^2/c_\theta^2$ in terms of Wilson coefficients, specializing to the $\{\alpha, G_F, m_Z\}$ and $\{m_W, G_F, m_Z\}$ input schemes.

   (c) By comparing with the result of the previous exercise, verify that the following relation holds in both schemes

   $$\frac{\delta m_W^2}{m_W^2} = -\Delta m_Z^2 + \frac{\delta c_\theta^2}{c_\theta^2} + \frac{s_{4\theta}}{4c_\theta^2}\bar{c}_{HWB} \tag{2}$$

   Can you give an intuitive interpretation of this formula?

   (d) The ρ parameter can be defined *à la Veltman*, from the ratio of Z (neutral) and W (charged) currents, i.e.:

   $$\rho \equiv \frac{g_Z^2}{g^2}\frac{m_W^2}{m_Z^2} \tag{3}$$

   Where m_Z, m_W are the pole masses and g_Z is defined such that, in unitary gauge, the covariant derivative for a chiral fermionic field ψ contains the term

   $$D_\mu\psi = -ig_Z Z_\mu\left(T_3 - Qs_\theta^2\right)\psi + \dots \tag{4}$$

   being $T_3 = \pm 1/2$ and Q the isospin and electric charge of ψ respectively.

Compute ρ to order Λ^{-2} in the Warsaw basis.
Use the Lagrangian expression *before* defining the input parameters and write the result in terms of Wilson coefficients.
Hint: remember from lecture 1 that a generic covariant derivative contains the term

$$D_\mu \psi = -i\frac{g}{c_\theta} Z_\mu \left(T_3 - Q s_\theta^2\right)\left[1 + \frac{t_\theta}{2}\bar{C}_{HWB}\right]\psi + \dots \tag{5}$$

having defined the angle θ as in (1).

(e) Now repeat the calculation with the Lagrangian defined *after* defining inputs: write ρ as a function of hat quantities and $\delta g/g$ etc.

(f) Write ρ in terms of Wilson coefficients, specializing to the $\{\alpha, G_F, m_Z\}$ and $\{m_W, G_F, m_Z\}$ input schemes. Verify that, in both cases, you get the same result and that this also coincides with the result found at point (a).

(g) Deduce the relation between $\delta m_W^2/m_W^2$, $\delta c_\theta^2/c_\theta^2$ and $(\rho - 1)$.

The W decay width in SMEFT

3. In this exercise we will compute the SMEFT correction to the total decay width of the W boson. We will ignore all fermion masses and mixings.

   (a) Compute the *squared* amplitude for a decay $W^- \to e^-\nu_e$, averaged over the polarizations of the W boson, considering only one lepton flavor and expanding to linear order in the SMEFT. The relevant Feynman rule is

$$W_\mu^- \bar{e}\nu \qquad -\frac{ig}{\sqrt{2}}(\gamma^\mu P_L)\left[1 + \frac{\delta g}{g} + \bar{C}_{Hl}^{(3)}\right] \tag{6}$$

   (b) Repeat for $W^- \to \bar{u}d$. The Feynman rule in this case is

$$W_\mu^- \bar{d}u \qquad -\frac{ig}{\sqrt{2}}(\gamma^\mu P_L)\left[1 + \frac{\delta g}{g} + \bar{C}_{Hq}^{(3)}\right] \tag{7}$$

   (c) Compute the decay widths $\Gamma(W^- \to e^-\nu_e)$ and $\Gamma(W^- \to \bar{u}d)$.
   Remember that

$$\Gamma = \frac{|A|^2}{16\pi m_W} \tag{8}$$

   (d) Compute the total decay width Γ_W and express it as $\Gamma_W^{SM}\left[1 + \delta\Gamma_W/\Gamma_W\right]$.
   Then specialize the result to the $\{\alpha, G_F, m_Z\}$ and $\{m_W, G_F, m_Z\}$ input schemes.

Jacobian formulation of input shifts

⚠ For this exercise you will need Mathematica to invert matrices.
One can ask whether there is a simple way to translate between different EW input schemes. In fact, when working to $\mathcal{O}(\Lambda^{-2})$, the translation can be done quite easily using a Jacobian description of the whole inputs procedure:

Let's define the vector $\vec{G} = (g, g', v_T)$ of the 3 Lagrangian parameters in the EW sector of the SM. In order to fix their numerical values, we need to relate them to 3 observables, that we don't specify yet. We will call $\vec{O}$ the vector formed by them.
Each of these observables can be computed in the SMEFT using the canonically normalized Lagrangian. The prediction for observable O_n has the form

$$O_n(\vec{G}, C_i) = O_n^{SM}(\vec{G}) + \Delta O_n(\vec{G}, C_i) \tag{9}$$

where the first term is the SM prediction and the second is the $\mathcal{O}(\Lambda^{-2})$ correction.
What we do when fixing the input scheme is solving the system $\vec{O}(\vec{G}, C_i) = \hat{\vec{O}}$ for $\vec{G}$, where $\hat{\vec{O}}$ are the measured values for $\vec{O}$. The solution can be written

$$\vec{G} = \hat{\vec{G}} - J^{-1}\Delta\vec{O} \tag{10}$$

where $\hat{\vec{G}}$ is the SM solution and J is a Jacobian matrix defined by

$$J_{nk} = \frac{\partial O_n^{SM}}{\partial G_k} \tag{11}$$

4. Choose the input observables $\vec{O}_{mW} = \{m_W^2, m_Z^2, G_F\}$ and re-derive the shifts presented in class (and given in the appendix) using the formula (10).

   Remember the starting point:

$$m_W^2 = \frac{g^2 v_T^2}{4} \qquad m_Z^2 = \frac{(g^2 + (g')^2)v_T^2}{4}(1 + \Delta m_Z^2) \qquad G_F = \frac{1}{\sqrt{2}v_T^2}(1 + \Delta G_F) \tag{12}$$

   You don't need to open the Δ's at this stage.

5. Using Eq. (10) you can see that the solution for the scheme with $\vec{O}_a = \{\alpha, m_Z^2, G_F\}$ will be given by

$$\vec{G} = \hat{\vec{G}}_a - (J_\alpha^{-1})\Delta\vec{O}_\alpha \tag{13}$$

   where $\hat{\vec{G}}_a$ is the new SM solution (which is trivial to find). More interestingly, deriving by parts:

$$J_\alpha = \frac{\partial \vec{O}_\alpha^{SM}}{\partial \vec{G}} = \frac{\partial \vec{O}_\alpha^{SM}}{\partial \vec{O}_{mW}^{SM}} \frac{\partial \vec{O}_{mW}^{SM}}{\partial \vec{G}} = \frac{\partial \vec{O}_\alpha^{SM}}{\partial \vec{O}_{mW}^{SM}} J_{mW} \tag{14}$$

   where trivially

$$\frac{\partial \vec{O}_\alpha^{SM}}{\partial \vec{O}_{mW}^{SM}} = \begin{pmatrix} \frac{\partial\alpha}{\partial m_W^2} & \frac{\partial\alpha}{\partial m_Z^2} & \frac{\partial\alpha}{\partial G_F} \\ & 1 & \\ & & 1 \end{pmatrix} \tag{15}$$

   with all observable predictions computed in the SM. So the new Jacobian J_α in α scheme can be computed very easily, once the Jacobian J_{mW} in m_W scheme is known.

   Do this computation and verify that eq. (13) gives the result presented in the lecture for the α scheme.

   Hint: for the Jacobian, you will need to express α as a function of the $\mathcal{O}_{mW}$ observables. To do this, take $\alpha = \frac{1}{4\pi}\frac{(gg')^2}{g^2+(g')^2}$ and replace g, g' with the SM solutions in m_W scheme.

6. Let's use the Jacobian to define a new input scheme: $\vec{O}_{\text{new}} = \{m_W^2, m_Z^2, \alpha\}$.

   Start from the $\vec{O}_{mW}$ set, and replace G_F with α.

   (a) Compute the SM solutions $\vec{\bar{G}}$ in the new scheme.

   (b) Compute the jacobian as $J_{\text{new}} = \frac{\partial \vec{O}^{SM}_{\text{new}}}{\partial \vec{O}^{SM}_{mW}} J_{mW}$.

   (c) Put everything together in (10) to find the result for the parameters and their shfits.

A Input shift expressions

In the $\{\alpha, m_Z, G_F\}$ scheme:

$$\frac{\delta g}{g} = \frac{1}{2c_{2\theta}}\left[-c_\theta^2\left(\Delta m_Z^2 + \Delta G_F\right) + s_\theta^2 \Delta\alpha\right] \tag{16}$$

$$\frac{\delta g'}{g'} = \frac{1}{2c_{2\theta}}\left[s_\theta^2\left(\Delta m_Z^2 + \Delta G_F\right) - c_\theta^2 \Delta\alpha\right] \tag{17}$$

$$\frac{\delta v_T}{v_T} = \frac{\Delta G_F}{2} \tag{18}$$

In the $\{m_W, m_Z, G_F\}$ scheme:

$$\frac{\delta g}{g} = -\frac{\Delta G_F}{2} \tag{19}$$

$$\frac{\delta g'}{g'} = -\frac{1}{2}\left[\Delta G_F + \frac{\Delta m_Z^2}{s_\theta^2}\right] \tag{20}$$

$$\frac{\delta v_T}{v_T} = \frac{\Delta G_F}{2} \tag{21}$$

And the Δ's are

$$\Delta G_F = 2\bar{C}_{Hl}^{(3)} - \bar{C}'_{ll} \tag{22}$$

$$\Delta m_Z^2 = \frac{2gg'}{g^2 + (g')^2}\bar{C}_{HWB} + \frac{\bar{C}_{HD}}{2} \tag{23}$$

$$\Delta\alpha = -\frac{2gg'}{g^2 + (g')^2}\bar{C}_{HWB} \tag{24}$$

EW SMEFT – Exercise sheet 2 – Solutions

The W mass in SMEFT

1. (a)

$$m_W^2 = \frac{g^2}{v_T^2}4 = \frac{\hat{g}^2\hat{v}_T^2}{4}\left[1 + 2\frac{\delta g}{g} + 2\frac{\delta v_T}{v_T}\right] \tag{1}$$

   (b)

$$\begin{aligned} m_W^2 &= \frac{\hat{g}^2\hat{v}_T^2}{4}\left[1 - \frac{c_\theta^2}{c_{2\theta}}\Delta m_Z^2 + \frac{s_\theta^2}{c_{2\theta}}(\Delta\alpha - \Delta G_F)\right] &(2)\\ &= \frac{\hat{g}^2\hat{v}_T^2}{4}\left[1 - \frac{1}{c_{2\theta}}\left(s_{2\theta}\bar{C}_{HWB} + \frac{c_\theta^2}{2}\bar{C}_{HD} + s_\theta^2(2\bar{C}_{Hl}^{(3)} - \bar{C}'_{ll})\right)\right] &(3)\end{aligned}$$

   (c)

$$m_W^2 = \frac{\hat{g}^2\hat{v}_T^2}{4}\left[1 - \frac{\Delta G_F}{2} + \frac{\Delta G_F}{2}\right] = \frac{\hat{g}^2\hat{v}_T^2}{4} \tag{4}$$

2. (a) From the definition:

$$\begin{aligned} \cos^2\theta &= \frac{g^2}{g^2 + (g')^2} - \frac{s_{4\theta}}{4}\bar{C}_{HWB} &(5)\\ &= \frac{\hat{g}^2}{\hat{g}^2 + (\hat{g}')^2}\left[1 + 2s_\theta^2\left(\frac{\delta g}{g} - \frac{\delta g'}{g'}\right)\right] - \frac{s_{4\theta}}{4}\bar{C}_{HWB} &(6)\end{aligned}$$

   (b) $\{\alpha, G_F, m_Z\}$ scheme:

$$\begin{aligned} \cos^2\theta &= \frac{\hat{g}^2}{\hat{g}^2 + (\hat{g}')^2}\left[1 + \frac{s_\theta^2}{c_{2\theta}}(\Delta\alpha - \Delta G_F - \Delta m_Z^2) - \frac{s_{4\theta}}{4c_\theta^2}\bar{C}_{HWB}\right] &(7)\\ &= \frac{\hat{g}^2}{\hat{g}^2 + (\hat{g}')^2}\left[1 + \frac{s_\theta^2}{c_{2\theta}}\left(-\frac{1}{2}\bar{c}_{HD} - 2\bar{C}_{Hl}^{(3)} + \bar{C}'_{ll} - \frac{2}{s_{2\theta}}\bar{C}_{HWB}\right)\right] &(8)\end{aligned}$$

   $\{m_W, G_F, m_Z\}$ scheme:

$$\begin{aligned} \cos^2\theta &= \frac{\hat{g}^2}{\hat{g}^2 + (\hat{g}')^2}\left[1 + \Delta m_Z^2 - \frac{s_{4\theta}}{4c_\theta^2}\bar{C}_{HWB}\right] &(9)\\ &= \frac{\hat{g}^2}{\hat{g}^2 + (\hat{g}')^2}\left[1 + \frac{\bar{C}_{HD}}{2} + t_\theta\bar{C}_{HWB}\right] &(10)\end{aligned}$$

(c) The formula is easy to verify using the expressions in terms of Δ's.
As an intuitive interpretation, it can be derived differentiating the SM relation

$$m_W^2 = c_\theta^2 m_Z^2\,. \tag{11}$$

and noting that Δm_Z^2 enters with a minus sign because it's the contribution to the input quantity, and not a predicted shift. The term in $\bar{C}_{HWB}$ is practically subtracting the genuine correction to θ from $\delta c_\theta^2/c_\theta^2$. This makes sense because the c_θ we have in the SM relation actually stands for the ratio $g^2/(g^2+(g')^2)$, and is unrelated to the mass diagonalization in the neutral gauge sector. Since the $\bar{C}_{HWB}$ dependence comes from the latter, it has to cancel in this equation.

(d)

$$\begin{aligned}
\rho &= \frac{1}{g^2}\frac{g^2}{c_\theta^2}\left(1+\frac{t_\theta}{2}\bar{C}_{HWB}\right)^2\frac{g^2 v_T^2}{4}\frac{4}{(g^2+(g')^2)v_T^2(1+\Delta m_Z^2)} && (12)\\
&= 1-\Delta m_Z^2 + 2\frac{gg'}{g^2+(g')^2}\bar{C}_{HWB} && (13)\\
&= 1-\frac{\bar{C}_{HD}}{2} && (14)
\end{aligned}$$

(e)

$$\begin{aligned}
\rho &= \frac{\hat{g}^2+(\hat{g}')^2}{\hat{g}^2}\left[1+\frac{2(\hat{g}')^2}{\hat{g}^2+(\hat{g}')^2}\left(-\frac{\delta g}{g}+\frac{\delta g'}{g'}\right)+\frac{2\hat{g}\hat{g}'}{\hat{g}^2+(\hat{g}')^2}\bar{C}_{HWB}\right]\frac{\hat{g}^2(1+\delta m_W^2/m_W^2)}{\hat{g}^2+(\hat{g}')^2}\times\\
&\times\left[1+2c_\theta^2\frac{\delta g}{g}+s_\theta^2 2\frac{\delta g'}{g'}+2\frac{\delta v_T}{v_T}+\Delta m_Z^2\right]^{-1} && (15)\\
&= 1-2\left(\frac{\delta g}{g}+\frac{\delta v_T}{v_T}\right)-\Delta m_Z^2+s_{2\theta}\bar{C}_{HWB}+\frac{\delta m_W^2}{m_W^2} && (16)
\end{aligned}$$

(f) $\{\alpha, G_F, m_Z\}$ scheme:

$$\begin{aligned}
\rho &= 1+\frac{s_\theta^2}{c_{2\theta}}\left[\Delta m_Z^2+\Delta G_F-\Delta\alpha\right]+s_{2\theta}\bar{C}_{HWB}-\frac{c_\theta^2}{c_{2\theta}}\Delta m_Z^2+\frac{s_\theta^2}{c_{2\theta}}(\Delta\alpha-\Delta G_F) && (17)\\
&= 1-\Delta m_Z^2+s_{2\theta}\bar{C}_{HWB} = 1-\frac{\bar{C}_{HD}}{2} && (18)
\end{aligned}$$

$\{m_W, G_F, m_Z\}$ scheme:

$$\begin{aligned}
\rho &= 1-\Delta m_Z^2+s_{2\theta}\bar{C}_{HWB}+0 && (19)\\
&= 1-\frac{\bar{C}_{HD}}{2} && (20)
\end{aligned}$$

(g)

$$(\rho-1) = \frac{\delta m_W^2}{m_W^2}-\frac{\delta c_\theta^2}{c_\theta^2}+t_\theta\bar{C}_{HWB} = s_{2\theta}\bar{C}_{HWB}-\Delta m_Z^2 \tag{21}$$

This holds in any input scheme: in fact we can get it directly from (16) and (6).

The W decay width in SMEFT

3.

$$|A(W^- \to e^- \nu_e)|^2 = \frac{1}{3} g^2 m_W^2 \left[1 + 2\frac{\delta g}{g} + \bar{2}C_{Hl}^{(3)}\right] \tag{22}$$

4.

$$|A(W^- \to \bar{u}d)|^2 = g^2 m_W^2 \left[1 + 2\frac{\delta g}{g} + \bar{2}C_{Hq}^{(3)}\right] \tag{23}$$

where this time the 3 from the average over polarizations goes away with the quark color factor.

5.

$$\Gamma(W^- \to e^- \nu_e) = \frac{|A|^2}{16\pi m_W} = \frac{g^2 \hat{m}_W}{48\pi} \left[1 + 2\frac{\delta g}{g} + \bar{2}C_{Hl}^{(3)} + \frac{\delta m_W}{m_W}\right] \tag{24}$$

$$\Gamma(W^- \to \bar{u}d) = \frac{|A|^2}{16\pi m_W} = \frac{g^2 \hat{m}_W}{16\pi} \left[1 + 2\frac{\delta g}{g} + \bar{2}C_{Hq}^{(3)} + \frac{\delta m_W}{m_W}\right] \tag{25}$$

where we have noted that $|A|^2$ was computed in terms of the Lagrangian parameter m_W, and we have replaced it here with the $\hat{m}_W(1 + \delta m_W/m_W)$.

6.

$$\begin{aligned}
\Gamma_W &= 3\Gamma(W^- \to e^- \nu_e) + 2\Gamma(W^- \to \bar{u}d) && (26)\\
&= \frac{3g^2 \hat{m}_W}{16\pi} \left[1 + \frac{1}{3}\left(2\frac{\delta g}{g} + \bar{2}C_{Hl}^{(3)} + \frac{\delta m_W}{m_W}\right) + \frac{2}{3}\left(2\frac{\delta g}{g} + \bar{2}C_{Hq}^{(3)} + \frac{\delta m_W}{m_W}\right)\right] && (27)\\
&= \frac{3g^2 \hat{m}_W}{16\pi} \left[1 + 2\frac{\delta g}{g} + \frac{\delta m_W}{m_W} + \frac{2}{3}(\bar{C}_{Hl}^{(3)} + 2\bar{C}_{Hq}^{(3)})\right] && (28)
\end{aligned}$$

$\{\alpha, G_F, m_Z\}$ scheme:

$$\begin{aligned}
\frac{\delta\Gamma_W}{\Gamma_W} &= \frac{3}{2c_{2\theta}}\left[s_\theta^2 \Delta\alpha - c_\theta^2(\Delta m_Z^2 + \Delta G_F)\right] + \frac{1}{2}\Delta G_F + \frac{2}{3}\left(\bar{C}_{Hl}^{(3)} + 2\bar{C}_{Hq}^{(3)}\right) && (29)\\
&= -\frac{3t_{2\theta}}{2}\bar{C}_{HWB} - \frac{3}{4}\frac{c_\theta^2}{c_{2\theta}}\bar{C}_{HD} + \left(\frac{3c_\theta^2}{c_{2\theta}} - 1\right)\frac{\bar{C}_{ll}'}{2} - \left(\frac{3c_\theta^2}{c_{2\theta}} - \frac{5}{3}\right)\bar{C}_{Hl}^{(3)} + \frac{4}{3}\bar{C}_{Hq}^{(3)} && (30)
\end{aligned}$$

$\{m_W, G_F, m_Z\}$ scheme:

$$\begin{aligned}
\frac{\delta\Gamma_W}{\Gamma_W} &= -\Delta G_F + \frac{2}{3}\left(\bar{C}_{Hl}^{(2)} + 2\bar{C}_{Hq}^{(3)}\right) && (31)\\
&= \bar{C}_{ll}' + \frac{4}{3}\left(\bar{C}_{Hq}^{(3)} - \bar{C}_{Hl}^{(3)}\right) && (32)
\end{aligned}$$

Jacobian formulation of input shifts

7. The Jacobian and the input Δ's are

$$J_{mW} = \frac{g v_T^2}{2} \begin{pmatrix} 1 & & \frac{g}{v_T} \\ 1 & \frac{g'}{g} & \frac{g^2+(g')^2}{g v_T} \\ & & -\frac{2\sqrt{2}}{g v_T^5} \end{pmatrix} \tag{33}$$

$$\Delta\vec{O}_{mW} = \begin{pmatrix} 0 & \frac{v_T^2(g^2+(g')^2}{4}\Delta m_Z^2, & \frac{\Delta G_F}{\sqrt{2}v_T^2} \end{pmatrix}^T \tag{34}$$

Inverting the Jacobian:

$$J_{mW}^{-1} = \frac{2}{g v_T^2} \begin{pmatrix} 1 & & \frac{g^2 v_T^4}{2\sqrt{2}} \\ -\frac{g}{g'} & \frac{g}{g'} & \frac{g g' v_T^4}{2\sqrt{2}} \\ & & -\frac{g v_T^5}{2\sqrt{2}} \end{pmatrix} \tag{35}$$

hence

$$\vec{G} = \hat{\vec{G}} + \begin{pmatrix} -\frac{1}{2} g \Delta G_F \\ -\frac{(g^2+(g')^2)}{2g'}\Delta m_Z^2 - \frac{g'}{2}\Delta G_F \\ \frac{v_T}{2}\Delta G_F \end{pmatrix} = \begin{pmatrix} \hat{g}\,[1-\Delta G_F/2] \\ \hat{g}'[1-\Delta m_Z^2/(2s_\theta^2) - \Delta G_F/2] \\ \hat{v}_T[1+\Delta G_F/2] \end{pmatrix} \tag{36}$$

8.

$$\frac{\partial \vec{O}_\alpha}{\partial \vec{O}_{mW}} = \begin{pmatrix} \frac{\partial\alpha}{\partial m_W^2} & \frac{\partial\alpha}{\partial m_Z^2} & \frac{\partial\alpha}{\partial G_F} \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} -\frac{c_{2\theta}}{\pi v_T^2} & \frac{c_\theta^4}{\pi v_T^2} & \frac{g^2 v_T^2 s_\theta^2}{2\sqrt{2}\pi} \\ & 1 & \\ & & 1 \end{pmatrix} \tag{37}$$

then

$$J_\alpha = \frac{\partial \vec{O}_\alpha}{\partial \vec{O}_{mW}} J_{mW} = \frac{1}{2} \begin{pmatrix} \frac{g(g')^4}{(g^2+(g')^2)^2\pi} & \frac{g^4(g')}{(g^2+(g')^2)^2\pi} & \\ g v^2 & g' v^2 & (g^2+(g')^2)v \\ & & -\frac{2/\sqrt{2}}{v^3} \end{pmatrix} \tag{38}$$

The inverse Jacobian is

$$J_\alpha^{-1} = \frac{1}{c_{2\theta}} \begin{pmatrix} -\frac{2\pi}{g} & \frac{2c_\theta^4}{g v_T^2} & \frac{g v_T^2 c_\theta^2}{\sqrt{2}} \\ \frac{2\pi}{g t_\theta} & -\frac{s_{2\theta} s_\theta^2}{g v_T^2} & -\frac{g v_T^2 s_\theta^2 t_\theta}{\sqrt{2}} \\ 0 & 0 & -\frac{v_T^2 c_{2\theta}}{\sqrt{2}} \end{pmatrix} \tag{39}$$

and with the observables corrections

$$\Delta\vec{O}_\alpha = \begin{pmatrix} \frac{g^2(g')^2}{4\pi(g^2+(g')^2)}\Delta\alpha & \frac{g^2+(g')^2 v_T^2}{4}\Delta m_Z^2 & \frac{1}{\sqrt{2}v_T^3}\Delta G_F \end{pmatrix}^T \tag{40}$$

one finally obtains

$$\vec{G} = \hat{\vec{G}} + \begin{pmatrix} g(s_\theta^2\Delta\alpha - c_\theta^2(\Delta m_Z^2 + \Delta G_F)/(2c_{2\theta}) \\ g'(s_\theta^2(\Delta m_Z^2 + \Delta G_F) - c_\theta^2\Delta\alpha)/(2c_{2\theta}) \\ v_T \Delta G_F/2 \end{pmatrix} \tag{41}$$

which is the same result as in the appendix.

9. Let's use the Jacobian to define a new input scheme: $\vec{O}_{\text{new}} = \{m_W^2, m_Z^2, \alpha\}$.
   Start from the $\vec{O}_{mW}$ set, and replace G_F with α.

   (a) The SM solutions are

$$\hat{g} = 2\sqrt{\frac{\pi\alpha}{1 - m_W^2/m_Z^2}} \qquad \hat{g}' = \frac{2m_Z\sqrt{\pi\alpha}}{m_W} \qquad \hat{v} = m_W\sqrt{\frac{1 - m_W^2/m_Z^2}{\pi\alpha}} \tag{42}$$

   (b)

$$J_{\text{new}} = \begin{pmatrix} 1 & & \\ & 1 & \\ -\frac{c_{2\theta}}{\pi v_T^2} & \frac{c_\theta^4}{\pi v_T^2} & \frac{g^2 v_T^2 s_\theta^2}{2\sqrt{2}\pi} \end{pmatrix} J_{mW} = \frac{g v_T^2}{2}\begin{pmatrix} 1 & 0 & \frac{g}{v_T} \\ 1 & t_\theta & \frac{g}{v_T c_\theta^2} \\ \frac{s_\theta^4}{\pi v_T^2} & \frac{s_{2\theta} c_\theta^2}{2\pi v_T^2} & 0 \end{pmatrix} \tag{43}$$

   Therefore

$$J_{\text{new}}^{-1} = \frac{2}{g v_T^2}\begin{pmatrix} \frac{1}{t_\theta^2} & -\frac{c_\theta^4}{s_\theta^2} & \frac{\pi v_T^2}{s_\theta^2} \\ -t_\theta & \frac{s_{2\theta}}{2} & \frac{2\pi v_T^2}{s_{2\theta}} \\ -\frac{c_{2\theta}}{s_\theta^2}\frac{v_T}{g} & \frac{c_\theta^4}{s_\theta^2}\frac{v_T}{g} & -\frac{\pi v_T^3}{g s_\theta^2} \end{pmatrix} \tag{44}$$

   (c)

$$\vec{G} = \hat{\vec{G}} + \frac{1}{2}\begin{pmatrix} \hat{g}(\Delta m_Z^2/t_\theta^2 - \Delta\alpha) \\ -\hat{g}'(\Delta m_Z^2 + \Delta\alpha) \\ \hat{v}_T(\Delta\alpha - \Delta m_Z^2/t_\theta^2) \end{pmatrix} \tag{45}$$

   Hence, in this scheme

$$\frac{\delta g}{g} = \frac{1}{2}\left(\frac{\Delta m_Z^2}{t_\theta^2} - \Delta\alpha\right) = \frac{1}{t_\theta}\bar{C}_{HWB} + \frac{1}{4t_\theta^2}\bar{C}_{HD} \tag{46}$$

$$\frac{\delta g'}{g'} = -\frac{1}{2}(\Delta m_Z^2 + \Delta\alpha) = -\frac{\bar{C}_{HD}}{4} \tag{47}$$

$$\frac{\delta v_T}{v_T} = \frac{1}{2}\left(\Delta\alpha - \frac{\Delta m_Z^2}{t_\theta^2}\right) = -\frac{1}{t_\theta}\bar{C}_{HWB} - \frac{1}{4t_\theta^2}\bar{C}_{HD} \tag{48}$$