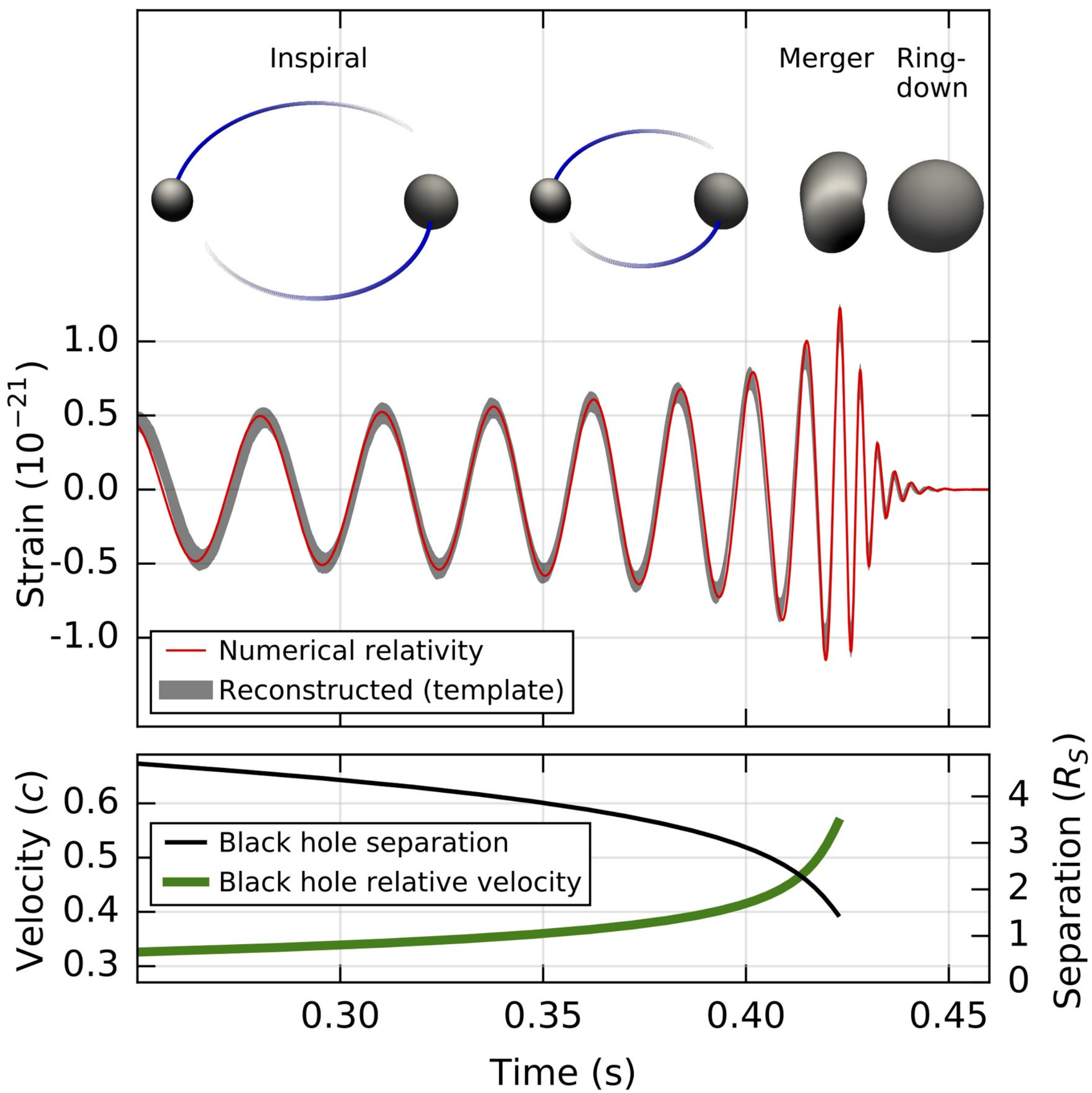
Cortona Young 2021

Gravitational waves from eccentric compact binaries: the Effective One-Body approach in the extreme mass-ratio limit

S. Albanesi

arXiv:2104.10559, S. Albanesi, A. Nagar & S. Bernuzzi

Two-body problem in general relativity



arXiv:1602.03837

GWs emission: loss of energy and angular momentum

In the early inspiral the loss of angular momentum leads to circularized orbits → LIGO/Virgo detect GWs form circularized binaries

LIGO/Virgo: last stages of binaries with comparable masses, circularized by angular momentum loss in the early inspiral

Why eccentricity?

- Comparable mass binaries
  - Perturbations from a third body (Kozai-Lidov)
  - dynamical captures

    → both detectable by LIGO/Virgo

    (dynamical capture maybe already detected, see GW190521)
- Extreme Mass Ratio Inspirals (**EMRI**s): compact stellar object orbiting around a supermassive black hole: <u>noncircular orbits</u> (neither planar)

  → ~mHz, not detectable by ground-based detectors

  → Laser Interferometer Space Antenna (**LISA**)

Two-body problem in general relativity

No exact analytical solution for binaries!

Different analytical approximations can be adopted:

- **Post-Newtonian (PN)** : ‘small’ velocities, EFE expansion in $\epsilon = \frac{v}{c} \ll 1$, near-zone
- **Post-Minkwoskian (PM)** : expansion in G, far-zone, but exists a matching region with PN expansion
- **Gravitational Self-Force (GSF)** : expansion in the mass ratio

Numerical Relativity (NR) simulations: highly technical, first binary merger performed only in 2005 (Pretorius arXiv:gr-qc/0507014)
Very interesting and technical: you have to write EFE in a 3+1 (conformal) decomposition, use smart gauge, take care of singularities, damp errors due to ID discretization…
HPC needed, some codes: SpEC (SXS), GRChombo, GR-Athena++, …

If $m_1 \ll m_2$, **Black Hole Perturbation Theory (BHPT)**: linearize EFE
Schwarzschild → Regge-Wheeler and Zerilli equations
Kerr → Teukolsky equation

Effective One-Body model

$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

One body moving in a ν-deformed effective metric,

$\nu = 0 \rightarrow$ Schwarzschild/Kerr

Based on PN approximation, but we can include also results from NR, GSF... EOB is very flexible!

Very accurate for circular binaries, now the challenge is to use this model also for noncircular binaries

We will focus on the **test mass limit**:
Eccentric equatorial orbits of a (nonspinning) test-particle around a Kerr black hole (we use geometrized units, $G = c = 1$)

Kerr
quasi-circular

- Hamiltonian: conservative dynamics

$$\hat{H}^{\text{eq.}}_{\text{Kerr}} = \frac{2\hat{a}p_\varphi}{r r_c^2} + \sqrt{A(r)\left(1 + \frac{p_\varphi^2}{r_c^2}\right) + p_{r_*}^2}$$

$$A(r) = \frac{1 + 2/r_c}{1 + 2/r_c}(1 - 2/r_c) \qquad r_c^2 = r^2 + \hat{a}^2 + 2\frac{\hat{a}^2}{r}$$

$$|\hat{a}| = J_{\text{BH}}/M^2 \in [0, 1]$$

- Radiation Reaction Force: non-conservative dynamics

$$\hat{\mathcal{F}}_\varphi = -\frac{32}{5}\nu {r_\Omega}^4 \Omega^5 \hat{f}$$

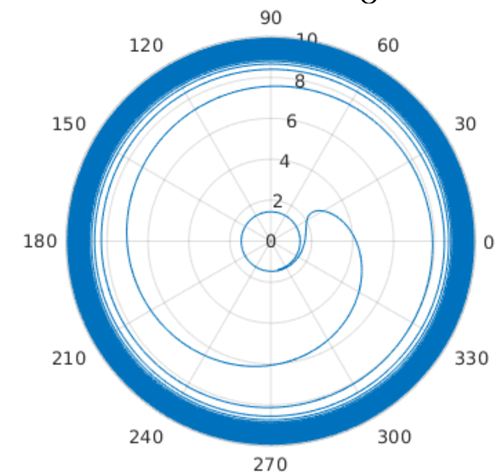
$${r_\Omega}^3 \Omega^2 = 1$$

Hamilton Equations $\rightarrow$ $\dot{q}^i = \frac{\partial \hat{H}}{\partial p_i}$, $\dot{p}_i = -\frac{\partial \hat{H}}{\partial q^i} + \hat{\mathcal{F}}_i$

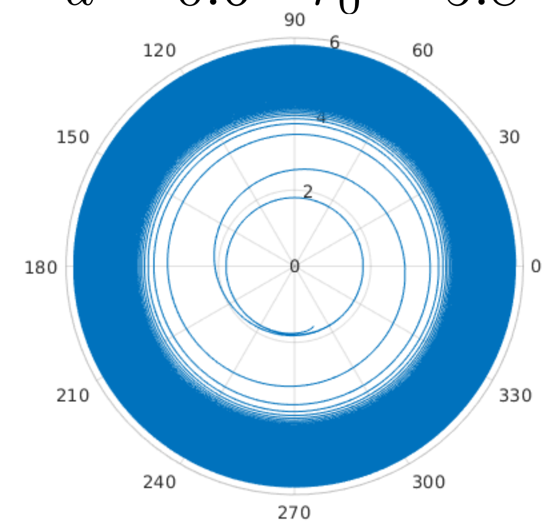
- Waveform

$$h_{\ell m} = h^{(N,\epsilon)}_{\ell m} \hat{S}^{(\epsilon)}_{\text{eff}} T_{\ell m} e^{i\delta_{\ell m}} \rho^{\ell}_{\ell m}$$

$a = -0.9 \quad r_0 = 10.7$



$a = 0.6 \quad r_0 = 5.8$



Kerr

quasi-circular

- Hamiltonian: conservative dynamics

$$\hat{H}^{\text{eq.}}_{\text{Kerr}} = \frac{2\hat{a}p_\varphi}{r r_c^2} + \sqrt{A(r)\left(1 + \frac{p_\varphi^2}{r_c^2}\right) + p_{r_*}^2}$$

$$A(r) = \frac{1+2/r_c}{1+2/r_c}(1-2/r_c) \qquad r_c^2 = r^2 + \hat{a}^2 + 2\frac{\hat{a}^2}{r}$$

$$|\hat{a}| = J_{\text{BH}}/M^2 \in [0,1]$$

- Radiation Reaction Force: non-conservative dynamics

$$\hat{\mathcal{F}}_\varphi = -\frac{32}{5}\nu {r_\Omega}^4 \Omega^5 \hat{f}$$

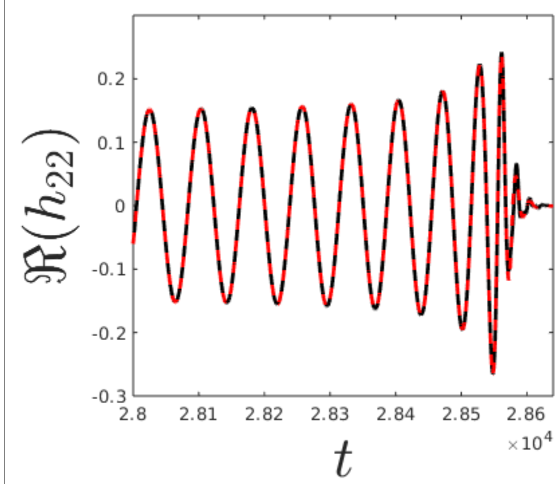
$${r_\Omega}^3 \Omega^2 = 1$$

Hamilton Equations $\rightarrow$ $\dot{q}^i = \dfrac{\partial \hat{H}}{\partial p_i}$, $\dot{p}_i = -\dfrac{\partial \hat{H}}{\partial q^i} + \hat{\mathcal{F}}_i$

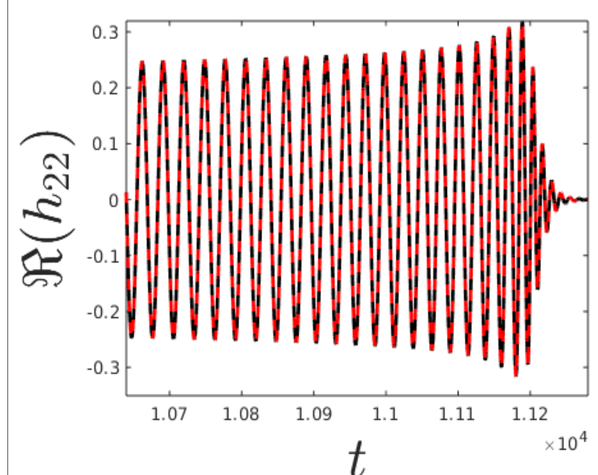
- Waveform

$$h_{\ell m} = h^{(N,\epsilon)}_{\ell m} \hat{S}^{(\epsilon)}_{\text{eff}} T_{\ell m} e^{i\delta_{\ell m}} \rho^{\ell}_{\ell m}$$

$a = -0.9 \quad r_0 = 10.7$



$a = 0.6 \quad r_0 = 5.8$



Definitions of eccentricity and semilatus rectum

- Newtonian mechanics:

$$r(\varphi) = \frac{p}{1 - e\cos\varphi}, \quad \rightarrow \quad r_p = \frac{p}{1+e}, \quad r_a = \frac{p}{1-e}$$

$$\rightarrow \quad e = \frac{r_a - r_p}{r_a + r_p}, \quad p = \frac{2 r_a r_p}{r_a + r_p}$$

- Effective-one-body model:

$$(E, p_\varphi) \Rightarrow E = \hat{H}_{\rm orb}\big|_{p_{r_*}=0} + a p_\varphi G_s^{\rm eq} = \sqrt{A(r)\left(1 + \frac{p_\varphi^2}{r_c^2}\right)} + 2a\frac{p_\varphi}{r r_c^2}$$

Two roots are the two turning radial points, then given the energy and the angular momentum we solve the systems and found (r_a, r_p). Then we define the eccentricity and the semilatus rectum as in Newtonian mechanics:

$$e = \frac{r_a - r_p}{r_a + r_p}, \quad p = \frac{2 r_a r_p}{r_a + r_p}$$

...not gauge invariant!

Kerr
eccentricity

- Hamiltonian: conservative dynamics

$$\hat{H}^{\text{eq.}}_{\text{Kerr}} = \frac{2\hat{a}p_\varphi}{r r_c^2} + \sqrt{A(r)\left(1+\frac{p_\varphi^2}{r_c^2}\right) + p_{r_*}^2}$$

$$A(r) = \frac{1+2/r_c}{1+2/r_c}(1-2/r_c) \qquad r_c^2 = r^2 + \hat{a}^2 + 2\frac{\hat{a}^2}{r}$$

$$|\hat{a}| = J_{\text{BH}}/M^2 \in [0,1]$$

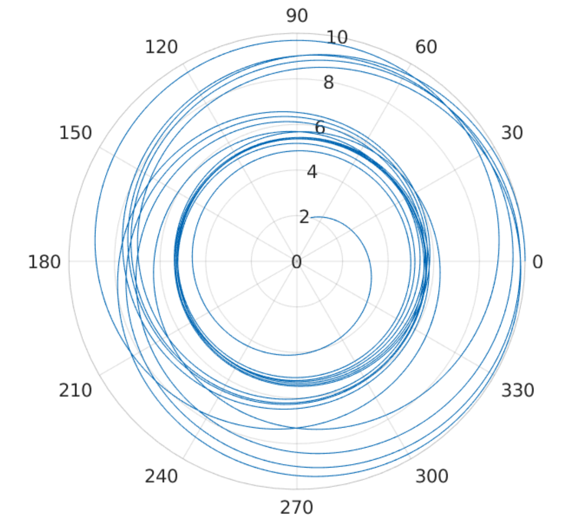
- Radiation Reaction Force: non-conservative dynamics

$$\hat{\mathcal{F}}_\varphi = -\frac{32}{5}\nu {r_\Omega}^4 \Omega^5 \hat{f}_{\text{nc}_{22}}$$

$${r_\Omega}^3\Omega^2 = 1$$

$$\hat{\mathcal{F}}_r = \frac{32}{3}\nu\frac{p_{r_*}}{r^4}P_2^0[\hat{f}_r^{\text{2PN}}]$$

(Bini-Damour)

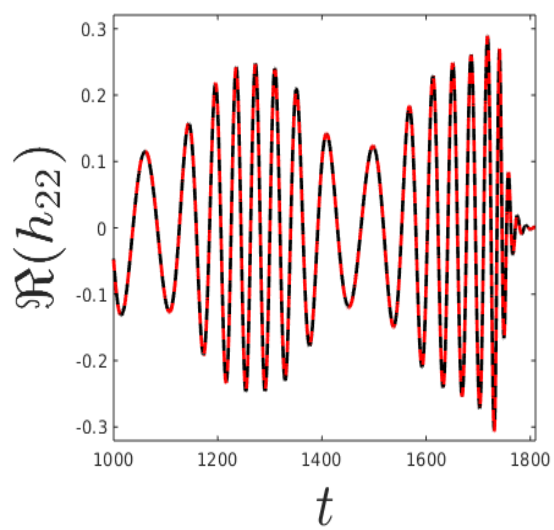


$$e_0 = 0.3 \quad a = 0 \quad p_0 = 7$$

Hamilton Equations → $\dot{q}^i = \frac{\partial \hat{H}}{\partial p_i}$, $\dot{p}_i = -\frac{\partial \hat{H}}{\partial q^i} + \hat{\mathcal{F}}_i$

- Waveform

$$h_{\ell m} = h_{\ell m}^{(N,\epsilon)}\hat{h}_{\ell m}^{(N,\epsilon)_{\text{nc}}}\hat{S}_{\text{eff}}^{(\epsilon)}T_{\ell m}e^{i\delta_{\ell m}}\rho_{\ell m}^{\ell}$$



Kerr

eccentricity

- Hamiltonian: conservative dynamics

$$\hat{H}^{\text{eq.}}_{\text{Kerr}} = \frac{2\hat{a}p_\varphi}{r r_c^2} + \sqrt{A(r)\left(1+\frac{p_\varphi^2}{r_c^2}\right)+p_{r_*}^2}$$

$$A(r) = \frac{1+2/r_c}{1+2/r_c}(1-2/r_c) \qquad r_c^2 = r^2 + \hat{a}^2 + 2\frac{\hat{a}^2}{r}$$

$$|\hat{a}| = J_{\text{BH}}/M^2 \in [0,1]$$

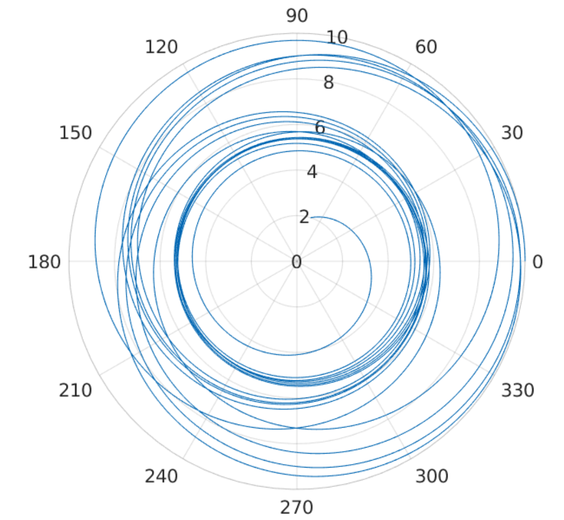
- Radiation Reaction Force: non-conservative dynamics

$$\hat{\mathcal{F}}_\varphi = -\frac{32}{5}\nu {r_\Omega}^4 \Omega^5 \hat{f}_{\text{nc}_{22}}$$

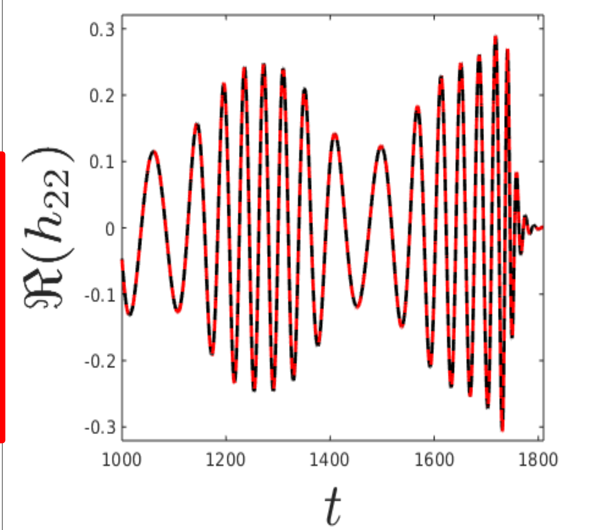
$${r_\Omega}^3\Omega^2 = 1$$

$$\hat{\mathcal{F}}_r = \frac{32}{3}\nu\frac{p_{r_*}}{r^4}P_2^0[\hat{f}_r^{\text{2PN}}]$$

(Bini-Damour)



$e_0 = 0.3 \quad a = 0 \quad p_0 = 7$



Hamilton Equations → $\dot{q}^i = \frac{\partial \hat{H}}{\partial p_i}$, $\dot{p}_i = -\frac{\partial \hat{H}}{\partial q^i} + \hat{\mathcal{F}}_i$

“only” Newtonian noncircular corrections

arXiv:2001.11736

- Waveform

$$h_{\ell m} = h_{\ell m}^{(N,\epsilon)}\hat{h}_{\ell m}^{(N,\epsilon)_{\text{nc}}}\hat{S}_{\text{eff}}^{(\epsilon)}T_{\ell m}e^{i\delta_{\ell m}}\rho_{\ell m}^{\ell}$$

Test of the analytical model

Consider geodesic motion (Hamilton equations without radiation reaction), then compare the analytical (EOB) and numerical results obtained solving the Teukolsky equation with **TEUKODE**

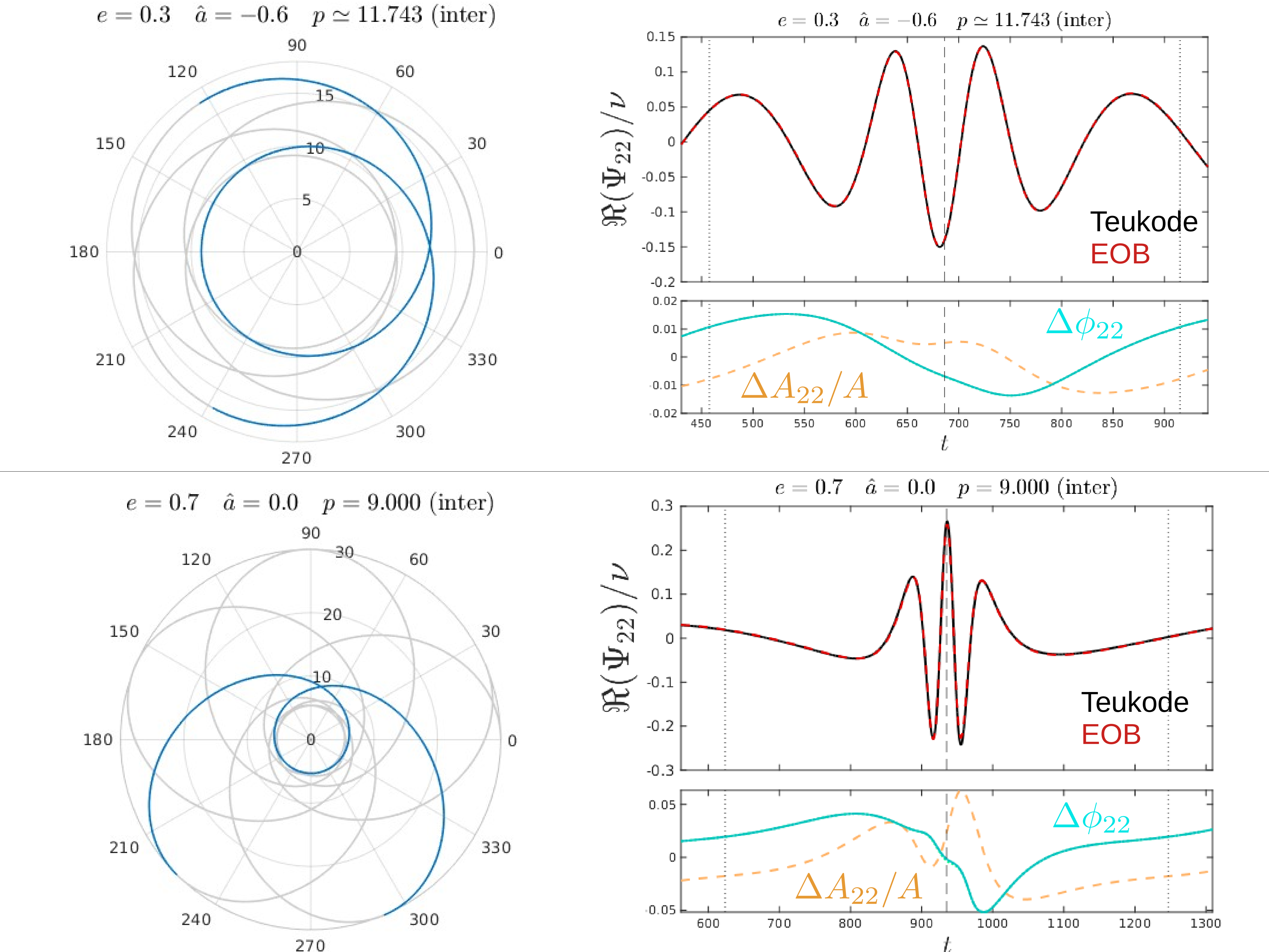
- Waveforms comparisons → reliability of the waveform prescription (not a big surprise)
- Fluxes comparisons → reliability of the radiation reaction (there are some subtleties!)

$$\dot{j}^{\infty} = -\mathcal{F}_{\varphi}$$

$$\dot{E}^{\infty} = -\dot{r}\mathcal{F}_{r} - \Omega\mathcal{F}_{\varphi} - \dot{E}_{\text{Schott}}$$

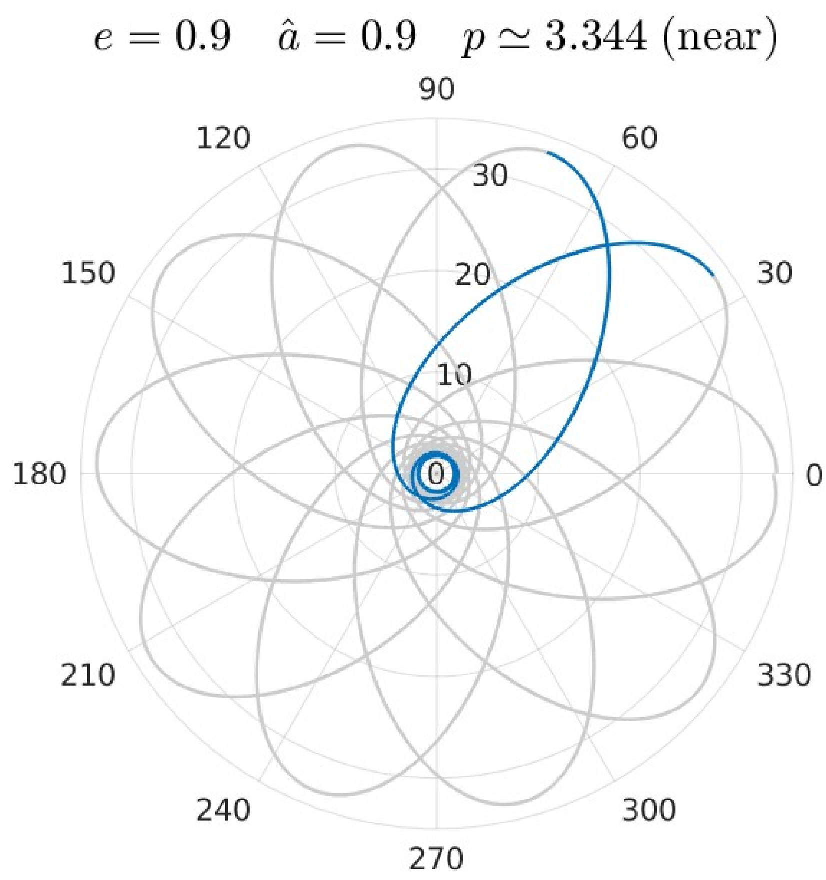
Small effect in the fluxes due to interaction of the source with the local field, but not present in the Hamilton equation

arXiv:2104.10559



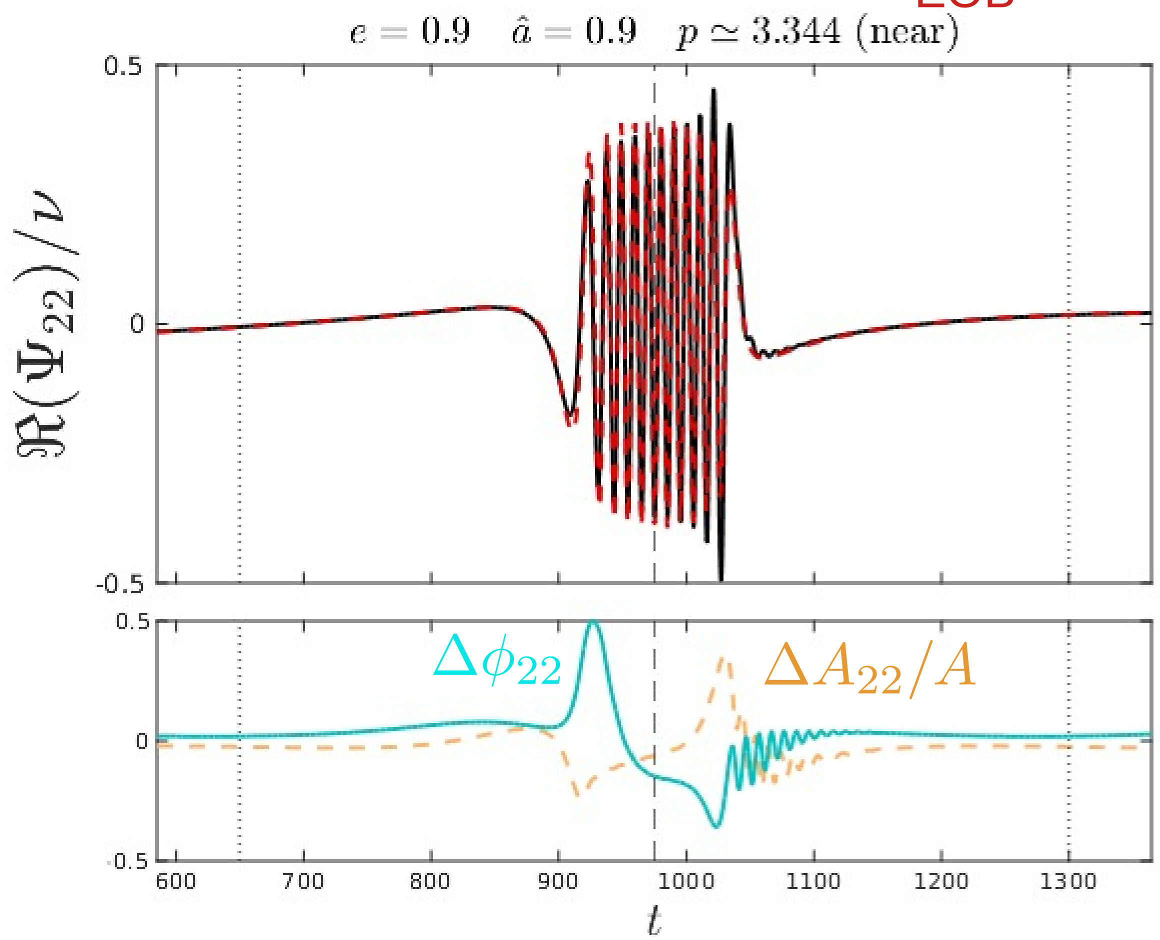
$e = 0.3 \quad \hat{a} = -0.6 \quad p \simeq 11.743$ (inter)
90
120
60
150
30
180
0
210
330
240
300
270
15
10
5
0
$e = 0.3 \quad \hat{a} = -0.6 \quad p \simeq 11.743$ (inter)
$\Re(\Psi_{22})/\nu$
Teukode
EOB
$\Delta\phi_{22}$
$\Delta A_{22}/A$
t
450
500
550
600
650
700
750
800
850
900
$e = 0.7 \quad \hat{a} = 0.0 \quad p = 9.000$ (inter)
30
20
10
$e = 0.7 \quad \hat{a} = 0.0 \quad p = 9.000$ (inter)
$\Re(\Psi_{22})/\nu$
Teukode
EOB
$\Delta\phi_{22}$
$\Delta A_{22}/A$
t
600
700
800
900
1000
1100
1200
1300

When the periastron is near to the light ring: **wiggles** (QNM excitations)



zoom-whirl orbit

Teukode
EOB



After merger: ringdown

The waveform is almost a Quasi-Normal-Modes superposition: $h_{\ell m} = \sum_n c_n(\tau) e^{-\sigma_n^{\ell m}\tau}, \qquad \sigma_n^{\ell m}(J) = \alpha_n^{\ell m} + i\omega_n^{\ell m}$

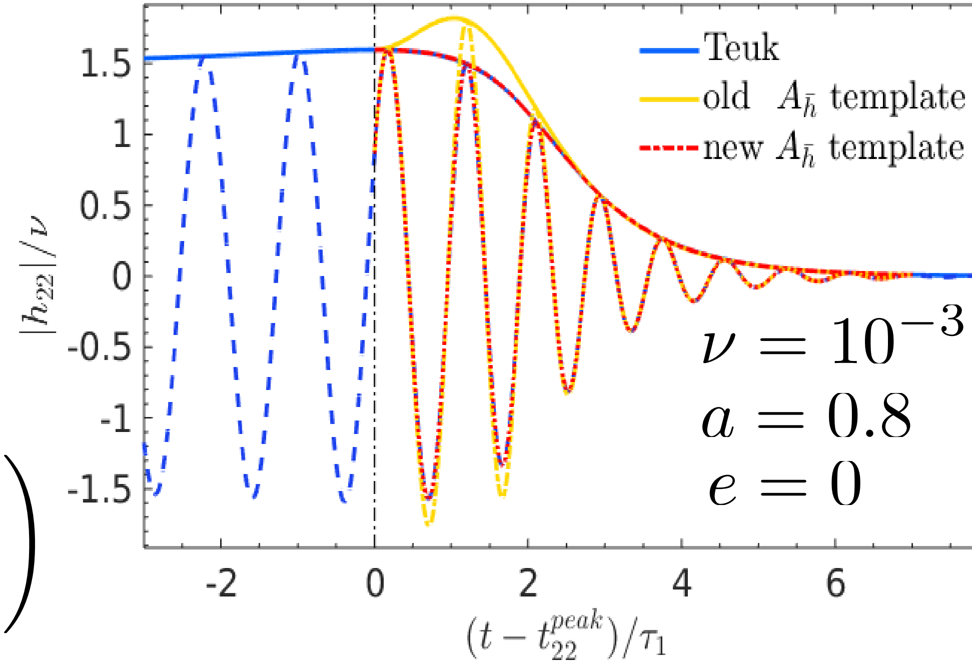
Multiplicative decompositon (for each multipole):

$$h(\tau) = e^{-\omega_1\tau - i\phi_{\ell m}^{\rm mrg}}\bar{h}(\tau)$$

$$\bar{h}(\tau) = A_{\bar{h}}(\tau)e^{i\phi_{\bar{h}}(\tau)}$$

$$A_{\bar{h}}(\tau) = \left(\frac{c_1^A}{1+e^{-c_2^A\tau + c_3^A}} + c_4^A\right)^{\frac{1}{c_5^A}}$$

$$\phi_{\bar{h}}(\tau) = -c_1^\phi \ln\left(\frac{1 + c_3^\phi e^{-c_2^\phi\tau} + c_4^\phi e^{-2c_2^\phi\tau}}{1 + c_3^\phi + c_4^\phi}\right)$$



Fits on the parameter space (spin and eccentricity) of the parameters obtained with the primary fits
→ analytical description of the ringdown (global fit)

Matching the inspiral+plunge solution to the ringdown:

Next-to-Quasi-Circular corrections

$$h_{\ell m} = h_{\ell m}^{\rm inspl} \hat{h}_{\ell m}^{\rm NQC}$$

$$\hat{h}_{\ell m}^{\rm NQC} = (1 + a_1 n_1 + a_2 n_2 + a_3 n_3) e^{i(b_1 m_1 + b_2 m_2 + b_3 m_3)}$$

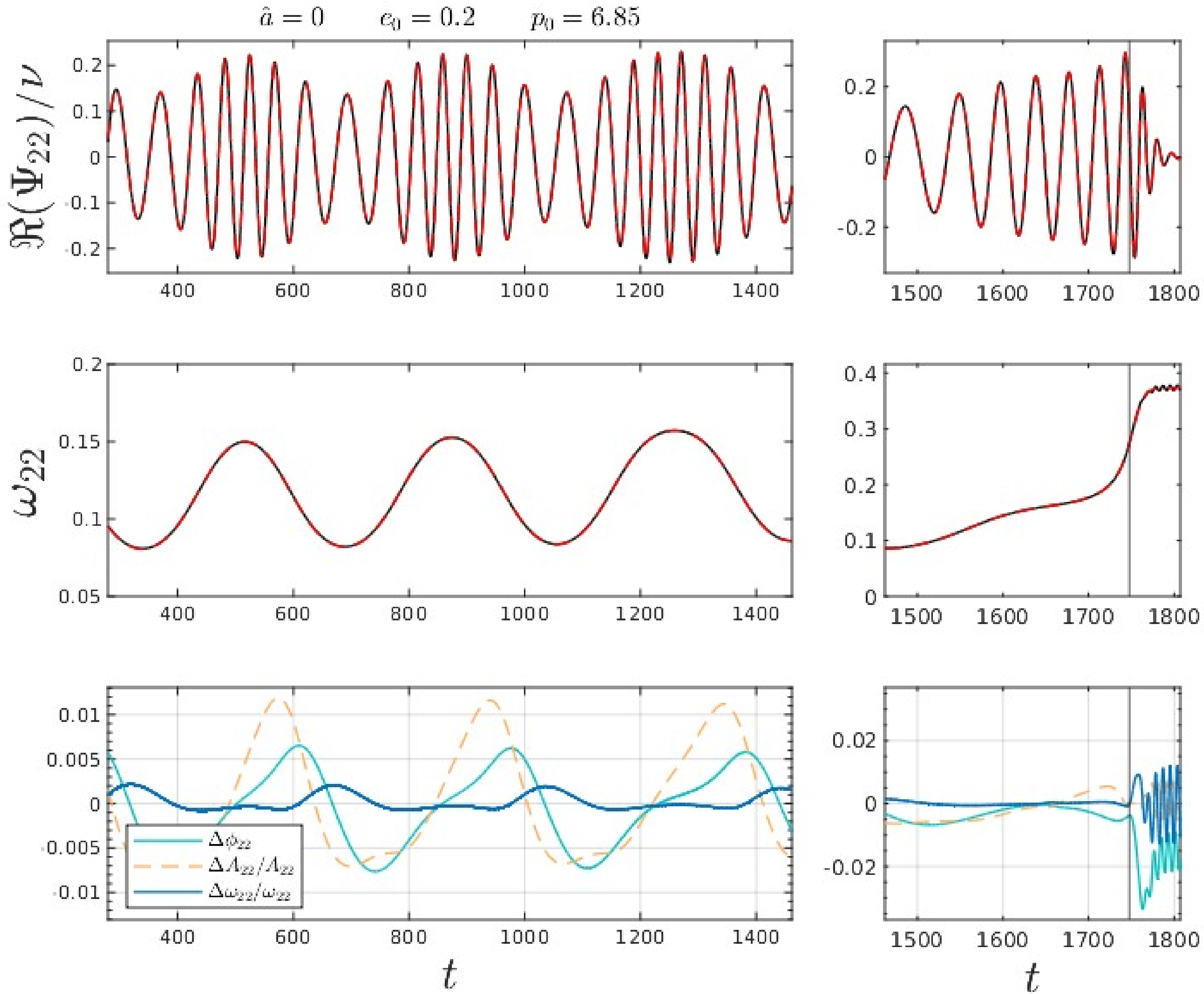
$$n_1 = \frac{p_{r_*}^2}{r^2 \Omega^2} \quad n_2 = \frac{\ddot{r}}{r \Omega^2} \quad n_3 = n_1 p_{r_*}^2$$

$$m_1 = \frac{p_{r_*}}{r \Omega} \quad m_2 = m_1 r^2 \Omega^2 \quad m_3 = m_2 p_{r_*}^2$$

NQC base

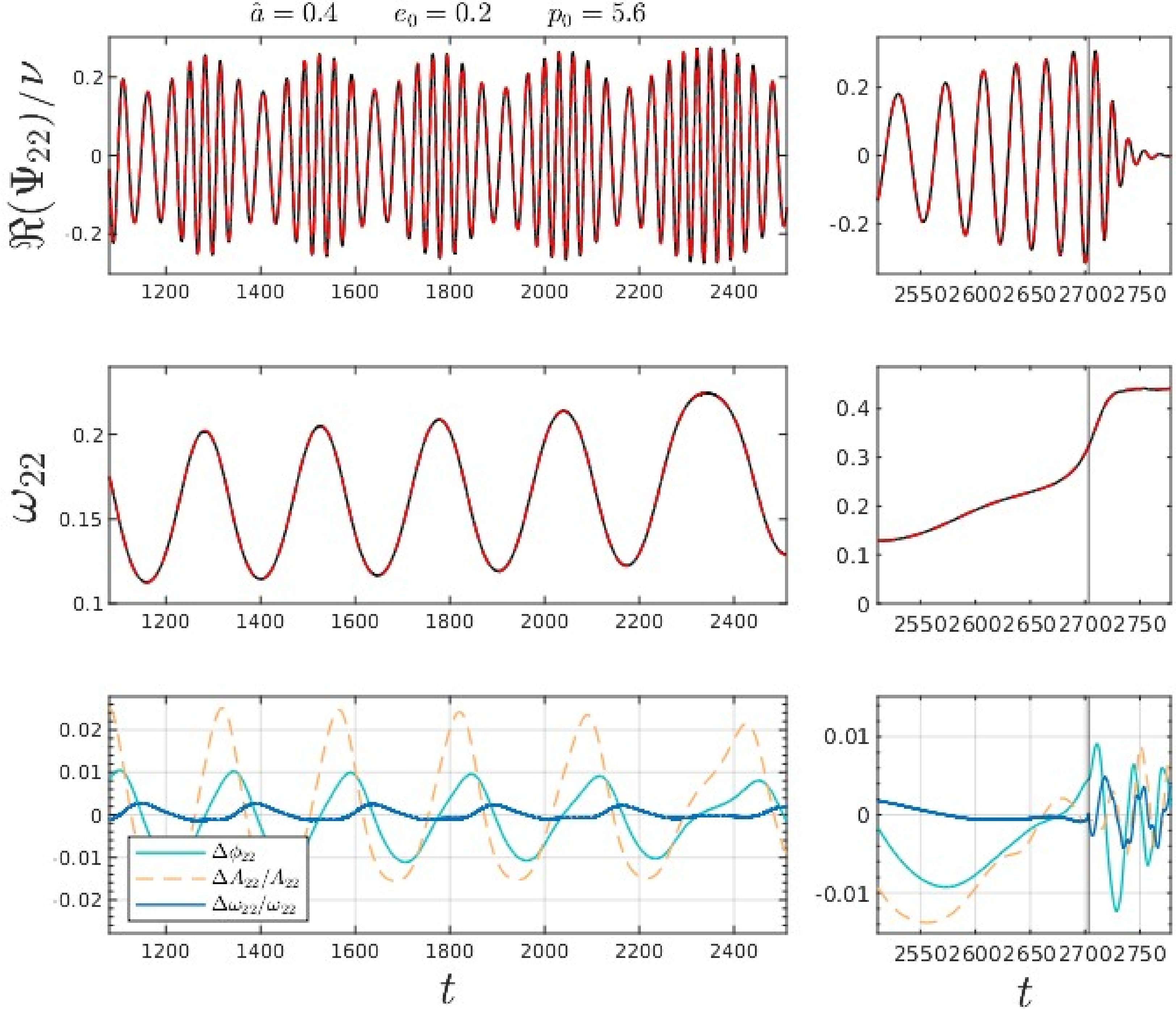
$a_i, \ b_j$ extracted from the ringdown

Complete quadrupolar waveform: nonspinning case



Also (3,3), (4,4), (2,1), (5,5) already avaliable

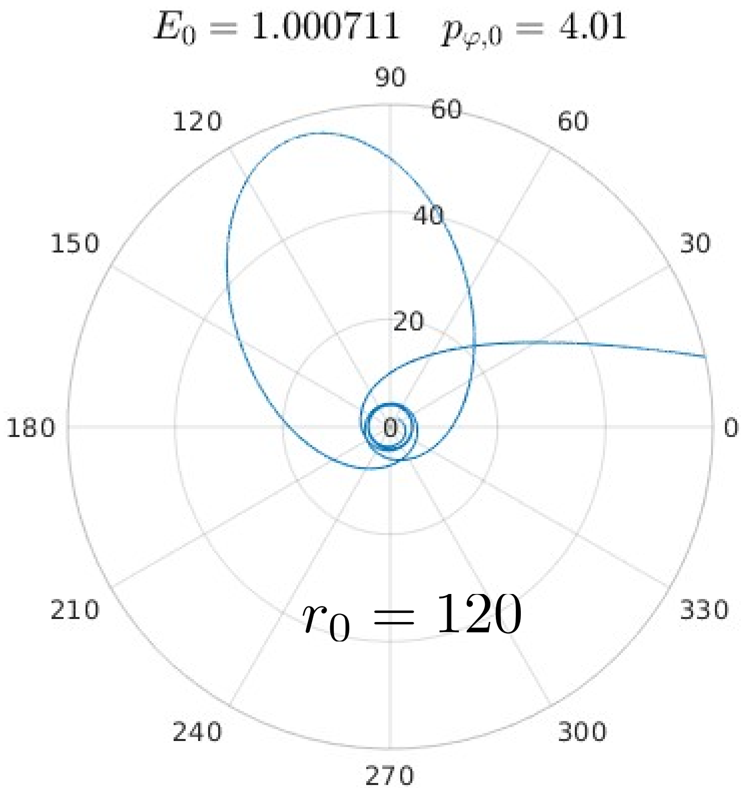
Complete quadrupolar waveform: spinning case

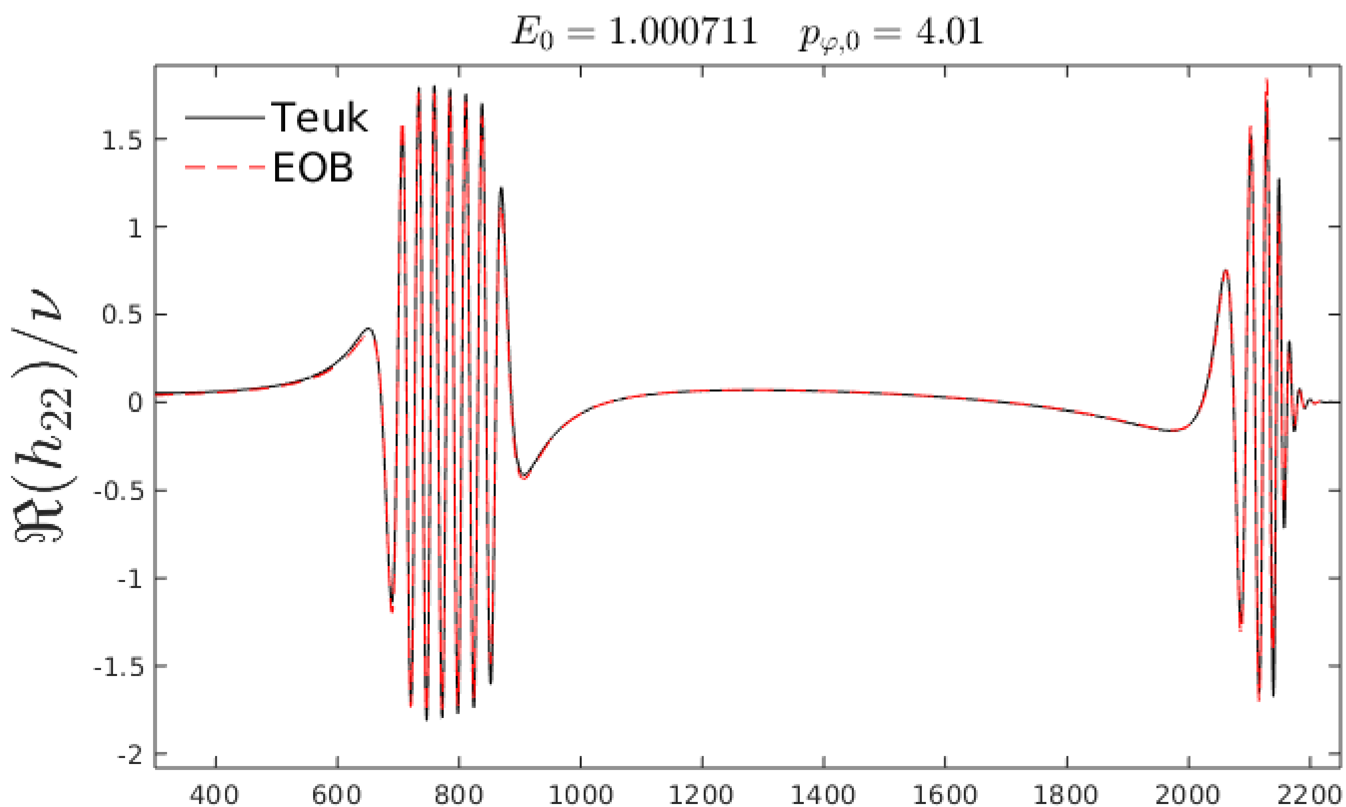


Also (3,3), (4,4), (2,1), (5,5) already avaliable

Also dynamical captures (hyperbolic)

$$\nu = 10^{-2} \quad \hat{a} = 0$$





For the comparable-masses case:

- **TEOBResumS**, see arXiv:2101.08624, arXiv:2101.08624
- Some full GR numerical simulations exist, but others will be avaliable! HYNTRHIGUE project: we will run **GR-ATHENA++** on Hawk (HRLS, Stut the next months (GR-ATHENA++ paper: arXiv:2101.08289)

Conclusions

The EOB model is a promising analytical framework also for EMRIs, but further improvements are needed:

- Noncircular information beyond Newtonian order to include in waveform and radiation reaction (2 PN terms already avaliable)
- Gravitational Self-Force effects to be included in the conservative part of the dynamics
- [distant future] Generalize for non-equatorial orbits

Thanks for your attention!